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M. M. Sheremeta ✉

GEOMETRIC PROPERTIES OF LAPLACE – STIELTJES INTEGRALS

The concepts of pseudostarlikeness and pseudoconvexity are introduced for Laplace – Stieltjes integrals. The criteria for pseudostarlikeness and pseudoconvexity are proved and applied to the study of a neighborhood of a function and convolutions of functions.

Key words: *Laplace – Stieltjes integral, pseudostarlikeness, pseudoconvexity, neighborhood of a function, convolution of functions.*

Introduction. Let V be a class of non-negative non-decreasing unbounded right-continuous functions F on $[0,+\infty)$. We assume that a non-negative bounded function f on $[0,+\infty)$ is such that Lebesque – Stieltjes integral $\int_0^A f(x)e^{sx}\,dF(x)$ exits for every $s\in\mathbb{C}$ and $A\in[0,+\infty)$. The integral

$$I(s)=\int_0^\infty f(x)e^{sx}\,dF(x),\qquad s=\sigma+it\,,$$

is called [4, 5] a Laplace – Stieltjes integral. Since Laplace – Stieltjes integrals are direct generalization of the Laplace integral $\int_0^\infty f(x)e^{sx}\,dx$ and the Dirichlet series $\sum_{n=0}^\infty f_n e^{\sigma\lambda_n}$ with non-negative coefficients and exponents (choosing $F(x)=n(x)=\sum_{\lambda_n\le x}1$), the investigation of the properties of Laplace – Stieltjes integrals is necessary and actual.

Let σ_c be the abscissa of the absolute convergence of $I(s)$ such, that $I(\sigma)$ exits for every $\sigma<\sigma_c$. In [22, p. 13] it is proved that if either $\ln F(x)=$ $=o(x)$ or $\ln F(x)=o(\ln f(x))$ as $x\to+\infty$, then $\sigma_c\ge\alpha:=\varliminf_{x\to+\infty}\frac{1}{x}\ln\frac{1}{f(x)}$. On the other hand, if there exists $\lim_{x\to+\infty}\frac{1}{x}\ln\frac{1}{f(x)}=\alpha$, then $\sigma_c=\alpha$ [22, p. 19]. Hence it follows that if

$$\ln F(x)=o(x),\qquad \ln\frac{1}{f(x)}=o(x) \tag{1}$$

as $x\to+\infty$, then $\sigma_c=0$.

Let $\lambda>0$. By V_λ we denote a class of the functions $F\in V$ such that $F(x)=0$ on $[0,\lambda)$, $F(\lambda)>0$ and (1) holds. Then

$$I(s)=f(\lambda)F(\lambda)e^{s\lambda}+\int_\lambda^\infty f(x)e^{sx}\,dF(x)\,. \tag{2}$$

For $f(\lambda)>0$, we denote the class of functions of the form (2) by LS^+. By LS^-, we denote the class of functions of the form

✉ m.m.sheremeta@gmail.com

$$J(s) = f(\lambda)F(\lambda)e^{s\lambda} - \int_{\lambda}^{\infty} f(x)e^{sx}dF(x)\,. \tag{3}$$

We will call function (3) the spoilt Laplace – Stieltjes integral.

The purpose of proposed article is to study the geometric properties of the functions from the classes LS^+ and LS^-.

1. Conformity and non-univalence. The following theorem is true.

Theorem 1. *The functions* $I \in LS^+$ *and* $J \in LS^-$ *are non-univalent in* $\Pi_0 = \{s : \operatorname{Re} s < 0\}$. *If*

$$\int_{\lambda}^{\infty} xf(x)\,dF(x) \le \lambda f(\lambda)F(\lambda)\,, \tag{4}$$

then the functions I, J *are conformal in* Π_0.

P r o o f. We put $g(s) = \int_{\lambda}^{\infty} f(x)e^{sx}\,dF(x)$. Then for every $\varepsilon > 0$

$$\left|\frac{g(s)}{\exp\{s\lambda\}}\right| \le \int_{\lambda}^{\lambda+\varepsilon} f(x)e^{\sigma(x-\lambda)}\,dF(x) + \int_{\lambda+\varepsilon}^{\infty} f(x)e^{\sigma(x-\lambda)}\,dF(x)\,.$$

In view of the right continuity of F, for $\sigma < 0$ we have that

$$\int_{\lambda}^{\lambda+\varepsilon} f(x)e^{\sigma(x-\lambda)}\,dF(x) \le \int_{\lambda}^{\lambda+\varepsilon} f(x)\,dF(x) \le$$

$$\le \sup\{f(x) : \lambda \le x \le \lambda + \varepsilon\}\big(F(\lambda+\varepsilon) - F(\lambda)\big) \to 0$$

as $\varepsilon \to 0$ and

$$\int_{\lambda+\varepsilon}^{\infty} f(x)e^{\sigma(x-\lambda)}\,dF(x) \to 0$$

as $\sigma \to -\infty$. Thus, there exists $\sigma_0 \in (-\infty, 0)$ such that $\left|\frac{g(s)}{f(\lambda)F(\lambda)\exp\{s\lambda\}}\right| < \frac{1}{4}$ for all s, $\operatorname{Re} s < \sigma_0$. Let

$$\sigma_1 = \sigma_0 - \frac{\ln 2}{\lambda}, \qquad \sigma_2 = \sigma_1 - \frac{\ln 2}{\lambda}, \qquad w = f(\lambda)F(\lambda)\exp\{\sigma_1\lambda\}\,.$$

If $\operatorname{Re} s = \sigma_0$, then

$$|f(\lambda)F(\lambda)\exp\{s\lambda\} - w| \ge f(\lambda)F(\lambda)\exp\{\sigma_0\lambda\} - f(\lambda)F(\lambda)\exp\{\sigma_1\lambda\} =$$

$$= \frac{f(\lambda)F(\lambda)\exp\{\sigma_0\lambda\}}{2} > 2|g(s)|\,,$$

and if $\operatorname{Re} s = \sigma_2$, then

$$|f(\lambda)F(\lambda)\exp\{s\lambda\} - w| \ge f(\lambda)F(\lambda)\exp\{\sigma_1\lambda\} - f(\lambda)F(\lambda)\exp\{\sigma_2\lambda\} =$$

$$= \frac{f(\lambda)F(\lambda)\exp\{\sigma_1\lambda\}}{2} > 2|g(s)|\,.$$

If $\sigma_2 \le \sigma \le \sigma_0$ and either $s = \sigma + i\pi/\lambda$ or $s = \sigma + 3i\pi/\lambda$, then

$$|f(\lambda)F(\lambda)\exp\{s\lambda\} - w| = f(\lambda)F(\lambda)\exp\{\sigma\lambda\} + f(\lambda)F(\lambda)\exp\{\sigma_1\lambda\} \ge$$

$$\ge f(\lambda)F(\lambda)\exp\{\sigma\lambda\} > 4|g(s)|\,.$$

Hence it follows that on the sides of the rectangle R with the vertices

$$\sigma_2 + \frac{i\pi}{\lambda}, \qquad \sigma_0 + \frac{i\pi}{\lambda}, \qquad \sigma_0 + \frac{3i\pi}{\lambda}, \qquad \sigma_2 + \frac{3i\pi}{\lambda}$$

the inequality $|g(s)| < |f(\lambda)F(\lambda)\exp\{s\lambda\} - w|/2$ holds. Since $I(s) - w = f(\lambda)F(\lambda)\exp\{s\lambda\} - w + g(s)$ and $J(s) - w = f(\lambda)F(\lambda)\exp\{s\lambda\} - w - g(s)$, by Rouche's theorem the functions $I(s)$, $J(s)$ and $\exp\{s\lambda\}$ have the same number of w-points in the interior of R. But in the interior of R the function $\exp\{s\lambda\}$ has only one w-point $s = \sigma_1 + 2i\pi/\lambda$. Therefore, I and J have one w-point in the interior of R.

By analogy one can show that in the domain bounded by the rectangle with the vertices

$$\sigma_2 + \frac{3i\pi}{\lambda}, \qquad \sigma_0 + \frac{3i\pi}{\lambda}, \qquad \sigma_0 + \frac{5i\pi}{\lambda}, \qquad \sigma_2 + \frac{5i\pi}{\lambda}$$

the functions I and J have one w-point, and thus, I and J are non-univalent in Π_0.

Further, for I, we have

$$I'(s) = \lambda f(\lambda)F(\lambda)e^{s\lambda}\left(1 + \int_{\lambda}^{\infty} \frac{xf(x)}{\lambda f(\lambda)F(\lambda)} \exp\{s(x-\lambda)\}\,dF(x)\right).$$

Therefore, for all $s \in \Pi_0$, $I'(s) \neq 0$ if and only if

$$1 + \int_{\lambda}^{\infty} \frac{xf(x)}{\lambda f(\lambda)F(\lambda)} \exp\{s(x-\lambda)\}dF(x) \neq 0, \qquad s \in \Pi_0 .$$

From condition (4) it follows that for all $s \in \Pi_0$

$$\left|\int_{\lambda}^{\infty} \frac{xa(x)}{\lambda f(\lambda)F(\lambda)} \exp\{s(x-\lambda)\}\,dF(x)\right| \leq \int_{\lambda}^{\infty} \frac{xf(x)}{\lambda f(\lambda)F(\lambda)} \exp\{\sigma(x-\lambda)\}\,dF(x) <$$

$$\leq \int_{\lambda}^{\infty} \frac{xf(x)}{\lambda f(\lambda)F(\lambda)}\,dF(x) \leq 1,$$

i.e. the function I is conformal in Π_0. The conformity of J can be proved similarly. Theorem 1 is proved. ♦

2. Pseudostarlikeness and pceudoconvexity. An analytic function $f(z) = \sum_{n=0}^{\infty} f_n z^n$ univalent in $\mathbb{D} = \{z : |z| < 1\}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [2, p. 203] that the condition $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$, is necessary and sufficient for the convexity of f. It is clear that f is convex in $\mathbb{D}$ if and only if $g(z) = (f(z) - f_0)/f_1 = z + \sum_{n=2}^{\infty} g_n z^n$ is convex in $\mathbb{D}$. The function g is said to be starlike in $\mathbb{D}$ if $\operatorname{Re}\{zg'(z)/g(z)\} > 0$, $z \in \mathbb{D}$ [2, p. 202]. A. W. Goodman [12] (see also [23, p. 9]) proved that if $\sum_{n=2}^{\infty} n|g_n| \leq 1$, then function g is starlike. The concept of the starlikeness got the series of generalizations. I. S. Jack [16] studied starlike functions of order $\alpha \in [0,1)$, i.e. such functions g, for which $\operatorname{Re}\{zg'(z)/g(z)\} > \alpha$, $z \in \mathbb{D}$. It is proved [16], [23,

p. 13] that if $\sum_{n=2}^{\infty}(n-\alpha)|g_n| \le 1-\alpha$, then function (1) is a starlike function of order α. V. P. Gupta [13] introduced the concept of a starlike function of order $\alpha \in [0,1)$ and type $\beta \in (0,1]$. This is a function g satisfying the inequality $\left|\frac{zf'(z)}{f(z)}-1\right| \cdot \left|\frac{zf'(z)}{f(z)}+1-2\alpha\right|^{-1} < \beta$.

If we put $z = e^s$, $G(s) = g(e^s)$ and $\Psi(s) = f(e^s)$ then the functions G and Ψ are analytic in Π_0,

$$\operatorname{Re}\{zg'(z)/g(z)\} > 0, \quad z \in \mathbb{D}, \quad \Leftrightarrow \quad \operatorname{Re}\{G'(s)/G(s)\} > 0, \quad s \in \Pi_0,$$

and

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0, \quad z \in \mathbb{D}, \quad \Leftrightarrow \quad \operatorname{Re}\{\Psi''(s)/\Psi'(s)\} > 0, \quad s \in \Pi_0.$$

Therefore, as in [23, p. 137] and [1], we will call the function G conformal in Π_0 pseudostarlike if

$$\operatorname{Re}\{G'(s)/G(s)\} > 0, \qquad s \in \Pi_0, \tag{5}$$

and pseudoconvex if

$$\operatorname{Re}\{G''(s)/G'(s)\} > 0, \qquad s \in \Pi_0. \tag{6}$$

It is clear that G is pseudoconvex if and only if G' is pseudostarlike.

We say that a curve $C = \{w = w(t) : -\infty < t < +\infty\}$ has the convexity property if the tangent to C revolves in the positive direction and has the starlikeness property if $\arg w(t)$ increases on $(-\infty, +\infty)$. From the proofs of Propositions 8.1 and 8.2 in [23, p. 138], it can be seen that a function G conformal in Π_0 is pseudoconvex if and only if each curve

$$C(\sigma_0) = \{w = G(\sigma_0 + it) : -\infty < t < +\infty\}, \qquad -\infty < \sigma_0 < 0,$$

has the convexity property, and G is pseudostarlike if and only if each curve $C(\sigma_0)$, $-\infty < \sigma_0 < 0$, has the starlikeness property.

A function G conformal in Π_0 is said [24] to be pseudostarlike of the order $\alpha \in [0,1)$ if

$$\operatorname{Re}\{G'(s)/G(s)\} > \alpha, \qquad s \in \Pi_0. \tag{7}$$

Since for every $\lambda > 0$ the inequality $|w-\lambda| < |w-(2\alpha-\lambda)|$ holds if and only if $\operatorname{Re} w > \alpha$, a function G is pseudostarlike of the order α if and only if

$$\left|\frac{G'(s)}{G(s)} - \lambda\right| < \left|\frac{G'(s)}{G(s)} - (2\alpha - \lambda)\right|, \qquad s \in \Pi_0. \tag{8}$$

In view of (8) a function G conformal in Π_0 we call pseudostarlike of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1]$ if [24]

$$\left|\frac{G'(s)}{G(s)} - \lambda\right| < \beta\left|\frac{G'(s)}{G(s)} - (2\alpha - \lambda)\right|, \qquad s \in \Pi_0. \tag{9}$$

Similarly [24], a function G conformal in Π_0 is said to be pseudoconvex of the order $\alpha \in [0,1)$, if

$$\operatorname{Re}\{G''(s)/G'(s)\} > \alpha, \qquad s \in \Pi_0.$$

and is said to be pseudoconvex of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1]$ if

$$\left|\frac{G''(s)}{G'(s)}-\lambda\right|<\beta\left|\frac{G''(s)}{G'(s)}-(2\alpha-\lambda)\right|,\qquad s\in\Pi_0.$$

Theorem 2. *Let* $\lambda\ge 1$, $F\in V_\lambda$, $f(\lambda)>0$, $\alpha\in[0,1)$ *and* $\beta\in(0,1]$. *If*

$$\int_\lambda^\infty \{(1+\beta)x-2\beta\alpha-\lambda(1-\beta)\}f(x)\,dF(x)\le 2\beta(\lambda-\alpha)f(\lambda)F(\lambda), \tag{10}$$

then the function $I\in LS^+$ *is pseudostarlike of the order* α *and the type* β.

P r o o f. It is clear that for the function $G(s)=I(s)$ condition (9) holds if and only if

$$|I'(s)-\lambda I(s)|-\beta|I'(s)-(2\alpha-\lambda)I(s)|<0,\qquad s\in\Pi_0. \tag{11}$$

On the other hand, in view of (2),

$$|I'(s)-\lambda I(s)|-\beta|I'(s)-(2\alpha-\lambda)I(s)|=\Bigg|\lambda f(\lambda)F(\lambda)e^{s\lambda}+$$

$$+\int_\lambda^\infty xf(x)e^{sx}\,dF(x)-\lambda f(\lambda)F(\lambda)e^{s\lambda}-\lambda\int_\lambda^\infty f(x)e^{sx}\,dF(x)\Bigg|-$$

$$-\beta\Bigg|\lambda f(\lambda)F(\lambda)e^{s\lambda}+\int_\lambda^\infty xf(x)e^{sx}dF(x)-(2\alpha-\lambda)f(\lambda)F(\lambda)e^{s\lambda}-$$

$$-(2\alpha-\lambda)\int_\lambda^\infty f(x)e^{sx}dF(x)\Bigg|=\left|\int_\lambda^\infty (x-\lambda)f(x)e^{sx}dF(x)\right|-$$

$$-\beta\left|2(\lambda-\alpha)f(\lambda)F(\lambda)e^{s\lambda}+\int_\lambda^\infty (x-2\alpha+\lambda)f(x)e^{sx}dF(x)\right|$$

Since $-|a+b|\le -|a|+|b|$ and $\sigma<0$, hence, in view of (10), we get

$$|I'(s)-\lambda I(s)|-\beta|I'(s)-(2\alpha-\lambda)I(s)|\le\int_\lambda^\infty (x-\lambda)f(x)e^{\sigma x}\,dF(x)+$$

$$+\beta\int_\lambda^\infty (x-2\alpha+\lambda)f(x)e^{\sigma x}\,dF(x)-2\beta(\lambda-\alpha)f(\lambda)F(\lambda)e^{\sigma\lambda}=$$

$$=e^{\sigma\lambda}\Bigg(\int_\lambda^\infty [(1+\beta)x-2\beta\alpha-\lambda(1-\beta)]f(x)e^{\sigma(x-\lambda)}\,dF(x)-$$

$$-2\beta(\lambda-\alpha)f(\lambda)F(\lambda)\Bigg)<\int_\lambda^\infty [(1+\beta)x-2\beta\alpha-\lambda(1-\beta)]\times$$

$$\times a(x)\,dF(x)-2\beta(\lambda-\alpha)f(\lambda)F(\lambda)\le 0,$$

i. e. (11) holds. Theorem 2 is proved. ♦

Since I is pseudoconvex of the order α and the type β if and only if I' is pseudostarlike of the order α and the type β, Theorem 2 implies the following statement.

Corollary 1. *Let* $\lambda\ge 1$, $F\in V_\lambda$, $f(\lambda)>0$, $\alpha\in[0,1)$ *and* $\beta\in(0,1]$. *If*

$$\int_\lambda^\infty \{(1+\beta)x-2\beta\alpha-\lambda(1-\beta)\}xf(x)\,dF(x)\le 2\beta\lambda f(\lambda)F(\lambda)(\lambda-\alpha), \tag{12}$$

then the function $I\in LS^+$ *is pseudoconvex of the order* α *and the type* β.

Theorem 2 and Corollary 1 imply the following statements.

Corollary 2. *Let* $\lambda \geq 1$, $F \in V_\lambda$, $f(\lambda) > 0$ *and* $\alpha \in [0,1)$. *If*

$$\int_\lambda^\infty (x-\alpha) f(x)\, dF(x) \leq f(\lambda)F(\lambda)(\lambda-\alpha), \tag{13}$$

then the function $I \in LS^+$ *is pseudostarlike of the order* α, *and if*

$$\int_\lambda^\infty (x-\alpha) x f(x)\, dF(x) \leq \lambda f(\lambda)F(\lambda)(\lambda-\alpha), \tag{14}$$

then the function $I \in LS^+$ *is pseudoconvex of the order* α.

Corollary 3. *Let* $\lambda \geq 1$, $F \in V_\lambda$ *and* $f(\lambda) > 0$. *If condition* (4) *holds, then the function* $I \in LS^+$ *is pseudostarlike, and if*

$$\int_\lambda^\infty x^2 f(x)\, dF(x) \leq \lambda^2 f(\lambda)F(\lambda), \tag{15}$$

then the function $I \in LS^+$ *is pseudoconvex.*

Remark 1. In theorem 2 and Corollaries 1 and 2, the condition $\lambda \geq 1$ can be replaced by a condition of $\lambda > 0$, but it should be considered that $0 \leq \alpha < \lambda$.

Let us turn to the spoilt Laplace − Stieltjes integral. As above we have

$$|J'(s) - \lambda J(s)| - \beta |J'(s) - (2\alpha-\lambda)J(s)| = \left| \int_\lambda^\infty (x-\lambda) f(x) e^{sx}\, dF(x) \right| -$$

$$- \beta \left| \int_\lambda^\infty (x - 2\alpha + \lambda) f(x) e^{sx}\, dF(x) - 2(\lambda-\alpha) f(\lambda) F(\lambda) e^{s\lambda} \right|,$$

and since $-|a-b| \leq a - b$, we get

$$|J'(s) - \lambda J(s)| - \beta |J'(s) - (2\alpha-\lambda)J(s)| \leq \left| \int_\lambda^\infty (x-\lambda) f(x) e^{sx}\, dF(x) \right| -$$

$$- \left| 2\beta(\lambda-\alpha) f(\lambda) F(\lambda) e^{s\lambda} \right| + \left| \beta \int_\lambda^\infty (x - 2\alpha + \lambda) f(x) e^{sx}\, dF(x) \right| \leq$$

$$\leq \int_\lambda^\infty (x-\lambda) f(x) e^{\sigma x}\, dF(x) - 2\beta(\lambda-\alpha) f(\lambda) F(\lambda) e^{\sigma\lambda} +$$

$$+ \beta \int_\lambda^\infty (x - 2\alpha + \lambda) f(x) e^{\sigma x}\, dF(x) = e^{\sigma\lambda} \left(\int_\lambda^\infty \left[(1+\beta)x - 2\beta\alpha - \right.\right.$$

$$\left.\left. - \lambda(1-\beta) \right] f(x) e^{\sigma(x-\lambda)}\, dF(x) - 2\beta(\lambda-\alpha) f(\lambda) F(\lambda) \right) <$$

$$< \int_\lambda^\infty \left[(1+\beta)x - 2\beta\alpha - \lambda(1-\beta) \right] a(x)\, dF(x) - 2\beta(\lambda-\alpha) f(\lambda) F(\lambda).$$

Therefore, if condition (10) holds, then $|J'(s) - \lambda J(s)| - \beta |J'(s) - (2\alpha-\lambda)J(s)| < 0$ and the function $J \in LS^-$ is pseudostarlike of the order α and the type β.

On the contrary, suppose that $J \in LS^-$ is pseudostarlike of the order α and the type β. Then

$$\left|\frac{-\int_{\lambda}^{\infty}(x-\lambda)f(x)e^{sx}\,dF(x)}{2(\lambda-\alpha)f(\lambda)F(\lambda)e^{s\lambda}-\int_{\lambda}^{\infty}(x-2\alpha+\lambda)f(x)e^{sx}\,dF(x)}\right| =$$

$$= \frac{|J'(s)-\lambda J(s)|}{|J'(s)-(2\alpha-\lambda)J(s)|} < \beta .$$

Therefore, for all $s \in \Pi_0$

$$\operatorname{Re}\left\{\frac{\int_{\lambda}^{\infty}(x-\lambda)f(x)e^{sx}\,dF(x)}{2(\lambda-\alpha)f(\lambda)F(\lambda)e^{s\lambda}-\int_{\lambda}^{\infty}(x-2\alpha+\lambda)f(x)e^{sx}\,dF(x)}\right\} < \beta ,$$

whence for all $\sigma < 0$ we get

$$\frac{\int_{\lambda}^{\infty}(x-\lambda)f(x)e^{\sigma x}\,dF(x)}{2(\lambda-\alpha)f(\lambda)F(\lambda)e^{\sigma\lambda}-\int_{\lambda}^{\infty}(x-2\alpha+\lambda)f(x)e^{\sigma x}\,dF(x)} < \beta .$$

Passing to the limit as $\sigma \to 0$ in this unequality we obtain

$$\frac{\int_{\lambda}^{\infty}(x-\lambda)f(x)\,dF(x)}{2(\lambda-\alpha)f(\lambda)F(\lambda)-\int_{\lambda}^{\infty}(x-2\alpha+\lambda)f(x)\,dF(x)} \le \beta ,$$

whence (10) follows.

Thus, the following theorem is correct.

Theorem 3. *Let* $\lambda \ge 1$, $F \in V_{\lambda}$, $f(\lambda) > 0$, $\alpha \in [0,1)$ *and* $\beta \in (0,1]$. *The function* $J \in LS^-$ *is pseudostarlike of the order* α *and the type* β *if and only if condition* (10) *holds.*

Theorem 3 implies the following statements.

Corollary 4. *Let* $\lambda \ge 1$, $F \in V_{\lambda}$, $f(\lambda) > 0$, $\alpha \in [0,1)$ *and* $\beta \in (0,1]$. *The function* $J \in LS^-$ *is pseudoconvex of the order* α *and the type* β *if and only if condition* (12) *holds.*

Corollary 5. *Let* $\lambda \ge 1$, $F \in V_{\lambda}$, $f(\lambda) > 0$ *and* $\alpha \in [0,1)$. *The function* $J \in LS^-$ *is pseudostarlike of the order* α *if and only if condition* (13) *holds, and the function* $J \in LS^-$ *is pseudoconvex of the order* α *if and only if condition* (14) *holds.*

Corollary 6. *Let* $\lambda \ge 1$, $F \in V_{\lambda}$ *and* $f(\lambda) > 0$. *The function* $J \in LS^-$ *is pseudostarlike if and only if condition* (4) *holds, and the function* $J \in LS^-$ *is pseudoconvex if and only if condition* (15) *holds.*

3. Neighborhoods of the spoilt Laplace – Stieltjes integrals. Following A. W. Goodman [12] and S. Ruscheweyh [21], for a function $f(z) = $ $= z + \sum_{k=2}^{\infty} f_k z^k$, analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$, a set

$$N_\delta(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} g_k z^k : \sum_{k=2}^{\infty} k\,|g_k - f_k| \le \delta \right\}$$

is called its neighborhood. The neighborhoods of functions analytical in $\mathbb{D}$ for various classes of functions were studied by many authors (we mention here only the articles [6, 10, 11, 19, 20, 25]).

Let $J \in LS^-$. We put $\hat{J}(s) = \dfrac{J(s)}{f(\lambda)F(\lambda)}$ and $a(x) = \dfrac{f(x)}{f(\lambda)F(\lambda)}$. Then

$$\hat{J}(s) = e^{s\lambda} - \int_\lambda^\infty a(x) e^{sx}\, dF(x). \tag{16}$$

We denote the class of such functions by $\hat{L}S^-$ and similarly to $N_\delta(f)$, for $m > 0$, $\delta > 0$ we define the neighborhood of the function $\hat{J}$ as follows

$$O_{m,\delta}(\hat{J}) = \left\{ \hat{Q}(s) = e^{s\lambda} - \int_\lambda^\infty b(x) e^{sx}\, dF(x) \in \right.$$

$$\left. \in \hat{L}S^- : \int_\lambda^\infty x^m\, |a(x) - b(x)|\, dF(x) \le \delta \right\}. \tag{17}$$

Lemma 1. *Let $\hat{J} \in \hat{L}S^-$. Then $\hat{Q} \in O_{2,\delta\lambda}(\hat{J})$ if and only if $\hat{Q}'/\lambda \in O_{1,\delta}(\hat{J}'/\lambda)$.*

P r o o f. Clearly

$$\frac{\hat{J}'(s)}{\lambda} = e^{s\lambda} - \int_\lambda^\infty \frac{x a(x)}{\lambda} e^{sx}\, dF(x), \qquad \frac{\hat{Q}'(s)}{\lambda} = e^{s\lambda} - \int_\lambda^\infty \frac{x b(x)}{\lambda} e^{sx}\, dF(x).$$

Therefore, $\hat{Q}'/\lambda \in O_{1,\delta}(\hat{J}'/\lambda)$ if and only if $\int_\lambda^\infty x \left| \dfrac{xa(x)}{\lambda} - \dfrac{xb(x)}{\lambda} \right| dF(x) \le \delta$, i. e. if and only if $\int_\lambda^\infty x^2\, |a(x) - b(x)|\, dF(x) \le \delta\lambda$. The last condition holds if and only if $\hat{Q} \in O_{2,\delta\lambda}(\hat{J})$. ♦

Now we prove the following theorem.

Theorem 4. *Let $1 \le \lambda \le 2$, $F \in V_\lambda$, $E(s) = e^{s\lambda}$.*

In order that the function $\hat{Q}$ is pseudostarlike in Π_0, it is sufficient that $\hat{Q} \in O_{1,2-\lambda}(E)$ and necessary that $\hat{Q} \in O_{1,\lambda b(\lambda)F(\lambda)}(E)$.

In order that the function $\hat{Q}$ is pseudoconvex, it is sufficient that $\hat{Q} \in O_{2,\lambda(2-\lambda)}(E)$ and necessary that $\hat{Q} \in O_{2,\lambda^2 b(\lambda)F(\lambda)}(E)$.

P r o o f. If $\hat{Q} \in O_{1,\delta}(E)$, then $\int_\lambda^\infty x b(x)\, dF(x) \le \delta$. Therefore, if $\delta = 2 - \lambda$, then

$$\left|\hat{Q}(s)-\hat{Q}(s)\right| = \left|\lambda e^{s\lambda} - \int\limits_{\lambda}^{\infty} x b(x) e^{sx}\, dF(x) - e^{s\lambda} + \int\limits_{\lambda}^{\infty} b(x) e^{sx}\, dF(x)\right| =$$

$$= \left|(\lambda-1)e^{s\lambda} - \int\limits_{\lambda}^{\infty} (x-1) b(x) e^{sx}\, dF(x)\right| \le$$

$$\le (\lambda-1)e^{\sigma\lambda} + \int\limits_{\lambda}^{\infty} (x-1) b(x) e^{\sigma x}\, dF(x) =$$

$$= (\lambda-1)e^{\sigma\lambda} + \int\limits_{\lambda}^{\infty} x b(x) e^{\sigma x}\, dF(x) - \int\limits_{\lambda}^{\infty} b(x) e^{\sigma x}\, dF(x) \le$$

$$\le (\lambda-1)e^{\sigma\lambda} + e^{\sigma\lambda}\int\limits_{\lambda}^{\infty} x b(x)\, dF(x) - \int\limits_{\lambda}^{\infty} b(x) e^{\sigma x}\, dF(x) \le$$

$$\le (\lambda-1+\delta)e^{\sigma\lambda} - \int\limits_{\lambda}^{\infty} b(x) e^{\sigma x}\, dF(x) \le e^{\sigma\lambda} - \int\limits_{\lambda}^{\infty} b(x) e^{\sigma x}\, dF(x).$$

On the other hand,

$$\left|\hat{Q}(s)\right| = \left|e^{s\lambda} + \int\limits_{\lambda}^{\infty} b(x) e^{sx}\, dF(x)\right| \ge e^{\sigma\lambda} - \int\limits_{\lambda}^{\infty} b(x) e^{\sigma x}\, dF(x).$$

Thus, $\left|\hat{Q}'(s)-\hat{Q}(s)\right| \le \hat{Q}(s)$, i.e. $\left|\dfrac{\hat{Q}'(s)}{\hat{Q}(s)}-1\right| \le 1$ for all $s \in \Pi_0$. From this it follows that $\operatorname{Re}\{\hat{Q}'(s)/\hat{Q}(s)\} > 0$, i.e. the function $\hat{Q}$ is pseudostarlike in Π_0. The sufficiency of the condition $\hat{Q} \in O_{1,2-\lambda}(E)$ is proved.

We prove the necessity of the condition $\hat{Q} \in O_{1,\lambda}(E)$. From Corollary 6 it follows that the function $\hat{Q}$ is pseudostarlike in Π_0 if and only if $\int\limits_{\lambda}^{\infty} x b(x)\, dF(x) \le \lambda b(\lambda) F(\lambda)$, i.e. $\hat{Q} \in O_{1,\delta}(E)$ with $\delta = \lambda b(\lambda) F(\lambda)$. The first part of Theorem 4 is proved.

Now, if $\hat{Q} \in O_{2,\lambda(2-\lambda)}(E)$, then by Lemma 1 $\hat{Q}'/\lambda \in O_{1,2-\lambda}(E'/\lambda) = = O_{1,2-\lambda}(E)$. Therefore, the functions $\hat{Q}'/\lambda$ and $\hat{Q}'$ are pseudostarlike. Thus, the function $\hat{Q}$ is pseudoconvex. On the other hand, if $\hat{Q}$ is pseudoconvex then $\hat{Q}/\lambda$ is pseudoconvex and $\hat{Q}'/\lambda$ is pseudostarlike, i. e. $\hat{Q}' \in O_{1,\lambda b(\lambda)F(\lambda)}(E)$ and by Lemma 1, $\hat{Q} \in O_{2,\lambda^2 b(\lambda)F(\lambda)}(E)$. The proof of Theorem 4 is complete. ♦

In the case where $\hat{J}$ is pseudostarlike of the order α the following statement is true.

Proposition 1. *Let* $\lambda \ge 1$, $F \in V_\lambda$ *and* $0 \le \alpha_1 \le \alpha < 1$.

If $\hat{J}$ *is pseudostarlike of the order* α *and* $\hat{Q}$ *is pseudostarlike of the order* α_1 *then* $\hat{Q} \in O_{1,\delta}(\hat{J})$ *with* $\delta = \lambda\big(a(\lambda)+b(\lambda)\big)F(\lambda)$.

If $\hat{J}$ *is pseudoconvex of the order* α *and* $\hat{Q}$ *is pseudoconvex of the order* α_1 *then* $\hat{Q} \in O_{2,\delta}(\hat{J})$ *with* $\delta = \lambda^2\big(a(\lambda)+b(\lambda)\big)F(\lambda)$.

P r o o f. If $\hat{J}$ is pseudostarlike of the order α and $\hat{Q}$ is pseudostarlike of the order α_1 then by Corollary 5 we have

$$\int_{\lambda}^{\infty} (x-\alpha)a(x)\,dF(x) \le a(\lambda)F(\lambda)(\lambda-\alpha),$$

$$\int_{\lambda}^{\infty} (x-\alpha_1)b(x)\,dF(x) \le b(\lambda)F(\lambda)(\lambda-\alpha_1).$$

Therefore,

$$\int_{\lambda}^{\infty} x\,|a(x)-b(x)|\,dF(x) = \int_{\lambda}^{\infty} \frac{x}{x-\alpha_1}(x-\alpha_1)\,|a(x)-b(x)|\,dF(x) \le$$

$$\le \frac{\lambda}{\lambda-\alpha_1}\int_{\lambda}^{\infty} (x-\alpha_1)\,|a(x)-b(x)|\,dF(x) \le$$

$$\le \frac{\lambda}{\lambda-\alpha_1}\left(\int_{\lambda}^{\infty}(x-\alpha_1)a(x)\,dF(x) + \int_{\lambda}^{\infty}(x-\alpha_1)b(x)\,dF(x)\right) \le$$

$$\le \frac{\lambda}{\lambda-\alpha_1}\left(\int_{\lambda}^{\infty}\frac{x-\alpha_1}{x-\alpha}(x-\alpha)a(x)\,dF(x) + b(\lambda)F(\lambda)(\lambda-\alpha_1)\right) \le$$

$$\le \frac{\lambda}{\lambda-\alpha}\int_{\lambda}^{\infty}(x-\alpha)a(x)\,dF(x) + \lambda b(\lambda)F(\lambda) \le$$

$$\le \lambda a(\lambda)F(\lambda) + \lambda b(\lambda)F(\lambda),$$

i.e. $\hat{Q} \in O_{1,\delta}(\hat{J})$ with $\delta = \lambda\big(a(\lambda)+b(\lambda)\big)F(\lambda)$. The first part of Proposition 1 is proved.

The second part of Proposition 1 can be easily proved using Lemma 1 and its first part. ♦

Remark 2. For the function $\hat{J}$ pseudostarlike of the order α, we could not find $\delta > 0$ such that from the condition $\hat{Q} \in O_{1,\delta}(\hat{J})$ it follows that $\hat{Q}$ is pseudostarlike of the order $\alpha_1 \in [0,\alpha)$.

For the neighborhoods of pseudostarlike and pseudoconvex functions of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1)$ the following theorem is true.

Theorem 5. *Let* $\lambda > 1$, $F \in V_{\lambda}$, $0 \le \alpha < 1$, $0 < \beta < \beta_1 \le 1$ *and* $\hat{J}$ *is pseudostarlike of the order* $\alpha \in [0,1)$ *and the type* $\beta \in (0,1)$. *Suppose that* $b(\lambda) \ge a(\lambda)$ *and put*

$$\omega = \frac{(1+\beta_1)\lambda - 2\alpha\beta_1 - \lambda(1-\beta_1)}{(1+\beta)\lambda - 2\alpha\beta - \lambda(1-\beta)},$$

$$\delta_1 = \frac{2(\lambda-\alpha)F(\lambda)\big(\beta_1 b(\lambda) - \omega\beta a(\lambda)\big)}{1+\beta_1},$$

$$\delta_2 = \frac{2\lambda(\lambda-\alpha)F(\lambda)\big(\beta_1 b(\lambda) + \omega\beta a(\lambda)\big)}{(1+\beta_1)\lambda - 2\alpha\beta_1 - \lambda(1-\beta_1)}.$$

In order that $\hat{Q}$ is pseudostarlike of the order $\alpha \in [0,1)$ and the type $\beta_1 \in (0,1)$, it is sufficient that $\hat{Q} \in O_{1,\delta_1}(\hat{J})$, and it is necessary that $\hat{Q} \in O_{1,\delta_2}(\hat{J})$.

In order that $\hat{Q}$ is pseudoconvex of the order $\alpha \in [0,1)$ and the type $\beta_1 \in (0,1)$, it is sufficient that $\hat{Q} \in O_{2,\lambda\delta_1}(\hat{J})$, and it is necessary that $\hat{Q} \in O_{2,\lambda\delta_2}(\hat{J})$.

P r o o f. At first we note that

$$\max_{x\ge\lambda} \frac{(1+\beta_1)x - 2\alpha\beta_1 - \lambda(1-\beta_1)}{(1+\beta)x - 2\alpha\beta - (1-\beta)} = \omega, \qquad \beta_1 - \omega\beta > 0,$$

and in view of the condition $b(\lambda) \ge a(\lambda)$ we have $\delta_1 > 0$.

Since $\hat{J}$ is pseudostarlike of the order $\alpha \in [0,1)$ and the type $\beta \in (0,1)$, by Theorem 3 we have

$$\int_\lambda^\infty \{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)\} a(x)\, dF(x) \le 2\beta(\lambda-\alpha)a(\lambda)F(\lambda). \tag{18}$$

Therefore, if $\hat{Q} \in O_{1,\delta_1}(\hat{J})$, then for $0 < \beta < \beta_1 \le 1$ we get

$$\int_\lambda^\infty \{(1+\beta_1)x - 2\beta\alpha - \lambda(1-\beta_1)\} b(x)\, dF(x) \le$$

$$\le \int_\lambda^\infty \{(1+\beta_1)x - 2\beta\alpha - \lambda(1-\beta_1)\} |b(x) - a(x)|\, dF(x) +$$

$$+ \int_\lambda^\infty \{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)\} a(x)\, dF(x) \le$$

$$\le (1+\beta_1)\int_\lambda^\infty x\,|b(x) - a(x)|\, dF(x) +$$

$$+ \int_\lambda^\infty \frac{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)}{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)} \times$$

$$\times \{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)\} a(x)\, dF(x) \le$$

$$\le (1+\beta_1)\delta_1 + 2\beta\omega(\lambda-\alpha)a(\lambda)F(\lambda) =$$

$$= 2\beta_1(\lambda-\alpha)b(\lambda)F(\lambda),$$

i. e. by Theorem 3 the function $\hat{Q}$ is pseudostarlike of the order $\alpha \in [0,1)$ and the type $\beta_1 \in (0,1)$.

Now suppose that the function $\hat{Q}$ is pseudostarlike of the order $\alpha \in [0,1)$ and the type $\beta_1 \in (0,1)$. Then in view of (18) we have

$$\int_\lambda^\infty x\,|b(x) - a(x)|\, dF(x) = \int_\lambda^\infty \frac{x}{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)} \times$$

$$\times \{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)\} |b(x) - a(x)|\, dF(x) \le$$

$$\leq \frac{\lambda}{(1+\beta_1)\lambda - 2\alpha\beta_1 - \lambda(1-\beta_1)} \int_{\lambda}^{\infty} \{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)\} \times$$

$$\times |b(x) - a(x)|\, dF(x) \leq \frac{\lambda}{(1+\beta_1)\lambda - 2\alpha\beta_1 - \lambda(1-\beta_1)} \times$$

$$\times \left(\int_{\lambda}^{\infty} \{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)\} b(x) dF(x) + \right.$$

$$\left. + \int_{\lambda}^{\infty} \{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)\} a(x) dF(x) \right) \leq$$

$$\leq \frac{\lambda}{(1+\beta_1)\lambda - 2\alpha\beta_1 - \lambda(1-\beta_1)} \{2\beta_1(\lambda - \alpha) b(\lambda) F(\lambda) +$$

$$+ \int_{\lambda}^{\infty} \frac{(1+\beta_1)x - 2\beta_1\alpha - \lambda(1-\beta_1)}{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)} \{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)\} \times$$

$$\times a(x)\, dF(x) \leq \frac{\lambda}{(1+\beta_1)\lambda - 2\alpha\beta_1 - \lambda(1-\beta_1)} \times$$

$$\times \left(2\beta_1(\lambda - \alpha) b(\lambda) F(\lambda) + 2\omega\beta(\lambda - \alpha) a(\lambda) F(\lambda)\right) = \delta_2 ,$$

i.e. $\hat{Q} \in O_{1,\delta_2}(\hat{J})$. The first part of Theorem 5 is proved.

The second part of Theorem 5 can be easily proved using Lemma 1 and its first part. ♦

4. Convolutions of the spoilt Laplace – Stieltjes integrals. For power series $f_j(z) = \sum_{k=0}^{\infty} f_{k,j} z^k$, $j = 1, 2$, the series $(f_1 * f_2)(z) = \sum_{k=0}^{\infty} f_{k,1} f_{k,2} z^k$ is called the Hadamard composition (convolution) [14, 15]. The properties of this composition obtained by J. Hadamard found their applications in the theory of the analytic continuation of the functions represented by power series [8, 14]. Note also that singular points of the Hadamard composition are investigated in the article [3].

L. Zalzman [26] studied convolutions of univalent functions in $\mathbb{D}$. For the functions $f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} f_{k,j} z^k \in \Sigma$, $j = 1, 2$, M. L. Mogra [18] defined the convolution as $(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} f_{k,1} f_{k,2} z^k$ and proved, for example, that if the functions f_j are meromorphically starlike of the order $\alpha_j \in [0, 1)$ and $f_{k,j} \geq 0$ for all $k \geq 1$ then $f_1 * f_2$ is meromorphically starlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$. Convolutions of analytic functions in $\mathbb{D}$ were studied also by J. H. Choi, Y. C. Kim, and S. Owa [9], M. K. Aouf and H. Silverman [7], J. Liu and R. Srivastava [17] and many other mathematicians.

For Dirichlet series with positive exponents increasing to $+\infty$ and absolutely convergent in half-plane $\Pi_0 = \{s : \operatorname{Re} s < 0\}$, a convolution was studied in [24].

Here for the functions $\hat{J}_k(s) = e^{s\lambda} - \int_{\lambda}^{\infty} a_k(x) e^{sx}\, dF(x)$, $k = 1, 2$, we define the convolution as

$$(\hat{J}_1 * \hat{J}_2)(s) = e^{s\lambda} - \int_{\lambda}^{\infty} a_1(x) a_2(x) e^{sx}\, dF(x).$$

Corollary 5 implies the following statement.

Corollary 8. *Let* $\lambda \geq 1$, $F \in V_\lambda$, $a_k(x) \leq a_k(\lambda)$ *for all* $x \geq \lambda$ *and* $\alpha_k \in [0,1)$ *for* $k = 1, 2$.

If the functions $\hat{J}_k$ *are pseudostarlike of the order* α_k, *respectively, then the convolution* $\hat{J}_1 * \hat{J}_2$ *is pseudostarlike of the order* $\alpha = \max\{\alpha_1, \alpha_2\}$.

If the functions $\hat{J}_k$ *are pseudoconvex of the order* α_k, *respectively, then the convolution* $\hat{J}_1 * \hat{J}_2$ *is pseudoconvex of the order* $\alpha = \max\{\alpha_1, \alpha_2\}$.

P r o o f. Indeed, if $\hat{J}_k$ are pseudostarlike of the order α_k then, in view of (13),

$$\int_{\lambda}^{\infty} (x - \alpha_1) a_1(x) a_2(x)\, dF(x) \leq a_2(\lambda) \int_{\lambda}^{\infty} (x - \alpha_1) a_1(x)\, dF(x) \leq$$

$$\leq a_2(\lambda)(\lambda - \alpha_1) a_1(\lambda) F(\lambda),$$

i.e. (7) holds with $G = \hat{J}_1 * \hat{J}_2$ and $\alpha = \alpha_1$. Similarly, (7) holds with $G = \hat{J}_1 * \hat{J}_2$ and $\alpha = \alpha_2$. Therefore, (7) holds with $G = \hat{J}_1 * \hat{J}_2$ and $\alpha = \max\{\alpha_1, \alpha_2\}$ and thus, the convolution $\hat{J}_1 * \hat{J}_2$ is pseudostarlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.

The proof of the pseudoconvexity of the convolution $\hat{J}_1 * \hat{J}_2$ is similar. ♦

Corollary 4 implies the following statement.

Corollary 9. *Let* $\lambda \geq 1$, $F \in V_\lambda$, $a_k(x) \leq a_k(\lambda)$ *for all* $x \geq \lambda$, $\alpha \in [0,1)$ *and* $\beta_k \in (0,1]$ *for* $k = 1, 2$.

If the functions $\hat{J}_k$ *are pseudostarlike of the order* α *and the type* β_k, *respectively, then the convolution* $\hat{J}_1 * \hat{J}_2$ *is pseudostarlike of the order* α *and the type* $\beta = \min\{\beta_1, \beta_2\}$.

If the functions $\hat{J}_k$ *are pseudoconvex of the order* α *and the type* β_k, *respectively, then the convolution* $\hat{J}_1 * \hat{J}_2$ *is pseudoconvex of the order* α *and the type* $\beta = \min\{\beta_1, \beta_2\}$.

P r o o f. Indeed, if $\hat{J}_k$ are pseudostarlike of the order α_k then, in view of (12),

$$\int_{\lambda}^{\infty} \{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)\} a_1(x) a_2(x)\, dF(x) \leq$$

$$\leq a_2(\lambda) \int_{\lambda}^{\infty} \{(1+\beta)x - 2\beta\alpha - \lambda(1-\beta)\} a_1(x)\, dF(x) \leq$$

$$\leq a_2(\lambda) 2\beta_1 a_1(\lambda)(\lambda - \alpha) F(\lambda),$$

i. e. (9) holds with $G = \hat{J}_1 * \hat{J}_2$ and $\beta = \beta_1$. Similarly, (9) holds with $G = \hat{J}_1 * \hat{J}_2$ and $\beta = \beta_2$. Therefore, (9) holds with $G = \hat{J}_1 * \hat{J}_2$ and $\beta = \min\{\beta_1, \beta_2\}$ and thus, the convolution $\hat{J}_1 * \hat{J}_2$ is pseudostarlike of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.

The proof of the pseudoconvexity of the convolution $\hat{J}_1 * \hat{J}_2$ is similar. ♦

Remark 3. Since $\hat{J}(s) = \dfrac{J(s)}{f(\lambda)F(\lambda)}$ and $a(x) = \dfrac{f(x)}{f(\lambda)F(\lambda)}$, from Corollaries 8 and 9 it is easy to obtain their analogues for the integrals $J(s)$.

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ГЕОМЕТРИЧНІ ВЛАСТИВОСТІ ІНТЕГРАЛІВ ЛАПЛАСА – СТІЛТЬЄСА

Для інтегралів Лапласа – Стілтьєса введено поняття псевдозірковості та псевдоопуклості. Доведено критерії для псевдозірковості та псевдоопуклості і застосовано їх до вивчення околу функції та згортки функцій.

Ключові слова: *інтеграл Лапласа – Стілтьєса, псевдозірковість, псевдоопуклість, окіл функції, згортка функцій.*

Ivan Franko National University of Lviv, Lviv

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