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ABSORBING SYSTEMS IN THE HILBERT CUBE RELATED TO HAUSDORF AND COVERING DIMENSION

The topology of the system $\left(D_{\geq k}^{>\gamma_n}(Q)\right)_{k\in\mathbb{N}\cup\{0\},\gamma_n\in\Gamma,\gamma_{n+1}\geq k}$ *, where* $D_{\geq k}^{>\gamma_n}(Q)=\{A\in \exp(Q)\mid \dim_H(A)>\gamma_n, \dim(A)\geq k\}$ *and* $\Gamma=\{\gamma_i\}_{i=1}^{\infty}$ *is a countable ordered set with* $0<\gamma_1<\gamma_2<\ldots<\infty$ *, is described.*

Introduction. The theory of absorbing systems is one of powerful tools of modern infinite-dimensional topology that received numerous applications to the general problem of recognition of infinite-dimensional model spaces in topology, functional analysis, measure theory etc.

One of the directions of investigations concerns the hyperspaces of compacta with prescribed dimensional properties lying in the Euclidean spaces or, more generally, in ANR-spaces (see [3, 6, 8]). The author first considered the hyperspaces of compacta of given Hausdorf dimension [7]. It turned out that these hyperspaces lead to absorbing systems ordered by some linearly ordered subsets of real line.

In this paper, we consider the hyperspaces of compacta in the Hilbert cube with given both Hausdorf and covering dimension. Our main result (theorem 2) states that, in some cases, these hyperspaces form absorbing systems in the Hilbert cube ordered by partially ordered set. This allows us to describe the topology of the hyperspaces under consideration.

The paper is organized as follows. In section 1 we recall the necessary definitions and facts from the theory of absorbing systems. In particular, we construct a model absorbing system ordered by product of linearly ordered sets. Theorem 1 is proved in subsection 1.3.

1. Preliminaries. A typical metric will be denoted by d. By $\operatorname{diam}(A)$ we denote the diameter of a subset A in a metric space. Given a cover $\mathcal{U}$ of a metric space, we define $\operatorname{mesh}(\mathcal{U})$ as $\sup\{\operatorname{diam}(U)\mid U\in\mathcal{U}\}$. For $x\in X$ and $\varepsilon>0$ the set $O_\varepsilon(x)=\{y\in X|d(x,y)<\varepsilon\}$ is an *open* ε*-ball* centered at x. Further, all spaces are separable metrizable, all maps are continuous.

By Q we denote the Hilbert cube, $Q=\prod\limits_{i=1}^{\infty}[-1,1]_i$. The class of absolute neighborhood retracts is denoted by ANR. A closed subset A of $X\in ANR$ is called a Z*-set* in X if for every continuous function $\varepsilon\colon X\longrightarrow(0,\infty)$ there exists a map $f\colon X\longrightarrow X\backslash A$ which is ε-close to the identity in the sense that $d(x,f(x))<\varepsilon(x)$, for every $x\in X$. An embedding $g\colon Y\longrightarrow X$ is called a Z*-embedding* if its image $g(Y)$ is a Z-set in X. By $B(Q)$ we denote the *pseudoboundary* of Q, $B(Q)=Q\backslash\prod\limits_{i=1}^{\infty}(-1,1)_i$.

1.1. Hyperspaces. Let X be a metric space. The *hyperspace* of X is the space $\exp X$ of nonempty compact subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets

$$\langle V_1,\ldots,V_n\rangle=\{A\in\exp X\mid A\subset\bigcup_{i=1}^{n}V_i \text{ and for every } i\in\{1,2,...,n\}\ A\cap V_i\neq\emptyset\},$$

where $V_1, ..., V_n$ run over the topology of X. The Vietoris topology is generated by the Hausdorf metric d_H,

$$d_H(A, B) = \inf\{\varepsilon > 0 | A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

For $n \in \mathbb{N}$, we denote by $\exp_n X$ the subspace of $\exp X$ consisting of sets of cardinality $\leq n$. Let $\exp_\omega X = \cup\{\exp_n X \mid n \in \mathbb{N}\}$.

1.2. Hausdorf dimension. Let X be a complete separable metric space, let F be a compact nonempty subset of X and s a non-negative number. For $\varepsilon > 0$ define

$$\mathcal{H}^s_\varepsilon(F) = \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} (\mathrm{diam} B)^s,$$

where the infimum is taken over all covers $\mathcal{B}$ of F with $\mathrm{mesh}(\mathcal{B}) < \varepsilon$.

Let $\mathcal{H}^s(F) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(F)$. There exists a unique number s_0, the *Hausdorf dimension* of F, such that $\mathcal{H}^s(F) = \infty$ whenever $0 \leq s < s_0$ and $\mathcal{H}^s(F) = 0$ whenever $s_0 < s < \infty$. We write $\dim_H F = s_0$.

Proposition 1 [7]. *Let X be a complete separable metric space. For every $\alpha \geq \geq 0$ the set $C_\alpha = \{A \in \exp(X) \mid \dim_H(A) \leq \alpha\}$ is a G_δ-subset of* $\exp(X)$.

1.3. Absorbing systems. We briefly recall some definitions from the theory of absorbing systems; see [1, 2, 6, 8] for details.

A space X has the *Z-approximation property* if for every compact space B, every map $f\colon B \to X$ that restricts to a Z-embedding on some compact subset K of B, can be approximated arbitrarily closely by a Z-embedding $g\colon B \to X$ such that $g|K = f|K$.

Let Γ be an ordered set and $\mathcal{M}_\gamma$ a class of metric spaces for $\gamma \in \Gamma$. Put $\mathcal{M}_\Gamma = (\mathcal{M}_\gamma)_{\gamma \in \Gamma}$. An *$\mathcal{M}_\Gamma$-system* in a space X is an order preserving (with respect to inclusion) indexed collection $(A_\gamma)_{\gamma \in \Gamma}$ of subsets of X such that $A_\gamma \in \mathcal{M}_\gamma$ for every γ.

An $\mathcal{M}_\Gamma$-system $\mathfrak{X}$ in $X \in ANR$ is called *strongly $\mathcal{M}_\Gamma$-universal* in X if for every $\mathcal{M}_\Gamma$-system (A_γ) in Q, every map $f\colon Q \longrightarrow X$ that restricts to a Z-embedding on some compact subset K of Q can be approximated by a Z-embedding $g\colon Q \longrightarrow X$ such that $g|K = f|K$ and for every $\gamma \in \Gamma$ we have $g^{-1}(X_\gamma)\backslash K = = A_\gamma \backslash K$.

An $\mathcal{M}_\Gamma$-system $\mathfrak{X}$ is called *$\mathcal{M}_\Gamma$-absorbing* in X if the set $\bigcup_{\gamma \in \Gamma} X_\gamma$ is contained in a σ-compact σ-Z-set in X and $\mathfrak{X}$ is strongly $\mathcal{M}_\Gamma$-universal in X.

By $\mathcal{F}_\sigma$ we denote the class of σ-compact spaces.

We shall now consider a special case when the system $\mathfrak{X}$ is a decreasing sequence of absorbers (so Γ is ordered by the relation $\geq$) and we assume that all the classes $\mathcal{M}_\gamma$ are equal to a fixed class $\mathcal{M}$. In this situation we shall use the term $\mathcal{M}$-absorbing system.

Let $\mathfrak{X} = (X_\alpha)_{\alpha \in A}$, $\mathfrak{Y} = (Y_\beta)_{\beta \in B}$ be decreasing systems of sets (the sets A and B are ordered by the relation $\geq$). The system $\mathfrak{X} \times \mathfrak{Y} = (X_\alpha \times Y_\beta)_{\alpha \in A, \beta \in B}$ of sets is partially ordered by the relation $\supseteq$: $X_\alpha \times Y_\beta \supseteq X_{\alpha'} \times Y_{\beta'}$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$ where $\alpha, \alpha' \in A$, $\beta, \beta' \in B$.

Theorem 1. *For $i = 1, 2$ let E_i be a topologically complete ANR with the Z-approximation property and Γ_i be a countable ordered by the relation $\geq$ set with the first element. If $\mathfrak{X}^i = (X^i_\gamma)_{\gamma \in \Gamma_i}$ is a strongly $\mathcal{M}$-universal system in E_i for $i = 1, 2$, then the system $\mathfrak{X}^1 \times \mathfrak{X}^2 = \left(X^1_\gamma \times X^2_{\gamma'}\right)_{(\gamma, \gamma') \in \Gamma_1 \times \Gamma_2}$ is strongly $\mathcal{M}$-universal in $E = E_1 \times E_2$.*

P r o o f. For $i = 1, 2$, let d_i be an admissible metric on E_i that is bounded by 1. Then $\rho(x, y) = \max\{d_i(x_i, y_i)\}$ is an admissible metric on E.

Consider a map $f\colon Q \to E$ that restricts to a Z-embedding on some compact subset $K \subseteq Q$, and let $\varepsilon > 0$. In addition, let $\mathfrak{A} = (\mathcal{A}_{(\gamma,\gamma')})_{(\gamma,\gamma')\in\Gamma_1\times\Gamma_2}$ be an order preserving (with respect to inclusion) indexed system of subsets of Q consisting of elements of $\mathcal{M}$ (that is $\mathcal{A}_{(\gamma,\gamma')} \supseteq \mathcal{A}_{(\overline{\gamma},\overline{\gamma}')}$ if and only if $\gamma \geq \overline{\gamma}$ and $\gamma' \geq \overline{\gamma}'$). It is obvious that the system $\mathfrak{A}$ in addition should satisfy the condition:

(*1*) if $x \in \mathcal{A}_{(\gamma,\gamma')} \backslash \mathcal{A}_{(\overline{\gamma},\overline{\gamma}')}$ for any $\gamma \geq \overline{\gamma}$ and $\gamma' \geq \overline{\gamma}'$ then $x \in \mathcal{A}_{(\gamma_0,\gamma')} \backslash \mathcal{A}_{(\gamma_0,\overline{\gamma}')}$ and, similarly, $x \in \mathcal{A}_{(\gamma,\gamma_0')} \backslash \mathcal{A}_{(\overline{\gamma},\gamma_0')}$, where γ_0 and γ_0' are initial elements of sets Γ_1 and Γ_2 respectively.

We may assume without loss of generality that f is a Z-embedding. Write $Q\backslash K = \bigcup_{i=0}^{\infty} F_i$, where $F_0 = \emptyset$, each F_i is compact, and $F_i \subset \mathrm{Int}(F_{i+1})$ for every $i \in \mathbb{N}$. For every $i \geq 0$ put

$$\varepsilon_i = \min\left\{2^{-i} \cdot \varepsilon, \frac{1}{2}\rho\left(f[K], f[F_i]\right)\right\}$$

and observe that

$$\varepsilon_0 \geq \varepsilon_1 \geq \ldots \geq \varepsilon_i \geq \ldots > 0, \lim_{i\to\infty} \varepsilon_i = 0.$$

Now consider the kth component function $f_k\colon Q \to E_k$ for $k = 1, 2$. Put $\alpha_0 = f_k$ and assume that we have constructed $\alpha_i\colon Q \to E_k$ such that

(*2*) $\hat{d}_k(\alpha_i, \alpha_{i-1}) < \varepsilon_i, \alpha_i|F_{i-1} = \alpha_{i-1}|F_{i-1};$

(*3*) $\alpha_i|Q\backslash F_{i+1} = f_k|Q\backslash F_{i+1}$ and $\alpha_i|F_i$ is a Z-imbedding;

(*4*) $\alpha_i^{-1}[X_\gamma^1] \cap F_i = \mathcal{A}_{(\gamma,\gamma_0')} \cap F_i$ for every $\gamma \in \Gamma_1$ and $\alpha_i^{-1}[X_\gamma^2] \cap F_i = \mathcal{A}_{(\gamma_0,\gamma)} \cap F_i$ for every $\gamma \in \Gamma_2$.

For the construction of α_{i+1} satisfying our inductive hypotheses, use the strong $\mathcal{M}$-universality of the system $\mathfrak{X}^k$ for $k = 1, 2$. We can find a Z-imbedding $\beta\colon F_{i+1} \to E_k$, close to $\alpha_i|F_{i+1}$, with $\beta|F_i = \alpha_i|F_i$ and $\beta^{-1}[X_\gamma^1] = \mathcal{A}_{(\gamma,\gamma_0')} \cap F_{i+1}$ for every $\gamma \in \Gamma_1$ (similarly, $\beta^{-1}[X_\gamma^2] = \mathcal{A}_{(\gamma_0,\gamma)} \cap F_{i+1}$ for every $\gamma \in \Gamma_2$). Using the fact that E_k is an ANR, we can assume that β and $\alpha|F_{i+1}$ are sufficiently close so we could extend a map $\beta \cup (\alpha_i|\overline{Q\backslash F_{i+2}})\colon F_{i+1} \cup \overline{Q\backslash F_{i+2}} \to E_k$ to a map $\alpha_{i+1}\colon Q \to E_k$ such that $\hat{d}_k(\alpha_{i+1}, \alpha_i) < \varepsilon_{i+1}$.

The maps α_i obviously form a Cauchy sequence and thus the function

$$g_k = \lim_{i\to\infty} \alpha_i$$

is continuous. It is easy to verify that g_k has the following properties (for $k = 1, 2$)

(*5*) $\hat{d}_k(g_k, f_k) < \varepsilon$;

(*6*) if $x \in F_{i+1}\backslash F_i$ then $d_k(g_k(x), f_k(x)) < \rho(f[K], f[F_{i+1}])$;

(*7*) $g_k|K = f_k|K$, $g_k|F_i$ is a Z-imbedding for every i;

(*8*) $g_1^{-1}[X_\gamma^1]\backslash K = \mathcal{A}_{(\gamma,\gamma_0')}\backslash K$ for every $\gamma \in \Gamma_1$ and $g_2^{-1}[X_\gamma^2]\backslash K = \mathcal{A}_{(\gamma_0,\gamma)}\backslash K$ for every $\gamma \in \Gamma_2$.

Define $g = (g_1, g_2)\colon Q \to E_1 \times E_2$. It is easily seen that g is one-to-one, hence an embedding. The compact set $g[Q]$ is contained in the Z-set $f[K] \cup \bigcup\limits_{i=0}^{\infty} \bigcup\limits_{j=0}^{\infty} g_1[F_i] \times \times g_2[F_j]$ and is therefore a Z-set. Moreover, the maps f and g are clearly ε-close and satisfy $g|K = f|K$.

If x is an element of $\mathcal{A}_{(\gamma,\gamma')} \setminus K$ for any $(\gamma, \gamma') \in \Gamma_1 \times \Gamma_2$ then it is easy to see that $x \in \mathcal{A}_{(\gamma_0,\gamma')}$ and $x \in \mathcal{A}_{(\gamma,\gamma'_0)}$. Therefore, taking into account condition (***8***), we obtain that $g_1(x) \in X^1_\gamma$, $g_2(x) \in X^2_{\gamma'}$ and hence $g(x) \in X^1_\gamma \times X^2_{\gamma'}$.

If x is not an element of $\mathcal{A}_{(\gamma,\gamma')} \setminus K$ for any $(\gamma, \gamma') \in \Gamma_1 \times \Gamma_2$ then we consider two possible cases:

(***a***) $x \in Q \setminus \bigcup\limits_{(\gamma,\gamma') \in \Gamma_1 \times \Gamma_2} \mathcal{A}_{(\gamma,\gamma')}$. Then from condition (***8***) it follows that $g(x) \in \in E \setminus \bigcup\limits_{(\gamma,\gamma') \in \Gamma_1 \times \Gamma_2} (X^1_\gamma \times X^2_{\gamma'})$. This implies that $g(x) \notin X^1_\gamma \times X^2_{\gamma'}$.

(***b***) $x \in \mathcal{A}_{(\overline{\gamma},\overline{\gamma}')}$, where $\overline{\gamma} > \gamma$ or (and) $\overline{\gamma}' > \gamma'$. Without loss of generality we may assume that $\overline{\gamma} > \gamma$. Then, by condition (***1***), $x \in \mathcal{A}_{(\overline{\gamma},\gamma'_0)} \setminus \mathcal{A}_{(\gamma,\gamma'_0)}$ and by condition (***8***), clearly, $g_1(x) \notin X^1_\gamma$. Therefore $g(x) \notin X^1_\gamma \times X^2_{\gamma'}$. $\diamondsuit$

Corollary 1. *If $\mathcal{M} = \mathcal{F}_\sigma$ and a system $\mathfrak{X}^i$ is $\mathcal{F}_\sigma$-absorbing in E_i (for $i = = 1, 2$) then the system $\mathfrak{X}^1 \times \mathfrak{X}^2$ is $\mathcal{F}_\sigma$-absorbing in $E_1 \times E_2$.*

P r o o f follows from Theorem 1 and the definition of absorbing system. $\diamondsuit$

Corollary 2. *The system $\left(B(Q)^k \times Q \times \cdots \times B(Q)^n \times Q \times \cdots\right)_{k,n \in \mathbb{N} \cup \{0\}}$ is $\mathcal{F}_\sigma$-absorbing in $Q^\infty \times Q^\infty$.*

P r o o f follows from the standard results of the theory of absorbing sets in Q (see [6, 8]) and Corollary 1. $\diamondsuit$

2. Main result. For any non-negative integer number k and non-negative real number γ denote by $D^{>\gamma}_{\geq k}(Q)$ the collection of all compacta in Q for which the Hausdorf dimension is $> \gamma$ and the covering dimension is $\geq k$. In this section we consider a system $\left(D^{>\gamma}_{\geq k}(Q)\right)$ indexed by the partially ordered set $\Gamma \times (\mathbb{N} \cup \{0\})$ where $\Gamma = \{\gamma_i\}_{i=1}^\infty$ with $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n < \ldots < \infty$ and the set $\mathbb{N} \cup \{0\}$ is ordered naturally. It is easy to verify that the system $\left(D^{>\gamma_i}_{\geq k}(Q)\right)_{(\gamma_i,k) \in \Gamma \times (\mathbb{N} \cup \{0\})}$ is $\mathcal{F}_\sigma$-system if and only if a condition $\gamma_{i+1} \geq k$ holds. We shall prove that the system $\mathfrak{D} = \left(D^{>\gamma_i}_{\geq k}(Q)\right)_{(\gamma_i,k) \in \Gamma \times (\mathbb{N} \cup \{0\}), \gamma_{i+1} \geq k}$ is an absorbing system in $\exp(Q)$ for the class of all σ-compact spaces. Hence the pair

$$\left(\exp(Q), D^{>\gamma_n}_{\geq k}(Q)\right)_{(\gamma_n,k) \in \Gamma \times (\mathbb{N} \cup \{0\}), \gamma_{n+1} \geq k}$$

is homeomorphic to the pair

$$\left(Q^\infty \times Q^\infty, B(Q)^n \times Q \times \cdots \times B(Q)^k \times Q \times \cdots\right)_{k,n \in \mathbb{N} \cup \{0\}, \gamma_{n+1} \geq k}.$$

A topological characterization of the pair $(Q^\infty \times Q^\infty, B(Q)^n \times Q \times \cdots \times \mathcal{B}(Q)^k \times \times Q \times \cdots)_{k,n \in \mathbb{N} \cup \{0\}, \gamma_{n+1} \geq k}$ is given in Theorem 1.

Now we have to prove the strong $\mathcal{F}_\sigma$-universality of the system $\mathfrak{D}$ in $\exp(Q)$.

Theorem 2. *The system* $\mathfrak{D}$ *is strongly* $\mathcal{F}_\sigma$*-universal in* $\exp(Q)$.

P r o o f. Let $\varepsilon > 0$ and $f\colon Q \to \exp(Q)$ be a continuous map that restricts to a Z-embedding on some compact subset K of Q. Without loss of generality we may assume that f is a Z-embedding because $\exp(Q)$ is a Hilbert cube.

We consider the sets Γ and $\mathbb{N}\cup\{0\}$ with inverted order. Now choose an any order preserving (with respect to inclusion) system of σ-compact subsets $\mathcal{A} = \big(\mathcal{A}_{(\gamma_n,k)}\big)_{(\gamma_n,k)\in\Gamma\times(\mathbb{N}\cup\{0\}),\gamma_{n+1}\geq k}$ in Q. It is obvious that the system $\mathcal{A}$ should satisfy a condition analogical to condition (*1*) in the proof of Theorem 1:

(*1*) if $x \in \mathcal{A}_{(\gamma,k)}\setminus\mathcal{A}_{(\overline{\gamma},\overline{k})}$ for any $\gamma \leq \overline{\gamma}$ and $k \leq \overline{k}$ then $x \in \mathcal{A}_{(\gamma_{k'},k)}\setminus\mathcal{A}_{(\gamma_{\overline{k}'},\overline{k})}$ and similarly $x \in \mathcal{A}_{(\gamma,0)}\setminus\mathcal{A}_{(\overline{\gamma},0)}$, where $k' = \max\{i \in \mathbb{N} \mid \gamma_i \leq k\}$. (Here, for technical reasons, we accept that $\max\{\emptyset\} = 1$.) Clearly, $\gamma_{k'} \leq \gamma$ and $\gamma_{\overline{k}'} \leq \overline{\gamma}$.

Define $\mu\colon Q \to \mathbb{I}$ by

$$\mu(x) = \frac{1}{3}\cdot\min\{\varepsilon, d_H(f(x), f[K])\}.$$

The set $\exp(Q)\setminus\exp_\omega(Q)$ is locally homotopy negligible in $\exp(Q)$ (see [6]), therefore there is a homotopy $H\colon \exp(Q)\times\mathbb{I}\to\exp(Q)$ such that

(*2*) $H_0 = 1_{\exp(Q)}$;

(*3*) for every $t \in (0,1]$, $H_t(\exp(Q)) \subseteq \exp_\omega(Q)$.

It is clear that we may additionally assume that

(*4*) for every $t \in [0,1]$, $\hat{d}_H(H_t, 1_{\exp(Q)}) \leq 2t$;

(*5*) for every $t \in (0,1]$ and $A \in \exp(Q)$, $H_t(A) \in \exp_\omega([-1+t, 1-t]^\infty)$.

For every $x \in Q$ let $F(x) = H(f(x), \mu(x))$. Then if $\mu(x) > 0$, $F(x)$ is a finite approximation of the set $f(x)$.

Consider a sequence of compact subsets $\{B_p^n\}_{p=1}^\infty$ in the finite-dimensional cube $[0,1]^n\times\{(0,0,\ldots)\}$ in Q defined as follows:

$$\begin{aligned}
B_1^n &= \tfrac{1}{2}[0,1]^n \times \{(0,0,\ldots)\},\\
B_2^n &= \left[\tfrac{1}{2^2}[0,1]^n + \tfrac{1}{2}(\underbrace{1,1,\ldots,1}_{n})\right] \times \{(0,0,\ldots)\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,,\\
B_p^n &= \left[\tfrac{1}{2^p}[0,1]^n + \big(1-\tfrac{1}{2^{p-1}}\big)(\underbrace{1,1,\ldots,1}_{n})\right] \times \{(0,0,\ldots)\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,.
\end{aligned}$$

It is easy to see that the sets B_p^n are smaller copies of the cube $[0,1]^n\times\{(0,0,\ldots)\}$ located on its diagonal, and the sequence of these sets converges to the point $(\underbrace{1,1,\ldots,1}_{n},0,0,\ldots)$. Let $\beta_p^n\colon [0,1]^n\times\{(0,0,\ldots)\}\to B_p^n$ any homeomorphism that is a similarity. Observe, that in this case the homeomorphism β_p^n does not change the Hausdorf dimension (see [4]).

For every $k \in \mathbb{N}\cup\{0\}$ and $\gamma\in\Gamma$ write $\mathcal{A}_{(\gamma,k)} = \bigcup\limits_{p=1}^{\infty} A_{(\gamma,k)}^p$, where for every $p \in \mathbb{N}$ $A_{(\gamma,k)}^p$ is a compact subset of Q. Denote for every $x \in Q$, $t_{(\gamma,k)}^p(x) = d(x, A_{(\gamma,k)}^p)$.

For every $n \in \mathbb{N}$, using locally homotopy negligibility of the set $\exp([0,1]^n) \setminus \exp_\omega([0,1]^n)$, we can construct a homotopy $H^n\colon \exp([0,1]^n) \times \mathbb{I} \to \exp([0,1]^n)$ that satisfies conditions similar to conditions (***2***)–(***4***). Besides, obviously, for every $n \in \mathbb{N}$ and for every real $\gamma \in [0,n)$, there is a set $A_\gamma \in \exp([0,1]^n)$ such that the following condition is satisfied (see [4, 5]):

(***6***) $\dim_H(A_\gamma) = \gamma$ and $\dim(A_\gamma) = 0$.

Let $\varphi_\gamma^n\colon [0,1] \to \exp(Q)$ be a map defined by the formula

$$\varphi_\gamma^n(t) = H^n(A_\gamma, t) \times \{(0,0,\dots)\}.$$

Clearly, the map φ_γ^n is defined for all $\gamma \le n$.

Now we define maps $h_{\gamma_i}, h_\Gamma\colon Q \to \exp(Q)$ by the formulas

$$h_{\gamma_i}(x) = \bigcup_{p=1}^{\infty} \beta_p^{[\gamma_{i+1}]+1} \left(\varphi_{\gamma_{i+1}}^{[\gamma_{i+1}]+1} \left(t^p_{(\gamma_i,0)}(x) \right) \right)$$

and

$$h_\Gamma(x) = \bigcup_{i=1}^{\infty} \left[\frac{1}{2^i} h_{\gamma_i}(x) + \left(1 - \frac{1}{2^{i-1}}\right)(1,0,0,\dots) \right] \cup \{(1,0,0,\dots)\}.$$

It is easy to verify that the map $h_\Gamma\colon Q \to \exp(Q)$ satisfies the following conditions:

(***7***) h_Γ is continuous;

(***8***) $h_\Gamma(Q) \subseteq \exp([0,1]^\infty)$;

(***9***) if $x \in \mathcal{A}_{(\gamma,k)} \setminus \mathcal{A}_{(\gamma',k')}$ then $\gamma < \dim_H(h_\Gamma(x)) \le \gamma'$;

(***10***) for every $x \in Q$, $\dim(h_\Gamma(x)) = 0$.

The condition (***9***) is easy to verify, using the condition (***1***) and the construction of the maps h_γ and h_Γ. Condition (***10***) follows from condition (***6***).

Now we construct maps $h_k, h_{\mathbb{N}\cup\{0\}}\colon Q \to \exp(Q)$ by the formulas

$$h_k(x) = \bigcup_{p=1}^{\infty} \beta_p^k \left(H^k \left([0,1]^k, t^p_{(\gamma_{k'},k)}(x) \right) \times \{(0,0,\dots)\} \right),$$

where k' is defined in condition (***1***) and

$$h_{\mathbb{N}\cup\{0\}}(x) = \bigcup_{k=1}^{\infty} \left[\frac{1}{2^k} h_k(x) + \left(1 - \frac{1}{2^{k-1}}\right)(1,0,0,\dots) \right] \cup \{(1,0,0,\dots)\}.$$

The map $h_{\mathbb{N}\cup\{0\}}\colon Q \to \exp(Q)$ satisfies the following conditions:

(***11***) $h_{\mathbb{N}\cup\{0\}}$ is continuous;

(***12***) $h_{\mathbb{N}\cup\{0\}}(Q) \subseteq \exp([0,1]^\infty)$;

(***13***) if $x \in \mathcal{A}_{(\gamma,k)} \setminus \mathcal{A}_{(\gamma',k+1)}$ for some $\gamma, \gamma' \in \Gamma$ then $\dim(h_{\mathbb{N}\cup\{0\}}(x)) = k$.

Condition (***13***) can be easily verified by using condition (***1***) and the construction of the maps h_k and $h_{\mathbb{N}\cup\{0\}}$.

Now define the map $h\colon Q \to \exp(Q)$ as

$$h(x) = \frac{1}{2} h_{\mathbb{N}\cup\{0\}}(x) \cup \frac{1}{2}\left[h_\Gamma(x) + (1,0,0,\dots)\right].$$

Taking into account conditions (***7***)–(***13***), we can conclude that

(*14*) h is continuous;

(*15*) $h(Q) \subseteq \exp([0,1]^\infty)$;

(*16*) for every $\gamma_i \in \Gamma$ and $k \in \mathbb{N} \cup \{0\}$ where $\gamma_{i+1} \geq k$, $h^{-1}\left[D^{>\gamma_i}_{\geq k}(Q)\right] = \mathcal{A}_{(\gamma_i,k)}$.

Actually, we have constructed a map which creates a set from $\exp(Q)$ with required covering and Hausdorf dimensions. Let us construct now an injective map. This is a final stage of our constructions.

Write the segment $\mathbb{I} = [0,1]$ as $\mathbb{I} = \bigcup\limits_{k=1}^{\infty} \mathbb{I}_k \cup \{1\}$, where $\mathbb{I}_k = \left[1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}\right]$. Let α_k be any homeomorphism of the segment $[-1,1]$ onto the segment $\mathbb{I}_k$. For every $x \in Q$, $x = (x_i)_{i=1}^\infty$ let $\hat{x} \in Q$ be defined as follows:

$$\hat{x} = (\underbrace{x_1}, \underbrace{x_1, x_2}, \underbrace{x_1, x_2, x_3}, \underbrace{x_1, x_2, x_3, x_4}, \ldots).$$

Define a map $\xi \colon Q \to \exp(Q)$ by the formula

$$\xi(x) = \left[\bigcup_{k=1}^{\infty} \{\alpha_k(\hat{x}_k)\} \cup \{1\}\right] \times \{(0,0,\ldots)\}.$$

It is clear that ξ is continuous and, for every $x \in Q$, $\xi(x) \in \exp(Q)$. On the other hand, $\xi(x)$ is a countable subset of Q, therefore, from properties of Hausdorf dimension (see [4, 5]) it follows that $\dim_H(\xi(x)) = 0$ for every $x \in Q$. Clearly, $\dim(\xi(x)) = 0$.

Fix any $x, x' \in Q$, $x = (x_i)_{i=1}^\infty$, $x' = (x'_i)_{i=1}^\infty$. If $x \neq x'$, then there is $i \in \mathbb{N}$ such that $x_i \neq x'_i$. In this case, for some $j \in \mathbb{N}$, $\alpha_j(\hat{x}_j) \neq \alpha_j(\hat{x}'_j)$. Therefore, $\xi(x) \neq \xi(x')$. This implies that the map ξ is an embedding.

Finally, we define the map $g \colon Q \to \exp(Q)$ by the formula

$$g(x) = \bigcup_{y \in F(x)} y + \frac{\mu(x)}{4}\left(\xi(x) \cup [h(x) + (1,0,0,\ldots)] \cup [\xi(x) + (2,0,0,\ldots)]\right).$$

We claim that this map, g, is an approximation of f which is required in the definition of strong $\mathcal{F}_\sigma$-universality. That g is continuous is a consequence of the continuity of all used maps. $\diamondsuit$

Claim 1. The map g is well-defined and satisfies $g|K = f|K$. Moreover, for every $x \in Q$, we have $d_H(f(x), g(x)) \leq \frac{11}{12}\min\{\varepsilon, d_H(f(x), f[K])\}$.

(**a**) Fix $x \in Q$. Then, by condition (*5*), $F(x) \subseteq [-1+\mu(x), 1-\mu(x)]^n$. For every $y \in F(x)$, the diameter of the set

$$y + \frac{\mu(x)}{4}\left(\xi(x) \cup [h(x) + (1,0,0,\ldots)] \cup [\xi(x) + (2,0,0,\ldots)]\right)$$

does not exceed $\mu(x)/4 + \mu(x)/4 + \mu(x)/4 = 3\mu(x)/4$, which implies that $g(x) \subseteq Q$.

(**b**) If $\mu(x) > 0$, then the set $g(x)$ is compact and non-empty, being a finite union of compact non-empty sets. If $\mu(x) = 0$, then the set $g(x) = f(x)$ which is also compact and non-empty. Therefore, for every $x \in Q$, $g(x) \in \exp(Q)$.

(**c**) Fix $x \in Q$. It is clear that $d_H(f(x), g(x)) \leq 2\mu(x) + 3\mu(x)/4 = 11\mu(x)/4$, from which it follows that $d_H(f(x), g(x)) \leq 11/12\min\{\varepsilon, d_H(f(x), f[K])\}$. So we are done because this inequality implies that $g|K = f|K$.

Claim 2. The map g is injective. Let us first observe that from the fact that f is an embedding and Claim 1 it follows that

$$g[Q \backslash K] \cap g[K] = \emptyset. \tag{1}$$

Now fix $x, x' \in Q$. If both x and x' belong to K, then since $g|K = f|K$ and since f is an imbedding, it is trivial that the equality $g(x) = g(x')$ implies the equality $x = x'$. If $x \notin K$ and $x' \in K$, then from (1) it follows that $g(x) \neq g(x')$. So without loss of generality we may assume that $x, x' \in Q \backslash K$.

Let $g(x) = g(x')$. Our aim is to show that $x = x'$. We will first prove that $\mu(x) = \mu(x')$. Assume the contrary, i.e. assume that $\mu(x) < \mu(x')$. Let $\pi\colon Q \to [-1, 1]$ be the projection onto the first coordinate. Choose a point $y = (a, y') \in g(x) = g(x')$ such that

$$a = \min \pi \circ g(x) = \min \pi \circ g(x').$$

Observe that the point y is an element of both sets $F(x)$ and $F(x')$. Since these sets are finite, by construction of the map g, it is easy to see that the set $\pi\left(y + \left[-\frac{\mu(x)}{4}, \frac{\mu(x)}{4}\right]^\infty\right) \cap \pi \circ g(x) = \pi\left(y + \frac{\mu(x)}{4}\xi(x)\right) \cup \mathcal{C}$ (where, clearly, $\mathcal{C}$ is a finite union of finite sets) is infinite (because $\xi(x)$ is infinite), while the set

$$\pi\left(y + \left[-\frac{\mu(x)}{4}, \frac{\mu(x)}{4}\right]^\infty\right) \cap \pi \circ g(x')$$

is finite being a finite union of finite sets. This contradiction establishes that $\mu(x) = \mu(x')$.

Again, consider a point $y_1 = (b, y_1') \in g(x) = g(x')$ such that

$$b = \max \pi \circ g(x) = \max \pi \circ g(x').$$

Since $F(x)$ and $F(x')$ are finite, we can choose $\lambda > 0$ such that

$$\pi(y_1 + [-\lambda, \lambda]^\infty) \cap \pi \circ g(x) = \pi\left(\frac{\mu(x)}{4}[\xi(x) \cap [1-\lambda, 1]^\infty] + \left(b - \frac{\mu(x)}{4}\right)\right) =$$

$$= \pi\left(\frac{\mu(x)}{4}[\xi(x') \cap [1-\lambda, 1]^\infty] + \left(b - \frac{\mu(x)}{4}\right)\right) = \pi(y_1 + [-\lambda, \lambda]^\infty) \cap \pi \circ g(x').$$

Since the coordinates of x appear infinitely often in the coordinates of $\hat{x}$ (at pregiven places), and the same is true for x', it now easily follows that $x = x'$.

Claim 3. The map g is a Z-embedding. Since $g[K] = f[K]$ is a Z-set, it suffices to show that $g[Y]$ is a Z-set whenever $Y \subseteq Q \backslash K$ is compact. But this is clear, because the map $g'\colon Q \longrightarrow \exp(Q)$ defined by the formula

$$g'(x) = \overline{O_\delta(g(x))}$$

maps Q into the complement of $g[Y]$, for every positive δ, and is δ-close to g.

Claim 4. We have $g^{-1}\left[D_{\geq k}^{\geq \gamma_i}(Q)\right] \backslash K = \mathcal{A}_{(\gamma_i, k)} \backslash K$ for $(\gamma_i, k) \in \Gamma \times (\mathbb{N} \cup \{0\})$ where $\gamma_{i+1} \geq k$.

The proof follows from condition (***16***) and from the fact that for every $x \in Q$, $\dim_H(\xi(x)) = \dim(\xi(x)) = 0$.

Corollary 3. *The pair* $\left(\exp(Q), D_{\geq k}^{\geq \gamma_n}(Q)\right)_{(\gamma_n, k) \in \Gamma \times (\mathbb{N} \cup \{0\}), \gamma_{n+1} \geq k}$ *is homeomorphic to the pair* $\left(Q^\infty \times Q^\infty, B(Q)^n \times Q \times \dots \times B(Q)^k \times Q \times \cdots\right)_{k, n \in \mathbb{N} \cup \{0\}, \gamma_{n+1} \geq k}$.

P r o o f. The set $\exp(Q) \backslash \exp_\omega(Q)$ is locally homotopy negligible in $\exp(Q)$ (see [6]) and obviously $D_{\geq k}^{\geq \gamma_n}(Q) \subseteq \exp(Q) \backslash \exp_\omega(Q)$ for every $(\gamma_n, k) \in \Gamma \times (\mathbb{N} \cup \{0\})$, therefore, taking into account Theorem 2 and Proposition 1, we can conclude that the system $\left(D^{>\gamma_n}_{\geq k}(Q)\right)_{(\gamma_n,k)\in\Gamma\times(\mathbb{N}\cup\{0\}),\gamma_{n+1}\geq k}$ is $\mathcal{F}_\sigma$ -absorbing in $\exp(Q)$. Now, the proof follows from Corollary 2 and the uniqueness theorem for absorbing systems (see [8]) in a Hilbert cube. $\diamondsuit$

3. Conclusions and remarks. The results of the paper demonstrate that the topology of the hyperspaces of compacta of given Hausdorf and covering dimension can be described by means of model absorbing system in the Hilbert cube in the case when the values of the Hausdorf dimension form a countable subset of the real line ordered by the type of natural numbers. A natural question arizes whether the results remain valid for other sets of values of the Hausdorf dimension (countable or noncountable).

1. *Banakh T., Radul T., Zarichnyi M.* Absorbing sets in infinite-dimensional manifolds. – Lviv: VNTL Publishers, 1996. – **1**. – 231 p.
2. *Bessaga C., Pełczyński A.* Selected topics in infinite-dimensional topology. – Warszawa, 2001. – 352 p.
3. *Cauty R.* Suites $\mathcal{F}_\sigma$ -absorbantes en theorie de la dimension // Fund. Math. – 1999. – **159**, No. 194/2. – P. 115–126.
4. *Falconer K. J.* The geometry of fractal sets. – Cambridge: Univ. Press, 1985. – 162 p.
5. *Edgar Gerald A.* Measure, topology and fractal geometry. – New York: Springer-Verlag, 1995. – 221 p.
6. *Gladdines H.* Absorbing systems in infinite-dimensional manifolds and applications. – Amsterdam: Vrije Univ., 1994. – 117 p.
7. *Mazurenko N.* Absorbing sets related to Hausdorf dimension // Вісн. Львів. ун-ту. Сер. мех.-мат. – 2003. – Вип. 61. – С. 121–128.
8. *Van Mill J.* The infinite-dimensional topology of function spaces. – Amsterdam: Elsevier, 2001. – 630 p.

ПОГЛИНАЮЧІ СИСТЕМИ В ГІЛЬБЕРТОВОМУ КУБІ, ПОВ'ЯЗАНІ З РОЗМІРНІСТЮ ГАУСДОРФА ТА З РОЗМІРНІСТЮ, ОЗНАЧЕНОЮ ЧЕРЕЗ ПОКРИТТЯ

Описано топологію системи $\left(D^{>\gamma_n}_{\geq k}(Q)\right)_{k\in\mathbb{N}\cup\{0\},\gamma_n\in\Gamma,\gamma_{n+1}\geq k}$, *де* $D^{>\gamma_n}_{\geq k}(Q)=\{A\in\exp(Q)\mid \dim_H(A)>\gamma_n, \dim(A)\geq k\}$ *і* $\Gamma=\{\gamma_i\}_{i=1}^{\infty}$ *– зліченна впорядкована множина така, що* $0<\gamma_1<\gamma_2<\ldots<\infty$.

ПОГЛОЩАЮЩИЕ СИСТЕМЫ В ГИЛЬБЕРТОВОМ КУБЕ, СВЯЗАННЫЕ С РАЗМЕРНОСТЬЮ ХАУСДОРФА И РАЗМЕРНОСТЬЮ, ОПРЕДЕЛЁННОЙ ПОСРЕДСТВОМ ПОКРЫТИЙ

Описана топология системы $\left(D^{>\gamma_n}_{\geq k}(Q)\right)_{k\in\mathbb{N}\cup\{0\},\gamma_n\in\Gamma,\gamma_{n+1}\geq k}$, *где* $D^{>\gamma_n}_{\geq k}(Q)=\{A\in\exp(Q)\mid \dim_H(A)>\gamma_n, \dim(A)\geq k\}$ *и* $\Gamma=\{\gamma_i\}_{i=1}^{\infty}$ *– счётное упорядоченное множество такое, что* $0<\gamma_1<\gamma_2<\ldots<\infty$.

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