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STABILITY OF GENERALIZED MEAN VALUE TYPE FUNCTIONAL EQUATION

In the paper we prove the stability theorem for the generalized mean value type functional equation

$$\frac{xf(y)-yf(x)}{x-y}=\Phi(x+y) \qquad \textit{for} \qquad x,\ y\in\mathbb{K}, \qquad x\neq y\,.$$

We will show that if $(X,\|\cdot\|)$ *is a Banach space over* $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ *and if function* $f:\mathbb{K}\to X$ *satisfies the inequality*

$$\left\|\frac{xf(y)-yf(x)}{x-y}-\Phi(x+y)\right\|<\varepsilon \qquad \textit{for} \qquad x,\ y\in\mathbb{K}, \qquad x\neq y\,,$$

then there exist constants $a,b\in X$ *such that*

$$\|f(x)-(xa+b)\|<2\varepsilon \qquad \textit{and} \qquad \|\Phi(x)-b\|<3\varepsilon \quad \textit{for} \quad x\in\mathbb{K}\,.$$

Following the famous problem by S. Ulam [5] many authors have considered a behavior of functions satisfying stability conditions for different functional equations (cf. [2, 3]). Jung Soon-Mo and P. K. Sahoo [4] considered the stability of the equation

$$f(x)-f(y)=(x-y)h(x+y)$$

connected with classical mean value property. J. Aczél and M. Kuczma considered in [1] another functional equation associated with a mean value properties, namely

$$\frac{xf(y)-yf(x)}{x-y}=\Phi(x+y) \qquad \text{for} \qquad x\neq y\,.$$

They have proved that if K is a commutative field of characteristic different from 2 and 3 and if functions f, Φ satisfy the above equation, then there exist constants $a,\ b\in K$ with $f(x)=ax+b$ and $\Phi(x)=b$. Apparently this statement (with the same proof) holds also true in the following case.

Let $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and let X be a vector space over $\mathbb{K}$. We have

Theorem 1 (cf. [1]). *The general solution* $f,\ \Phi:\mathbb{K}\to X$ *of the equation*

$$\frac{xf(y)-yf(x)}{x-y}=\Phi(x+y) \qquad \textit{for} \qquad x,y\in\mathbb{K}, \qquad x\neq y\,, \tag{1}$$

is given by

$$f(x)=ax+b \qquad \textit{for} \qquad x\in\mathbb{K}\,,$$

$$\Phi(x)=b \qquad \textit{for} \qquad x\in\mathbb{K}\,,$$

where $a,\ b\in X$ *are arbitrary constants.*

In the following we need a solution of a simply modification of the equation (1), i. e. we prove the following

Theorem 2. *If a function* $f:\mathbb{K}\to X$ *satisfies the equation*

$$\frac{xf(y)-yf(x)}{x-y}=0 \qquad \textit{for} \qquad x,y\in\mathbb{K}, \quad x\neq y, \quad x+y\neq 0\,, \tag{2}$$

then there exists a constant $a\in X$ *such that* $f(x)=xa$ *for* $x\in\mathbb{K}$.

P r o o f. Assume that a function f satisfy (2). Then putting in (2) $y = 1$ we obtain

$$f(x) = xf(1) \qquad \text{for} \qquad x \in \mathbb{K} \setminus \{-1,1\}. \tag{3}$$

Setting next in (2) $y = 2$ and using (3) we get

$$f(x) = xf(1) \qquad \text{for} \qquad x \in \mathbb{K} \setminus \{-2,2\},$$

which jointly (3) finishes the proof. ◊

Now we assume that X is a Banach space over $\mathbb{K}$. We will prove the following

Theorem 3. *Assume that functions* $f,\ \Phi : \mathbb{K} \to X$ *satisfy the inequality*

$$\left\| \frac{xf(y) - yf(x)}{x - y} - \Phi(x + y) \right\| < \varepsilon \qquad \textit{for} \qquad x, y \in \mathbb{K}, \quad x \neq y. \tag{4}$$

Then there are constants $a,\ b \in X$ *such that*

$$\| f(x) - (xa + b) \| < 2\varepsilon,$$

$$\| \Phi(x) - b \| < 3\varepsilon.$$

P r o o f. Put in (4) $y = 0$. Then we get

$$\| \Phi(x) - f(0) \| < \varepsilon \qquad \text{for} \qquad x \in \mathbb{K} \setminus \{0\}. \tag{5}$$

Next, for arbitrary $x,\ y \in \mathbb{K}$ such that $x \neq y$ and $x + y \neq 0$ using (5) we obtain

$$\left\| \frac{xf(y) - yf(x)}{x - y} - f(0) \right\| \leq \left\| \frac{xf(y) - yf(x)}{x - y} - \Phi(x + y) \right\| + \| \Phi(x + y) - f(0) \| < 2\varepsilon.$$

Consequently

$$\left\| \frac{xf(y) - yf(x)}{x - y} - f(0) \right\| < 2\varepsilon \qquad \text{for} \qquad x, y \in \mathbb{K}, \quad x \neq y, \quad x + y \neq 0. \tag{6}$$

Let $g(x) := f(x) - f(0)$ for $x \in \mathbb{K}$. Then from (6) we get

$$\left\| \frac{xg(y) - yg(x)}{x - y} \right\| < 2\varepsilon \qquad \text{for} \qquad x, y \in \mathbb{K}, \quad x \neq y, \quad x + y \neq 0. \tag{7}$$

Setting in (7) $y = 0$ we get

$$\| g(0) \| < 2\varepsilon. \tag{8}$$

Next, put in (7) $2x$ in the place of x and x in the place of y. Then we get

$$\left\| \frac{1}{2} g(2x) - g(x) \right\| < \varepsilon \qquad \text{for} \qquad x \in \mathbb{K} \setminus \{0\},$$

which with (8) gives

$$\left\| \frac{1}{2} g(2x) - g(x) \right\| < \varepsilon \qquad \text{for} \qquad x \in \mathbb{K}. \tag{9}$$

Fix $x \in \mathbb{K}$ and consider a sequence $\left(\frac{g(2^n x)}{2^n} \right)$. We will show that it is a Cauchy sequence. For $x = 0$ it is obvious. Now let $x \neq 0$ and take nonnegative integer numbers $m,\ n$ such that $m \leq n$. Then using (9) we get

$$\left\| \frac{g(2^n x)}{2^n} - \frac{g(2^m x)}{2^m} \right\| = \frac{1}{2^m} \left\| \frac{g(2^{n-m} 2^m x)}{2^{n-m}} - g(2^m x) \right\| =$$

$$= \frac{1}{2^m} \left\| \sum_{k=0}^{n-m-1} \frac{1}{2^k} \left(\frac{1}{2} g(2^{k+1} 2^m x) - g(2^k 2^m x) \right) \right\| \le$$

$$\le \frac{1}{2^m} \sum_{k=0}^{n-m-1} \frac{1}{2^k} \left\| \frac{1}{2} g(2^{k+1} 2^m x) - g(2^k 2^m x) \right\| <$$

$$< \frac{1}{2^m} \sum_{k=0}^{n-m-1} \frac{1}{2^k} \varepsilon \le \frac{1}{2^m} \sum_{k=0}^{\infty} \frac{1}{2^k} \varepsilon = \frac{\varepsilon}{2^{m-1}} .$$

Thus $\left(\frac{g(2^n x)}{2^n} \right)$ is a Cauchy sequence and moreover, for $m = 0$ we have

$$\left\| \frac{g(2^n x)}{2^n} - g(x) \right\| < \varepsilon \qquad \text{for} \qquad x \in \mathbb{K} \setminus \{0\} .$$

Since $\| g(0) \| < 2\varepsilon$, so we get

$$\left\| \frac{g(2^n x)}{2^n} - g(x) \right\| < \varepsilon \qquad \text{for} \qquad x \in \mathbb{K}, \quad n \in \mathbb{N} . \tag{10}$$

Then we can define a function $T : \mathbb{K} \to X$,

$$T(x) = \lim_{n \to \infty} \frac{g(2^n x)}{2^n} ,$$

which satisfies (cf. (10))

$$\| T(x) - g(x) \| < 2\varepsilon \qquad \text{for} \qquad x \in \mathbb{K} . \tag{11}$$

Put in (7) $2^n x$ and $2^n y$ in the place of x and y, respectively. We have

$$\left\| \frac{x\, g\left(\frac{2^n y}{2^n} \right) - y\, g\left(\frac{2^n x}{2^n} \right)}{x - y} \right\| < \frac{\varepsilon}{2^{n-1}} .$$

Take $n \to \infty$. Then we get

$$\frac{xT(y) - yT(x)}{x - y} = 0 \qquad \text{for} \qquad x, y \in \mathbb{K}, \quad x \ne y, \quad x + y \ne 0 . \tag{12}$$

From Theorem 2 we obtain that there exists a constant $a \in X$ such that $T(x) = xa$ for $x \in \mathbb{K}$. Let $b = f(0)$. Since $g(x) = f(x) - f(0)$, so from (11) we have

$$\| f(x) - (xa + b) \| < 2\varepsilon . \tag{13}$$

Fix $x \ne 0$. Then, by (13) we have

$$\| f(x) + f(-x) - 2b \| = \| f(x) - xa - x + f(-x) + xa - b \| \le$$

$$\le \| f(x) - (xa + b) \| + \| f(-x) - (-xa + b) \| \le 2\varepsilon + 2\varepsilon = 4\varepsilon .$$

Thus

$$\left\| \frac{1}{2} (f(x) + f(-x)) - b \right\| < 2\varepsilon . \tag{14}$$

Next, put in (4) $y = -1$. We have

$$\left\| \frac{1}{2} (f(x) + f(-x)) - \Phi(0) \right\| < \varepsilon . \tag{15}$$

From (14) and (15) we get

$$\| \Phi(0) - b \| \le \left\| \Phi(0) - \frac{1}{2}(f(x) + f(-x)) \right\| +$$

$$+ \left\| \frac{1}{2}(f(x) + f(-x)) - b \right\| < \varepsilon + 2\varepsilon = 3\varepsilon .$$

Consequently, from (5) we get $\| \Phi(x) - b \| < 3\varepsilon$, which completes the proof. ◊

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СТІЙКІСТЬ ФУНКЦІОНАЛЬНОГО РІВНЯННЯ ТИПУ УЗАГАЛЬНЕНОГО СЕРЕДНЬОГО ЗНАЧЕННЯ

Доведено теорему стійкості для функціонального рівняння типу узагальненого середнього значення

$$\frac{xf(y) - yf(x)}{x - y} = \Phi(x + y) \qquad \text{для} \qquad x,\ y \in \mathbb{K}, \qquad x \ne y .$$

Показано, що, якщо $(X, \| \cdot \|)$ *– банахів простір над* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *і якщо функція* $f : \mathbb{K} \to X$ *задовольняє нерівність*

$$\left\| \frac{xf(y) - yf(x)}{x - y} - \Phi(x + y) \right\| < \varepsilon \qquad \text{для} \qquad x,\ y \in \mathbb{K}, \qquad x \ne y ,$$

то існують сталі $a, b \in X$ *такі, що*

$$\| f(x) - (xa + b) \| < 2\varepsilon \qquad \text{і} \qquad \| \Phi(x) - b \| < 3\varepsilon \qquad \text{для} \qquad x \in \mathbb{K} .$$

УСТОЙЧИВОСТЬ ФУНКЦИОНАЛЬНОГО УРАВНЕНИЯ ТИПА ОБОБЩЕННОГО СРЕДНЕГО ЗНАЧЕНИЯ

Доказана теорема устойчивости для функционального уравнения типа обобщенного среднего значения

$$\frac{xf(y) - yf(x)}{x - y} = \Phi(x + y) \qquad \text{для} \qquad x,\ y \in \mathbb{K}, \qquad x \ne y .$$

Показано, что, если $(X, \| \cdot \|)$ *– банахово пространство над* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *и если функция* $f : \mathbb{K} \to X$ *удовлетворяет неравенству*

$$\left\| \frac{xf(y) - yf(x)}{x - y} - \Phi(x + y) \right\| < \varepsilon \qquad \text{для} \qquad x,\ y \in \mathbb{K}, \qquad x \ne y ,$$

то существуют постоянные $a, b \in X$ *такие, что*

$$\| f(x) - (xa + b) \| < 2\varepsilon \qquad \text{и} \qquad \| \Phi(x) - b \| < 3\varepsilon \qquad \text{для} \qquad x \in \mathbb{K} .$$

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