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METHOD OF SOLVING THE CAUCHY PROBLEM FOR EVOLUTIONARY EQUATION IN BANACH SPACE

We propose a method of solving the Cauchy problem for evolutionary equation with infinite order abstract operator in the Banach space. For the right-hand side of initial condition, from a special subspace of the Banach space, in which vectors are represented as Stieltjes integrals over a certain measure, the solution of the problem is represented as certain Stieltjes integral over the same measure. We give examples of applying the method to solving the Cauchy problem for partial differential equations in the class of entire analytical functions of certain orders.

1. Statement of the problem. Let A be a given linear operator acting in the Banach space $\mathfrak{H}$, and, for this operator, arbitrary powers A^j, $j = 2, 3, \ldots$, be defined in $\mathfrak{H}$, i.e. an arbitrary vector $h \in \mathfrak{H}$ be a C^∞-vector of the operator A [1]. Denote by $x(\lambda)$ the eigenvector of the operator A, which corresponds to its eigenvalue λ, i.e. nonzero solution in $\mathfrak{H}$ of the equation

$$Ax(\lambda) = \lambda x(\lambda), \qquad \lambda \in \Lambda,$$

where Λ is an arbitrary subset of the set $\mathbb{C}$. If λ is not an eigenvalue of the operator A then $x(\lambda) = 0$.

Consider an analytical on Λ function $b(\lambda)$, which would be a symbol of the abstract operator $b(A)$, in general, of infinite order, assuming that

$$b(A)x(\lambda) = b(\lambda)x(\lambda).$$

We shall investigate the Cauchy problem as follows:

$$\frac{dU}{dt} = b(A)U, \qquad t \in \mathbb{R}_+, \tag{1}$$

$$U\big|_{t=0} = h, \tag{2}$$

where h is a given vector in $\mathfrak{H}$, $U : \mathbb{R}_+ \to \mathfrak{H}$ is a sought function.

Investigation of the problem (1), (2) originates from the case $b(A) = A$. A special place in those investigations is taken by the semigroup theory, i.e. the theory of evolution differential equations in Banach spaces. Important results of this theory could be found in the fundamental monographs by S. G. Krein [6], E. Hille and R. Phillips [5], A. Pazy [7], K. Yosida [8].

In the last years, new approaches to studying a Cauchy problem, both for differential-operator equations and for partial differential equations, have been appearing. In particular, the work [2] deals with the problem (1), (2) in the case when $b(A)$ is an infinite order differential operator, where A is a Bessel operator. The problem (1), (2) for the infinite order operator $b(A)$, where $A = \frac{d}{dx}$, has been studied in [3]. In the work [2], by means of the Fourier-Bessel integral transform, an integral representation of a solution of the problem (1), (2) have been obtained. In [3], by means of the proposed by the authors differential-symbol method, a solution of the problem (1), (2) is represented in a differential form as an action of, in general, infinite order differential operator, whose symbol is an initial function, onto a certain entire function of a parameter.

In the present paper we propose a method of solving the Cauchy problem (1), (2), which seems to embrace, as particular cases, various above mentioned approaches.

2. Constructing the formal solution of the problem.

Definition. We shall say that vector h from $\mathfrak{H}$ belongs to $\mathfrak{L}$, where $\mathfrak{L} \subseteq \mathfrak{H}$, if on Λ there exist depending on h linear operator $R_h(\lambda) : \mathfrak{H} \to \mathfrak{H}$ and measure $\mu_h(\lambda)$, such that

$$h = \int\limits_{\Lambda} R_h(\lambda) x(\lambda) d\mu_h(\lambda). \tag{3}$$

So, each vector h from $\mathfrak{L}$ can be represented as a Stieltjes integral (3) over a certain measure.

Lemma 1. *On the set* $\Lambda \times \mathbb{R}_+$ *the following identity holds:*

$$\left[\frac{d}{dt} - b(A)\right] \Big\{\exp[b(\lambda)t] x(\lambda)\Big\} \equiv 0. \tag{4}$$

P r o o f. As supposed, for the operator A, arbitrary powers A^n, for $n \in \mathbb{N}$, are defined in $\mathfrak{H}$. Then for any $\lambda \in \Lambda$ and $t \in \mathbb{R}_+$ we find:

$$\left[\frac{d}{dt} - b(A)\right] \Big\{\exp[b(\lambda)t] x(\lambda)\Big\} \equiv \frac{d}{dt}\Big\{\exp[b(\lambda)t] x(\lambda)\Big\} -$$

$$- b(A)\Big\{\exp[b(\lambda)t] x(\lambda)\Big\} \equiv b(\lambda)\Big\{\exp[b(\lambda)t] x(\lambda)\Big\} -$$

$$- \exp[b(\lambda)t]\Big\{b(A) x(\lambda)\Big\} \equiv b(\lambda)\Big\{\exp[b(\lambda)t] x(\lambda)\Big\} -$$

$$- \exp[b(\lambda)t]\Big\{b(\lambda) x(\lambda)\Big\} \equiv 0.$$

This completes our proof. ♢

Theorem 1. *Let in the problem* (1), (2) *the vector* h *belong to* $\mathfrak{L}$, *i.e.* h *can be represented in the form* (3). *Then the formula*

$$U(t) = \int\limits_{\Lambda} R_h(\lambda) \Big\{\exp[b(\lambda)t] x(\lambda)\Big\} d\mu_h(\lambda) \tag{5}$$

defines a formal solution U *of the problem* (1), (2).

P r o o f. According to the formulas (3)–(5), we have

$$\left[\frac{d}{dt} - b(A)\right] U(t) = \left[\frac{d}{dt} - b(A)\right] \int\limits_{\Lambda} R_h(\lambda) \Big\{\exp[b(\lambda)t] x(\lambda)\Big\} d\mu_h(\lambda) =$$

$$= \int\limits_{\Lambda} R_h(\lambda) \left[\frac{d}{dt} - b(A)\right] \Big\{\exp[b(\lambda)t] x(\lambda)\Big\} d\mu_h(\lambda) = \int\limits_{\Lambda} R_h(\lambda) \cdot 0 \, d\mu_h(\lambda).$$

Since the operator $R_h(\lambda)$ is linear, then the last integral is equal to zero, i.e. $U(t)$ formally satisfies the equation (1).

We shall prove the realization of the initial condition (2). Formula (3) implies the following equality:

$$U|_{t=0} = \int\limits_{\Lambda} R_h(\lambda) \Big\{\exp[b(\lambda)t] x(\lambda)\Big\} d\mu_h(\lambda) \Bigg|_{t=0} = \int\limits_{\Lambda} R_h(\lambda) x(\lambda) d\mu_h(\lambda) = h.$$

This completes our proof. ♢

Remark. The formula (5) defines a solution of the problem (1), (2) just formally, since the following equalities are not justified:

$$\begin{gathered}\left[\frac{d}{dt} - b(A)\right] \int\limits_{\Lambda} R_h(\lambda)\Big\{\exp[b(\lambda)t]x(\lambda)\Big\}d\mu_h(\lambda) = \\ = \int\limits_{\Lambda} R_h(\lambda)\left[\frac{d}{dt} - b(A)\right]\Big\{\exp[b(\lambda)t]x(\lambda)\Big\}d\mu_h(\lambda),\end{gathered} \tag{6}$$

$$U\big|_{t=0} = \int\limits_{\Lambda} R_h(\lambda)\left.\Big\{\exp[b(\lambda)t]x(\lambda)\Big\}\right|_{t=0} d\mu_h(\lambda), \tag{7}$$

neither is the convergence of the Stieltjes integrals in the right-hand sides of the formulas (5)–(7).

The proposed method of solving a Cauchy problem for evolution equation is quite general. In case of special space $\mathfrak{H}$, operator A and measure $\mu(\lambda)$, one can refine the obtained result and prove theorems concerning the existence and uniqueness of the Cauchy problem solution in the corresponding spaces of functions.

3. Examples of application of the method.

Example 1. Let us take as $\mathfrak{H}$ the class of entire analytical on $\mathbb{R}$ functions, i.e. $\mathfrak{H} \equiv \mathfrak{A}(\mathbb{R})$, as the operator A take the differentiating operator $\frac{d}{dx}$, $\Lambda = \mathbb{R}$. Then $\exp[\lambda x]$ is an eigenvector of the operator A on $\mathbb{R}$. Let $b(\lambda)$ be arbitrary polynomial of degree $p > 1$ with real coefficients. As a space $\mathfrak{L}$, we shall take the class of entire analytical functions with the order less than p', where $\frac{1}{p} + \frac{1}{p'} = 1$, i.e. $\mathfrak{L} = \mathfrak{A}_{p'}$. Note that the operator $b\Big(\frac{d}{dx}\Big)$ acts in $\mathfrak{A}_{p'}$ invariantly (cr. [4]).

The equality (3) for $\varphi(x) \in \mathfrak{L} \equiv \mathfrak{A}_{p'}$, in case of the Dirac measure, becomes as follows:

$$\varphi(x) = R_\varphi(\lambda)\exp[\lambda x]\Big|_{\lambda=0}.$$

It is easily seen that the operator $R_\varphi(\lambda)$ on $\mathbb{R}$ for arbitrary function $\varphi(x) \in \mathfrak{A}_{p'}$ is defined as the infinite order differential operator

$$R_\varphi(\lambda) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}\left(\frac{d}{d\lambda}\right)^k \quad \text{or} \quad R_\varphi(\lambda) = \varphi\left(\frac{d}{d\lambda}\right),$$

since $\forall k \in \mathbb{Z}_+$ the following equality holds:

$$\left(\frac{d}{d\lambda}\right)^k \exp[\lambda x]\Bigg|_{\lambda=0} = x^k.$$

For the solution of the problem

$$\begin{gathered}\left[\frac{\partial}{\partial t} - b\left(\frac{\partial}{\partial x}\right)\right]U(t,x) = 0, \\ U(0,x) = \varphi(x),\end{gathered} \tag{8}$$

the formula (5) gets the form

$$U(t,x) = \varphi\left(\frac{d}{d\lambda}\right)\left.\Big\{\exp[b(\lambda)t + \lambda x\Big\}\right|_{\lambda=0}. \tag{9}$$

So, we have obtained the representation (9) of solution of the problem (8), which has been found in [3] by means of the differential-symbol method.

Note that in brackets of the formula (9) there is an entire analytical function of order p. The infinite order operator $\varphi\left(\frac{d}{d\lambda}\right)$ is applicable to $\exp[b(\lambda)t + \lambda x]$, if $\varphi(x) \in \mathfrak{A}_{p'}$, moreover, the action of $\varphi\left(\frac{d}{d\lambda}\right)$ in $\mathfrak{A}_{p'}$ is invariant [4].

In the work [3] it is proved that for any $\varphi(x) \in \mathfrak{A}_{p'}$, in the class of analytical in t functions $U(t,x)$, which for fixed $t > 0$ belong to $\mathfrak{A}_{p'}$, there exists a unique solution of the problem (8), which could be found with the formula (9). $\triangleright$

Example 2. In the previous example, as the operator A we chose the first order differentiating operator. This time, for the operator A, we shall take a second order operator, namely the Bessel operator:

$$A = \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}, \quad \nu > -\frac{1}{2}.$$

As the spaces $\mathfrak{H}$ and $\mathfrak{L}$ we shall take $\mathfrak{H} = \mathfrak{L} = \overset{\circ}{W}{}^{\Omega}_{M}(\mathbb{R})$, i.e. the set of even entire analytical functions $h(x)$, which as functions of complex variable $z \in \mathbb{C}$ admit the estimate

$$h(z) \le c\exp\left\{-M(ax) + \Omega(by)\right\},$$

where $z = x+iy$, $a,b,c \in \mathbb{R}$, $a > 0$, $b > 0$, $c > 0$, moreover Ω, M are differentiable and even on $\mathbb{R}$ functions, increasing and convex on $\mathbb{R}_+$, for which $M(0) = \Omega(0) = 0$, $\lim\limits_{x\to+\infty} M(x) = +\infty$, $\lim\limits_{x\to+\infty} \Omega(x) = +\infty$.

Consider the problem (1), (2), in which $b(\lambda)$ is entire analytical even function. As an eigenfunction of the Bessel operator we shall take J_ν, i.e. the normalized Bessel function, and, besides, as a measure we shall take a Lebesgue measure ($d\mu(\lambda) = d\lambda$), $\Lambda = \mathbb{R}_+$.

The equality (3), for this case, becomes $h(x) = \int\limits_{\mathbb{R}_+} R_h(\lambda)J_\nu(\lambda x)d\lambda$, and the operator $R_h(\lambda)$ is as follows:

$$c_\nu \int\limits_{\mathbb{R}_+} h(x)J_\nu(\lambda x)x^{2\nu+1}dx,$$

where $c_\nu = \dfrac{1}{2^{2\nu}\Gamma^2(\nu+1)}$, Γ is an Euler gamma-function.

The formula (5) defines the solution

$$U(t,x) = \int\limits_{\mathbb{R}_+} R_h(\lambda)\Big\{\exp[b(\lambda)t]J_\nu(\lambda x)\Big\}d\lambda$$

of the problem (1), (2), which has been obtained in [2] by means of the Fourier–Bessel integral transform. $\triangleright$

4. Conclusions. In the present paper, we propose a method of solving a Cauchy problem for the evolution equation. The problem solution is represented in a special class of functions in the form of Stieltjes integral over a certain measure. Such a representation includes, as particular cases, the representations of solutions of the Cauchy problem for the evolution equation with the infinite order Bessel operator, in an integral form [2], and the representations of the Cauchy problem solutions in a differential form, obtained by means of the differential-symbol method [3].

The interconnections of the obtained representation of the Cauchy problem solution with another known representations, as well as obtaining similar representations of Cauchy problem solutions for more general equations or systems of equations and other boundary value problems for differential-operator equations, need further investigations.

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МЕТОД РОЗВ'ЯЗУВАННЯ ЗАДАЧІ КОШІ ДЛЯ ЕВОЛЮЦІЙНОГО РІВНЯННЯ У БАНАХОВОМУ ПРОСТОРІ

Запропоновано метод розв'язування задачі Коші для еволюційного рівняння з абстрактним оператором нескінченного порядку в банаховому просторі. Для правої частини початкової умови зі спеціального підпростору банахового простору, в якому вектори зображаються як інтеграли Стілтьєса за деякою мірою, розв'язок задачі зображено у вигляді деякого інтеграла Стілтьєса за тією ж мірою. Подано приклади застосування методу розв'язування задачі Коші для диференціальних рівнянь із частинними похідними у класі цілих аналітичних функцій певних порядків.

МЕТОД РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ ЭВОЛЮЦИОННОГО УРАВНЕНИЯ В БАНАХОВОМ ПРОСТРАНСТВЕ

Предложен метод решения задачи Коши для эволюционного уравнения с абстрактным оператором бесконечного порядка в банаховом пространстве. Для правой части начального условия из специального подпространства банахового пространства, в котором векторы представляются интегралами Стилтьеса по некоторой мере, решение задачи представлено в виде некоторого интеграла Стилтьеса по этой же мере. Приведены примеры использования метода решения задачи Коши для дифференциальных уравнений в частных производных в классе целых аналитических функций некоторых порядков.

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