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CLASSIFICATION OF THE POSETS OF MINMAX TYPES WHICH ARE SYMMETRIC OVERSUPERCRITICAL POSETS OF THE EIGHTH ORDER

Classification of the posets which are closely related (with respect to their Tits quadratic forms) to generalizations of the critical and supercritical posets which at first appeared in Kleiner's and Nazarova's criteria on representation types of posets is given. These criteria were first in a new representation theory which was initiated by L. O. Nazarova and A. V. Roiter in 1972. The method of minimax isomorphism (introduced by the first author) plays the main role in presented investigation.

Key words*: Hasse diagram, Tits quadratic form, critical, supercritical and oversupercritical posets, minmax type, coefficient of transitivity, nodal element, dense subposet.*

Introduction. Through this paper, all posets are finite with strict order relations $\prec$ and without elements denoted by 0 and they are identified with their Hasse diagrams. All considered subposets are assumed to be full and one-element subsets are identified with their elements. For fixed posets A, B, the phrases «A is included in B», «B contains a subposet of the form A», etc. mean that there is a subposet X of B isomorphic to A.

The direct sum of posets $S_1,\dots,S_k$, $k>1$ (i.e. pairwise disjoint union without comparable elements between them), is denoted by $S_1 \coprod S_2 \coprod \dots \coprod S_k$ or $(S_1,S_2,\dots,S_k)$. If S_i is a chain of length m (equivalently, a linearly ordered set of order m) then we often write m instead of S_i. A subposet A of a poset B which is its direct summand is called *isolated* (in B).

In the representation theory of posets (initiated by L. O. Nazarova and A. V. Roiter in [8]), the first criteria were criteria for posets to be of finite and tame representation types obtained respectively by M. M. Kleiner [6] and L. O. Nazarova [7].

Theorem 1. *A poset S is of finite representation type over a field K if and only if it does not contain subposets of the form* $K_1=(1,1,1,1)$, $K_2=(2,2,2)$, $K_3=(1,3,3)$, $K_4=(4,\mathcal{N})$ *and* $K_5=(1,2,5)$.

Theorem 2. *A poset S is of tame representation type over a field K if and only if it does not contain subposets of the form* $N_1=(1,1,1,1,1)$, $N_2=(1,1,1,2)$, $N_3=(2,2,3)$, $N_4=(1,3,4)$, $N_5=(5,\mathcal{N})$ *and* $N_6=(1,2,6)$.

Here $\mathcal{N}$ is the Hasse diagram of the poset with elements 1, 2, 3, 4 and relations $1\prec 2$, $3\prec 4$, $1\prec 4$.

In what follows, we will call $K_1,K_2,\dots,K_5$ and $N_1,N_2,\dots,N_6$ *critical* and *supercritical* posets, respectively.

The first author suggested to consider a family of (pairwise nonisomorphic) posets that differ from supercritical posets to the same extent as supercritical differ from critical ones, and to call these posets *1-oversupercritical* or simply *oversupercritical*. In more detail [1], if we take all five critical posets and consider all their one-element extensions such that either the new element is isolated or it forms a new isolated chain together with the elements of some old isolated chain, and then choose all minimal posets of the obtained class of posets (with respect to the full inclusion of posets $\subseteq$), then we obtain

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as a result all supercritical posets. If the same procedure is already done with the supercritical posets, then the resulting posets (which are called oversupercritical) will be the following:

$$B_1 = (1,1,1,1,1,1), \quad B_2 = (1,1,1,1,2), \quad B_3 = (1,1,2,2),$$

$$B_4 = (1,1,1,3), \quad B_5 = (2,3,3), \quad B_6 = (2,2,4), \quad B_7 = (1,4,4),$$

$$B_8 = (1,3,5), \quad B_9 = (1,2,7), \quad B_{10} = (6,\mathcal{N}).$$

Recall that a poset T is called *dual* to a poset S and is denoted by S^{op} if $T = S$ as usual sets and $x \prec y$ in T if and only if $x \succ y$ in S. We have from the above that all critical, supercritical and oversupercritical posets are self-dual.

In a number of previous papers we classified the posets which are closely related (with respect to their Tits quadratic forms) to oversupercritical ones with the group of automorphisms of order not equal to 2, and also studied their combinatorial properties.

In this paper we continue our investigations using the method of minimax isomorphism [9] as the main method.

1. Minimax isomorphism of posets. Let S be a poset, a be its minimal or maximal element and $T = \vec{S}_a$ denote the following new poset: $T = S$ as usual sets, $T \backslash a = S \backslash a$ as posets, the element a is, respectively, maximal or minimal element in T (i.e. vice versa), and a is comparable with another element in T if and only if they are incomparable in S.

A poset $\overline{S}$ is called *minimax equivalent* to a poset S, if it is obtained from S by successive application of a finite number m of such operations (the minimax equivalence is an equivalence, since the reflexivity follows from the case $m = 0$ and the symmetry follows from the fact that $T = \vec{S}_a$ implies $S = \vec{T}_a$). Then each poset T isomorphic to $\overline{S}$ is called *minimax isomorphic* to S.

For example, the posets T of the forms (Fig. 1, Fig. 2)

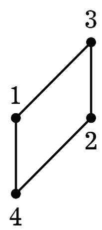


Fig. 1

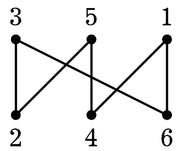


Fig. 2

are, respectively, minimax equivalent to the posets S of the forms (see Fig. 3, Fig. 4).

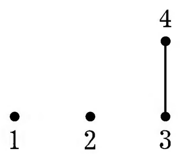


Fig. 3

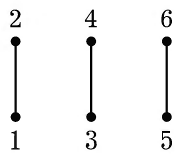


Fig. 4

In the first case the operations must be applied to the minimal point 3 of S and the maximal point 4 of $\vec{S}_3$, and in the second case to the minimal points 1, 3, 5 of S, $\vec{S}_1$, $\vec{U}_3$, respectively, with $U = \vec{S}_1$.

The main motivation for introducing [9] the notion of minimax isomorphism was the fact that the Tits quadratic form $qs(z)$ of a poset S given, by definition, by the equality

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{\substack{i<j \\ i,j \in S}} z_i z_j - z_0 \sum_{i \in S} z_i$$

is $\mathbb{Z}$-equivalent to the Tits quadratic form of any poset T which is minimax isomorphic to S.

The concept of minimax isomorphism is used by the authors to solve many problems.

Let's give two examples.

A poset S is called *positive* (respectively, *non-negative*) if so is its Tits quadratic form.

We call a poset P-*critical* (respectively, NP-*critical*) if it is not positive (respectively, non-negative), but all its proper subposets already are (an empty poset is positive since $q_z(z) = z_0^2$).

Theorem 3 [2]. *A poset S is P-critical if and only if it is minimax isomorphic to some critical poset.*

Theorem 4 [3]. *A poset S is NP-critical if and only if it is minimax isomorphic to some supercritical poset.*

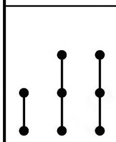
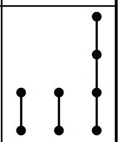
These theorems allowed the authors to describe all P-critical and NP-critical posets. Up to isomorphism and duality, their number is equal to 75 (see [2] or [18]) and 115 (see [4]), respectively.

An efficient algorithm for describing all posets that are minimax isomorphic to a given one is presented in Section 3.

2. Main result. Theorems 3 and 4, as well as other results by the authors (see, e.g., the description of the positive posets [2]) show the importance of the problem on this topic describing the posets that are minimax isomorphic to posets from a natural class. For a fixed poset P, we call a poset S of *minimax type* (briefly, *MM-type*) P if S is minimax isomorphic to P [14]. In this paper, continuing research of [1, 14, 15] (for $P = B_8, B_9$), [13] (for $P = B_1, B_2, B_3, B_4$), [16] (for $P = B_{10}$) we consider the case of oversupercritical posets B_i of order 8 that are symmetric (i.e. have automorphism of order 2). Such are the posets $B_5 = (2,3,3)$ and $B_6 = (2,2,4)$.

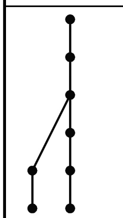
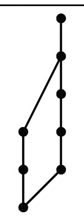
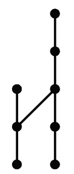
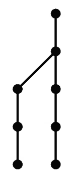
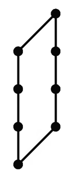
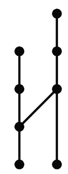
For formal reasons we denote the posets B_5 and B_6 by E_0 and F_0, respectively:

Table 1

E_0	F_0
	

Theorem 5. *Up to isomorphism and duality, the complete set of posets of MM-types E_0, F_0 consist of, in addition to E_0, F_0 themselves, respectively, the posets indicated in the following tables*:

Table 2

E_1	E_2	E_3	E_4	E_5	E_6
					

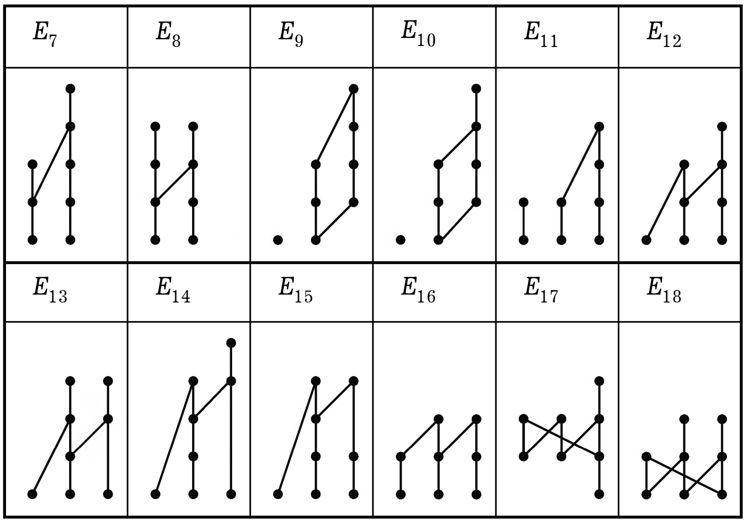
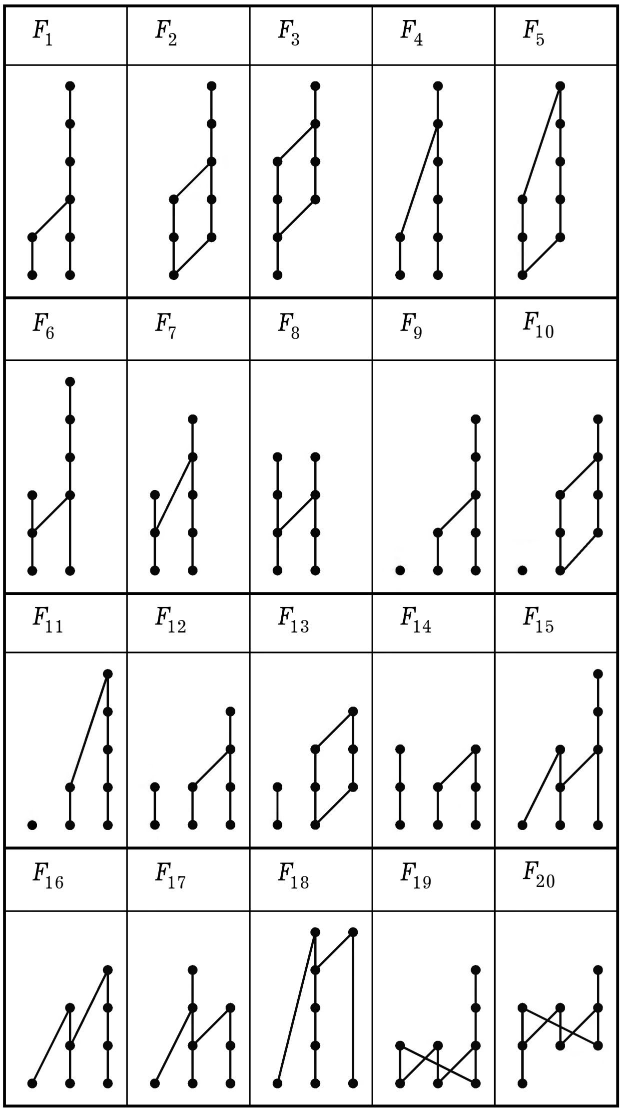

E_7 E_8 E_9 E_{10} E_{11} E_{12}
E_{13} E_{14} E_{15} E_{16} E_{17} E_{18}

Table 3


F_1 F_2 F_3 F_4 F_5
F_6 F_7 F_8 F_9 F_{10}
F_{11} F_{12} F_{13} F_{14} F_{15}
F_{16} F_{17} F_{18} F_{19} F_{20}

3. Proof of Theorem 5. We denote the poset $\vec{S}_a$ for a minimal (respectively, maximal) element of a poset S (see Section 1) by $S_a^{\uparrow}$ (respectively, $S_a^{\downarrow}$). In this definition an element a can be replaced by a lower (respectively, upper) subposet A, i.e. such A that $x \in A$ whenever $x \prec y$ (respectively, $x \succ y$) and $y \in A$. In more detail, the poset $T = S_A^{\uparrow}$ (respectively, $T = S_A^{\downarrow}$) is defined as follows:

1°) $T = S$ as usual sets;

2°) partial orders on A and $S \setminus a$ are the same as before;

3°) comparability and incomparability between elements $x \in A$ and $y \in S \setminus A$ are interchanged, and the new comparability can only be of the form $x \succ y$ (respectively, $x \prec y$).

Two subposets X and X' of a poset S are called *strongly isomorphic* if there exists an automorphism φ of S such that $\varphi(X) = X'$. Similarly, pairs (Y, X) and (Y', X') of subposets of S are called *strongly isomorphic* if $\varphi(Y) = Y'$ and $\varphi(X) = X'$ for some automorphism φ. The inequality $X \prec Y$ for subposets X, Y of S means that $x \prec y$ for any $x \in X$, $y \in Y$.

In [2], the authors proposed the following algorithm for finding (up to isomorphism) all posets that are minimax isomorphic to a given poset S.

I. Describe, up to strong isomorphism, all lower subposets $P \neq S$ in S, and for each of them, construct the poset $S_P^{\uparrow}$ ($P = \varnothing$ is not excluded).

II. Describe, up to strong isomorphism, all pairs (Q, P) consisting of a proper lower subposet Q in S and a nonempty lower subposet P in Q such that $P \prec S \setminus Q$, and for every such pair, construct the poset $(S_Q^{\uparrow})_P^{\uparrow}$.

III. Among the posets obtained in **I** and **II**, choose one from each class of isomorphic posets.

Now we apply this three-step algorithm to the posets E_0 and F_0.

For each posets E_0, F_0 (see the Table 1), we number the points with $1, 2, \ldots, 8$ in such a way that $i < j$ whenever $i \prec j$ or i is to the left of j. Then $1 \prec 2$, $3 \prec 4 \prec 5$, $6 \prec 7 \prec 8$ for E_0 and $1 \prec 2$, $3 \prec 4$, $5 \prec 6 \prec 7 \prec 8$ for F_0.

Step I. First, we describe (up to strong isomorphism) all lower subposets. They are the following

for E_0:

$X_0 = \varnothing$, $X_1 = \{1\}$, $X_2 = \{3\}$,

$X_3 = \{1, 2\}$, $X_4 = \{1, 3\}$, $X_5 = \{3, 4\}$, $X_6 = \{3, 6\}$,

$X_7 = \{1, 2, 3\}$, $X_8 = \{1, 3, 4\}$, $X_9 = \{1, 3, 6\}$,

$X_{10} = \{3, 4, 5\}$, $X_{11} = \{3, 4, 6\}$,

$X_{12} = \{1, 2, 3, 4\}$, $X_{13} = \{1, 2, 3, 6\}$,

$X_{14} = \{1, 3, 4, 5\}$, $X_{15} = \{1, 3, 4, 6\}$,

$X_{16} = \{3, 4, 5, 6\}$, $X_{17} = \{3, 4, 6, 7\}$,

$X_{18} = \{1, 2, 3, 4, 5\}$, $X_{19} = \{1, 2, 3, 4, 6\}$, $X_{20} = \{1, 3, 4, 5, 6\}$,

$X_{21} = \{1,3,4,6,7\}$, $X_{22} = \{3,4,5,6,7\}$,

$X_{23} = \{1,2,3,4,5,6\}$, $X_{24} = \{1,2,3,4,6,7\}$,

$X_{25} = \{1,3,4,5,6,7\}$, $X_{26} = \{3,4,5,6,7,8\}$,

$X_{27} = \{1,2,3,4,5,6,7\}$, $X_{28} = \{1,3,4,5,6,7,8\}$;

for F_0:

$Y_0 = \varnothing$, $Y_1 = \{1\}$, $Y_2 = \{5\}$,

$Y_3 = \{1,2\}$, $Y_4 = \{1,3\}$, $Y_5 = \{1,5\}$,

$Y_6 = \{5,6\}$, $Y_7 = \{1,2,3\}$, $Y_8 = \{1,2,5\}$,

$Y_9 = \{1,3,5\}$, $Y_{10} = \{1,5,6\}$, $Y_{11} = \{5,6,7\}$,

$Y_{12} = \{1,2,3,4\}$, $Y_{13} = \{1,2,3,5\}$, $Y_{14} = \{1,2,5,6\}$,

$Y_{15} = \{1,3,5,6\}$, $Y_{16} = \{1,5,6,7\}$, $Y_{17} = \{5,6,7,8\}$,

$Y_{18} = \{1,2,3,4,5\}$, $Y_{19} = \{1,2,3,5,6\}$, $Y_{20} = \{1,2,5,6,7\}$,

$Y_{21} = \{1,3,5,6,7\}$, $Y_{22} = \{1,5,6,7,8\}$,

$Y_{23} = \{1,2,3,4,5,6\}$, $Y_{24} = \{1,2,3,5,6,7\}$,

$Y_{25} = \{1,2,5,6,7,8\}$, $Y_{26} = \{1,3,5,6,7,8\}$,

$Y_{27} = \{1,2,3,4,5,6,7\}$, $Y_{28} = \{1,2,3,5,6,7,8\}$.

Let's denote by $K_{1,j}$ (respectively, $K_{2,j}$), where $1 \le j \le 28$ the poset $S_V^{\uparrow}$ for $S = E_0$, $V = X_j$ (respectively, $S = F_0$, $V = Y_j$):

$K_{1,0} \cong E_0$, $K_{1,1} \cong E_{10}$, $K_{1,2} \cong E_{11}$, $K_{1,3} \cong E_4$, $K_{1,4} \cong E_{15}$,

$K_{1,5} \cong E_8$, $K_{1,6} \cong E_{16}$, $K_{1,7} \cong E_6^{\mathrm{op}}$, $K_{1,8} \cong E_{14}$, $K_{1,9} \cong E_{18}^{\mathrm{op}}$,

$K_{1,10} \cong E_1$, $K_{1,11} \cong E_{12}$, $K_{1,12} \cong E_3^{\mathrm{op}}$, $K_{1,13} \cong E_{13}^{\mathrm{op}}$,

$K_{1,14} \cong E_7$, $K_{1,15} \cong E_{17}$, $K_{1,16} \cong E_3$, $K_{1,17} \cong E_{13}$,

$K_{1,18} \cong E_1^{\mathrm{op}}$, $K_{1,19} \cong E_{12}^{\mathrm{op}}$, $K_{1,20} \cong E_{14}^{\mathrm{op}}$, $K_{1,21} \cong E_{18}$

$K_{1,22} \cong E_6$, $K_{1,23} \cong E_8^{\mathrm{op}}$, $K_{1,24} \cong E_{16}^{\mathrm{op}}$, $K_{1,25} \cong E_{15}^{\mathrm{op}}$,

$K_{1,26} \cong E_4^{\mathrm{op}}$, $K_{1,27} \cong E_{11}^{\mathrm{op}}$, $K_{1,28} \cong E_{10}^{\mathrm{op}}$;

$K_{2,0} \cong F_0$, $K_{2,1} \cong F_{11}$, $K_{2,2} \cong F_{14}$, $K_{2,3} \cong F_4$, $K_{2,4} \cong F_{18}$,

$K_{2,5} \cong F_{16}$, $K_{2,6} \cong F_{12}$, $K_{2,7} \cong F_6^{\mathrm{op}}$, $K_{2,8} \cong F_7$, $K_{2,9} \cong F_{19}^{\mathrm{op}}$,

$K_{2,10} \cong F_{17}$, $K_{2,11} \cong F_9$, $K_{2,12} \cong F_1^{\mathrm{op}}$, $K_{2,13} \cong F_{15}^{\mathrm{op}}$, $K_{2,14} \cong F_8$,

$K_{2,15} \cong F_{20}$, $K_{2,16} \cong F_{15}$, $K_{2,17} \cong F_1$, $K_{2,18} \cong F_9^{\mathrm{op}}$, $K_{2,19} \cong F_{17}^{\mathrm{op}}$,

$$K_{2,20} \cong F_7^{\mathrm{op}},\ \ K_{2,21} \cong F_{19},\ \ K_{2,22} \cong F_6,$$

$$K_{2,23} \cong F_{12}^{\mathrm{op}},\ \ K_{2,24} \cong F_{16}^{\mathrm{op}},\ \ K_{2,25} \cong F_4^{\mathrm{op}},$$

$$K_{2,26} \cong F_{18}^{\mathrm{op}},\ \ K_{2,27} \cong F_{14}^{\mathrm{op}},\ \ K_{2,28} \cong F_{11}^{\mathrm{op}}.$$

Step II. Describe (up to strong isomorphism) all pairs of lower subposets (see the algorithm). They are the following

for E_0:

$$X_1' = (X_{23}, \{6\}),\ \ X_2' = (X_{27}, \{6\}),\ \ X_3' = (X_{27}, \{6,7\}),\ \ X_4' = (X_{28}, \{1\});$$

for F_0:

$$Y_1' = (Y_{18}, \{5\}),\ \ Y_2' = (Y_{23}, \{5\}),\ \ Y_3' = (Y_{23}, \{5,6\}),\ \ Y_4' = (Y_{27}, \{5\}),$$

$$Y_5' = (Y_{27}, \{5,6\}),\ \ Y_6' = (Y_{27}, \{5,6,7\}),\ \ Y_7' = (Y_{28}, \{3\}).$$

Let's denote by $K_{1,j}'$ the poset $(S_V^{\uparrow})_W^{\uparrow}$ for $S = E_0$, $(V, W) = X_j'$ and by $K_{2,j}'$ the poset $(S_V^{\uparrow})_W^{\uparrow}$ for $S = F_0$, $(V, W) = Y_j'$:

$$K_{1,1}' \cong E_2^{\mathrm{op}},\ \ K_{1,2}' \cong E_9,\ \ K_{1,3}' \cong E_2,\ \ K_{1,4}' \cong E_5,$$

$$K_{2,1}' \cong F_2^{\mathrm{op}},\ \ K_{2,2}' \cong F_{10}^{\mathrm{op}},\ \ K_{2,3}' \cong F_3,\ \ K_{2,4}' \cong F_{13},$$

$$K_{2,5}' \cong F_{10},\ \ K_{2,6}' \cong F_2,\ \ K_{2,7}' \cong F_5.$$

Step III. It is easy to verify that in **I** and **II** each of the posets E_i, F_i, indicated in Tables 2 and 3 in the formulation of Theorem 5 or dual to it (in the non-dual case) occurs only once.

Thus Theorem 5 is proved. ♦

4. Applications. Posets arise in the study of various problems in mathematics and its applications. Among such problems an important role is played by combinatorial ones associated with the study of various parameters (see, e.g., [5, 11, 12, 19–30]). The present section is devoted to the analysis of some combinatorial properties of the posets of *MM*-types E_0 and F_0, namely, the calculation of their coefficient of transitivity; such properties, for some other classes of posets, were studied by the authors earlier in [13–16].

For a poset S, let $S_{\prec}^2 := \{(x,y) \mid x, y \in S,\ x \prec y\}$. Elements x and y of S are called *neighboring*, if $(x,y) \in S_{\prec}^2$ and there is no z satisfying $x \prec z \prec$ $\prec y$. Put $n_w = n_w(S) := |S_{\prec}^2|$ and denote by $n_e = n_e(S)$ the number of pairs (x,y) of neighboring elements of S. On the language of the Hasse diagram of S, n_e is equal to the number of all its edges and n_w is equal to the number of all its paths, up to parallelism, going from bottom to top (two paths are called *parallel* if they start and end at the same points). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w is called the *coefficient of transitivity of* S; for $n_w = 0$, it is assumed that $k_t = 0$ [10]. Obviously, dual posets have the same coefficient of transitivity.

It is clear that the coefficient of transitivity of S is the probability that its comparable elements are not neighboring.

The main result of this section is the following theorem.

Theorem 6. *The following holds for posets* E_i, F_j *shown in Tables* 4, 5 *and* 6. *The coefficients of transitivity* k_t *are calculated up to the fifth decimal place. If the number of decimal places is less than five, then the decimal fraction is finite, and if it is five, then infinite. If two decimal fractions are equal up to five digits, then they are exactly equal.*

Table 4

N	n_e	n_w	k_t
E_0	5	7	0.28471
F_0	5	8	0.375

Table 5

N	n_e	n_w	k_t	N	n_e	n_w	k_t
E_1	7	22	0.68182	E_{10}	6	12	0.5
E_2	8	22	0.63636	E_{11}	6	10	0.4
E_3	7	19	0.63158	E_{12}	7	14	0.5
E_4	7	19	0.63158	E_{13}	7	15	0.53333
E_5	8	19	0.57895	E_{14}	7	16	0.5625
E_6	7	18	0.61111	E_{15}	7	13	0.46154
E_7	7	21	0.66667	E_{16}	7	11	0.36364
E_8	6	15	0.6	E_{17}	8	13	0.38462
E_9	7	15	0.53333	E_{18}	8	12	0.33333

Table 6

N	n_e	n_w	k_t	N	n_e	n_w	k_t
F_1	7	24	0.70833	F_{11}	6	13	0.53846
F_2	8	24	0.66667	F_{12}	6	12	0.5
F_3	8	24	0.66667	F_{13}	7	12	0.41667
F_4	7	20	0.65	F_{14}	6	9	0.33333
F_5	8	20	0.6	F_{15}	7	16	0.5625
F_6	7	21	0.66667	F_{16}	7	12	0.41667
F_7	7	17	0.58824	F_{17}	7	13	0.46154
F_8	7	16	0.5625	F_{18}	7	16	0.5625
F_9	6	17	0.64706	F_{19}	8	13	0.38462
F_{10}	7	17	0.58824	F_{20}	8	12	0.33333

P r o o f. For proving Theorem 6 we need some lemmas [17].

We will represent a poset as a direct sum $\coprod_i^k X_i$ of chains X_i, $i = 2,\dots,k$, with additional relations $y \prec z$ for y and z belonging to the different chains (this is possible due to Dilworth's theorem). The chains $a_1 \prec \dots \prec a_s$, $b_1 \prec \dots \prec b_s$, $c_1 \prec \dots \prec c_s$ are denoted by A_s, B_s, C_s, respectively.

Lemma 1. *Let* $S = S_1 \coprod S_2$. *Then*

(**a**) $n_e(S) = n_e(S_1) + n_e(S_2)$;

(**b**) $n_w(S) = n_w(S_1) + n_w(S_2)$.

Lemma 2. *Let* $S = A_m$. *Then*

(**a**) $n_e(S) = m - 1$;

(**b**) $n_w(S) = \frac{(m-1)m}{2}$.

Lemma 3. *Let* $S = \{A_m \coprod B_n,\ a_i \prec b_j\}$. *Then*

(**a**) $n_e(S) = m + n - 1$;

(**b**) $n_w(S) = \frac{(m-1)m + (n-1)n}{2} + i(n - j + 1)$.

Lemma 4. *Let* $S = \{A_m \coprod B_n,\ a_i \prec b_j,\ a_{i'} \prec b_{j'}\}$, *where* $i < i'$, $j < j'$. *Then*

(**a**) $n_e(S) = m + n$;

(**b**) $n_w(S) = \frac{(m-1)m + (n-1)n}{2} + i'(n - j' + 1) + i(j' - j)$.

Lemma 5. *Let* $S = \{A_m \coprod B_n \coprod C_s,\ a_i \prec b_j,\ b_{j'} \prec c_k\}$, *where* $j > j'$. *Then*

(**a**) $n_e(S) = m + n + s - 1$;

(**b**) $n_w(S) = \frac{(m-1)m + (n-1)n + (s-1)s}{2} + i(n - j + 1) + j'(s - k + 1)$.

Now Theorem 6 can be easily proven by direct calculations using Lemmas 1, 2 for $N = E_0, F_0$, Lemma 3 for $N = E_1, E_3, E_4, E_6, E_7, F_1, F_4, F_6, F_7, F_8$, Lemmas 1–3 for $N = E_8, E_{10}, E_{11}, F_9, F_{11}, F_{12}, F_{14}$, Lemma 4 for $N = E_2, E_5, F_2, F_3, F_5$, Lemmas 1, 2, 4 for $N = E_9, F_{10}, F_{13}$, Lemma 5 for $N = E_{12}, E_{13}, E_{14}, E_{15}, E_{16}, F_{15}, F_{16}, F_{17}, F_{18}$ and direct calculations for $N = E_{17}, E_{18}, F_{19}, F_{20}$.

Theorem 6 is proved. ♦

An element of a poset is called *nodal*, if it is comparable with all other elements. A subposet X of S is said to be *nodal* if all its elements are nodal and *dense* if there is no $x_1, x_2 \in X$, $y \in S \setminus X$ such that $x_1 \prec y \prec x_2$.

Since dual posets have the same coefficient of transitivity, Theorem 6 implies the following corollary.

Corollary 1. *Among all posets of MM-type* E_0 (*respectively,* F_0), *there is, up to isomorphism and duality, only one with largest dense nodal subposet. Any other poset of MM-type* E_0 (*respectively,* F_0), *that is not isomorphism neither to it nor to its dual has a smaller coefficient of transitivity.*

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КЛАСИФІКАЦІЯ ЧАСТКОВО ВПОРЯДКОВАНИХ МНОЖИН, МІНІМАКСНИМ ТИПОМ ЯКИХ Є СИМЕТРИЧНІ НАДСУПЕРКРИТИЧНІ ЧАСТКОВО ВПОРЯДКОВАНІ МНОЖИНИ ПОРЯДКУ 8

Наведено класифікацію частково впорядкованих множин, що тісно пов'язані (*стосовно своїх квадратичних форм Тітса*) *з узагальненнями критичних і суперкритичних частково впорядкованих множин, які вперше появилися у критеріях Клейнера та Назарової стосовно зображувальних типів частково впорядкованих множин. Ці критерії були першими в новій теорії зображень, започаткованій Л. О. Назаровою та А. В. Ройтером у* 1972 *р. Метод мінімаксного ізоморфізму* (*запроваджений першим автором*) *відіграє основну роль у поданому дослідженні.*

Ключові слова: *діаграма Гассе, квадратична форма Тітса, критичні, надкритичні та супернадкритичні множини, мінімаксний тип, коефіцієнт транзитивності, вузловий елемент, щільна підмножина.*

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