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CONCENTRATED LOADING IN TWO-DIMENSIONAL NONLOCAL ELASTIC MEDIUM

According to the nonlocal elasticity theory, the stress at a reference point in the body depends not only on the strains at this point but also on the strains at all other points of the solid. Using the fundamental solution of the Helmholtz equation as an appropriate nonlocal modulus (the weight function in the integral relation between stresses and strains), the stress distribution in a two-dimensional nonlocal elastic medium has been found under the concentrated loading. The nonlocal stress does not contain nonphysical singularities, in contrast to the solution obtained within the frame-work of classical elasticity.

1. Well-developed methods of the classical elasticity break down in a case of concentrated loading as they lead to nonphysical singularities. Considerable research efforts has been expended to develop the nonlocal theory of elasticity and to solve various problems of continuum mechanics using this theory. The essentials of the nonlocal theory were established by Pidstryhach [6], Kunin [2], Kröner [13], Eringen [9], Edelen [8] and others. According to the nonlocal theory of elasticity, the stress at a reference point $\mathbf{x}$ in the body depends not only on the strains at $\mathbf{x}$ but also on the strains at all other points $\mathbf{x}'$. The nonlocal theory reduces to the classical theory of elasticity in the long-wavelength limit and to the atomic lattice theory in the short-wavelength limit. The nonlocal theory is effective in removing nonphysical singularities occurring at dislocations, disclinations, points of applications of singular forces, cracks, etc.

For a linear isotropic nonlocal elastic solid the basic equations are [9]

$$\nabla \cdot \boldsymbol{\sigma}_{\mathrm{Nl}}(\mathbf{x}) = -\,\mathbf{F}(\mathbf{x})\,, \tag{1}$$

$$\boldsymbol{\sigma}_{\mathrm{Nl}}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|,\, \mathbf{c})\,\boldsymbol{\sigma}(\mathbf{x}')\,dV(\mathbf{x}')\,, \tag{2}$$

$$\boldsymbol{\sigma}(\mathbf{x}') = \lambda\,\mathrm{tr}\,\mathbf{e}(\mathbf{x}')\mathbf{I} + 2\mu\mathbf{e}(\mathbf{x}')\,, \tag{3}$$

$$\mathbf{e}(\mathbf{x}') = \mathrm{Def}\,\mathbf{u}(\mathbf{x}')\,. \tag{4}$$

Here $\boldsymbol{\sigma}_{\mathrm{Nl}}(\mathbf{x})$ is the nonlocal stress tensor at a reference point $\mathbf{x}$, $\boldsymbol{\sigma}(\mathbf{x}')$ is the classical stress tensor at the running point $\mathbf{x}'$ related to the linear strain tensor $\mathbf{e}(\mathbf{x}')$ by the classical Hooke law with λ and μ being Lamé constants, $\mathbf{F}$ is the nonlocal body force vector, $\mathbf{u}$ is the displacement vector, $\mathbf{I}$ denotes the unit tensor, ∇ is the gradient operator, Def is the deformator operator.

In a case of vanishing body forces Eringen [11] established conditions under which the displacement field of the boundary-value problem of nonlocal elasticity is identical to that of the classical elasticity theory. Considering the equilibrium equation of the classical elasticity

$$\nabla' \cdot \boldsymbol{\sigma}(\mathbf{x}') = -\,\mathbf{f}(\mathbf{x}')\,,$$

where $\mathbf{f}(\mathbf{x}')$ is the local body force vector, the following expression [4, 15]

$$\mathbf{F}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|,\, \mathbf{c})\,\mathbf{f}(\mathbf{x}')\,dV(\mathbf{x}') \tag{5}$$

gives the connection between the body forces causing the same displacement fields in local and nonlocal elastic media.

The volume integrals in (2) and (5) are over the region V occupied by the solid. The nonlocal modulus $\alpha(|\mathbf{x}' - \mathbf{x}|,\, \mathbf{c})$ includes the nonlocal parameter c such as

$$\lim_{c\to 0}\alpha(|\mathbf{x}'-\mathbf{x}|,\mathbf{c})=\delta(|\mathbf{x}'-\mathbf{x}|),$$

where $\delta(\mathbf{x})$ is the Dirac delta function.

Eringen [10] has ascertained the properties of the modulus $\alpha(|\mathbf{x}'-\mathbf{x}|,\mathbf{c})$ and found several different forms giving a perfect match with the Born – Kármán model of the atomic lattice dynamics and the atomic dispersion curves. In the present paper we use the two-dimensional nonlocal modulus

$$\alpha(|\mathbf{x}'-\mathbf{x}|,\mathbf{c})=\frac{1}{2\pi c^2}K_0\left(\frac{|\mathbf{x}'-\mathbf{x}|}{c}\right) \tag{6}$$

with $K_0(x)$ being the modified Bessel function.

As the kernel (6) is the fundamental solution of the two-dimensional Helmholtz operator

$$\frac{1}{2\pi}\left(\Delta-\frac{1}{c^2}\right)K_0\left(\frac{r}{c}\right)=\delta(x)\,\delta(y), \tag{7}$$

it follows from equations (2), (6) and (7) that

$$\Delta\boldsymbol{\sigma}_{\mathrm{Nl}}-\frac{1}{c^2}\boldsymbol{\sigma}_{\mathrm{Nl}}=-\frac{1}{c^2}\boldsymbol{\sigma}. \tag{8}$$

2. Let the concentrated loading act at the origin. Then the body force density $\mathbf{f}$ takes the form [1, 7]:

$$\mathbf{f}=\frac{\mathbf{P}}{n!}(-1)^n(\mathbf{I}\cdot\boldsymbol{\nabla})^n\delta(\mathbf{x}). \tag{9}$$

When $n=0$, we obtain the concentrated force

$$\mathbf{f}=\mathbf{P}\delta(\mathbf{x}). \tag{10}$$

For a case $n=1$ we have

$$\begin{aligned}\mathbf{f}=\mathbf{i}&\left[-\left(P_{11}\frac{\partial}{\partial x}+P_{12}\frac{\partial}{\partial y}+P_{13}\frac{\partial}{\partial z}\right)+\frac{1}{2}\left(M_3\frac{\partial}{\partial y}-M_2\frac{\partial}{\partial z}\right)\right]\delta(\mathbf{x})+\\ &+\mathbf{j}\left[-\left(P_{21}\frac{\partial}{\partial x}+P_{22}\frac{\partial}{\partial y}+P_{23}\frac{\partial}{\partial z}\right)+\frac{1}{2}\left(M_1\frac{\partial}{\partial z}-M_3\frac{\partial}{\partial x}\right)\right]\delta(\mathbf{x})+\\ &+\mathbf{k}\left[-\left(P_{31}\frac{\partial}{\partial x}+P_{32}\frac{\partial}{\partial y}+P_{33}\frac{\partial}{\partial z}\right)+\frac{1}{2}\left(M_2\frac{\partial}{\partial x}-M_1\frac{\partial}{\partial y}\right)\right]\delta(\mathbf{x}),\end{aligned} \tag{11}$$

where

$$P_{ij}=\frac{1}{2}(P_i\ell_j+P_j\ell_i),\qquad M_i=e_{ijk}\ell_iP_k$$

with e_{ijk} being the Levi – Civita tensor. The components

$$\mathbf{f}=-\,\mathbf{i}\,P_{11}\frac{\partial}{\partial x}\delta(\mathbf{x}),\qquad \mathbf{f}=-\,\mathbf{j}\,P_{22}\frac{\partial}{\partial y}\delta(\mathbf{x}),\qquad \mathbf{f}=-\,\mathbf{k}\,P_{33}\frac{\partial}{\partial z}\delta(\mathbf{x})$$

refer to the double forces acting in the corresponding direction.

In a case under consideration the force $\mathbf{P}$ has the components $\mathbf{P}=P_1\mathbf{i}+P_2\mathbf{j}$, while the arm is written as $\boldsymbol{\ell}=\ell_1\mathbf{i}+\ell_2\mathbf{j}$.

For the concentrated force (10) the classical solution of the problem is well-known and reads [3]

$$\begin{aligned}\sigma_{xx}&=\left[-(5-4\nu)A_1\cos\theta+(1-4\nu)A_2\sin\theta\right]\frac{1}{r}-\left[A_1\cos3\theta+A_2\sin3\theta\right]\frac{1}{r},\\ \sigma_{yy}&=\left[(1-4\nu)A_1\cos\theta-(5-4\nu)A_2\sin\theta\right]\frac{1}{r}+\left[A_1\cos3\theta+A_2\sin3\theta\right]\frac{1}{r},\\ \sigma_{xy}&=-(3-4\nu)\left[A_1\sin\theta+A_2\cos\theta\right]\frac{1}{r}+\left[-A_1\sin3\theta+A_2\cos3\theta\right]\frac{1}{r},\end{aligned} \tag{12}$$

where $A_i=\dfrac{P_i}{8\pi(1-\nu)}$, $i=1,2$; ν is the Poisson ratio.

Inserting the components of the classical stress tensor (12) into (8) and solving the obtained nonhomogeneous Helmholtz equation we arrive at the regular nonlocal solution

$$\sigma_{xx}^{\mathrm{Nl}} = \left[(5-4\nu)A_1\cos\theta - (1-4\nu)A_2\sin\theta\right]\frac{1}{c}k_1\left(\frac{r}{c}\right) - \\ - \left[A_1\cos 3\theta + A_2\sin 3\theta\right]\frac{1}{c}k_3\left(\frac{r}{c}\right),$$

$$\sigma_{yy}^{\mathrm{Nl}} = -\left[(1-4\nu)A_1\cos\theta - (5-4\nu)A_2\sin\theta\right]\frac{1}{c}k_1\left(\frac{r}{c}\right) + \\ + \left[A_1\cos 3\theta + A_2\sin 3\theta\right]\frac{1}{c}k_3\left(\frac{r}{c}\right),$$

$$\sigma_{xy}^{\mathrm{Nl}} = (3-4\nu)\left[A_1\sin\theta + A_2\cos\theta\right]\frac{1}{c}k_1\left(\frac{r}{c}\right) + \\ + \left[-A_1\cos 3\theta + A_2\sin 3\theta\right]\frac{1}{c}k_3\left(\frac{r}{c}\right). \qquad (13)$$

In this paper the following notation is used for the «regular parts» of the modified Bessel functions:

$$k_1(x) = K_1(x) - \frac{1}{x}, \qquad k_2(x) = K_2(x) - \frac{2}{x^2},$$

$$k_3(x) = K_3(x) + \frac{1}{x} - \frac{8}{x^3}, \qquad k_4(x) = K_4(x) + \frac{4}{x^2} - \frac{48}{x^4}.$$

We also present the solution for double force in the y -direction

$$\sigma_{xx}^{\mathrm{Nl}} = -2\nu A_2\frac{\cos 2\theta}{c^2}k_2\left(\frac{r}{c}\right) + A_2\frac{\cos 4\theta}{2c^2}k_4\left(\frac{r}{c}\right),$$

$$\sigma_{yy}^{\mathrm{Nl}} = -2(1-\nu)A_2\frac{\cos 2\theta}{c^2}k_2\left(\frac{r}{c}\right) - A_2\frac{\cos 4\theta}{2c^2}k_4\left(\frac{r}{c}\right),$$

$$\sigma_{xy}^{\mathrm{Nl}} = (1-2\nu)A_2\frac{\sin 2\theta}{c^2}k_2\left(\frac{r}{c}\right) + A_2\frac{\sin 4\theta}{2c^2}k_4\left(\frac{r}{c}\right) \qquad (14)$$

and for two-dimensional center of dilatation ($A_1 = A_2 = A$)

$$\sigma_{rr}^{\mathrm{Nl}} = -\sigma_{\theta\theta}^{\mathrm{Nl}} = 2(1-2\nu)\frac{A}{c^2}k_2\left(\frac{r}{c}\right). \qquad (15)$$

The nondimensional stress component $\bar{\sigma}_{rr} = \sigma_{rr}\dfrac{c^2}{(1-2\nu)A}$ is shown in fig. 1. The curve *1* corresponds to the classical result which has nonphysical singularity at the origin, the curve *2* represents the nonlocal solution for two-dimensional center of dilatation calculated from equation (15).

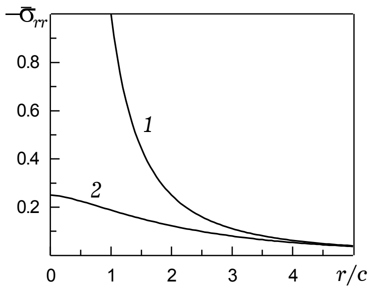

Fig. 1

In a similar manner we can interpret other terms in equation (9) which correspond to point loading of higher order.

It was pointed out in the paper [5] that the nonhomogeneous Helmholtz equation for the stress tensor with another structure constant c was obtained in the gauge theory of defects [14]; the «dual» nonhomogeneous Helmholtz equation in terms of strains arised in the gradient theory of elasticity [12]. From mathematical point of view the solution obtained in this paper can be also used in both these theories with corresponding changes in physical interpretation of the results.

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ЗОСЕРЕДЖЕНЕ НАВАНТАЖЕННЯ У ДВОВИМІРНОМУ НЕЛОКАЛЬНО ПРУЖНОМУ СЕРЕДОВИЩІ

У нелокальній теорії пружності напруження у відліковій точці тіла залежать не тільки від деформації у цій точці, але також від деформації у всіх інших точках твердого тіла. Використовуючи фундаментальний розв'язок рівняння Гельмгольца як відповідний модуль нелокальності (вагову функцію в інтегральному співвідношенні між напруженнями та деформаціями), визначено розподіл напружень у двовимірному нелокально пружному середовищі під дією зосередженого навантаження. Нелокальні напруження не містять нефізичних сингулярностей, на відміну від розв'язків, отриманих у рамках класичної теорії пружності.

СОСРЕДОТОЧЕННАЯ НАГРУЗКА В ДВУМЕРНОЙ НЕЛОКАЛЬНО УПРУГОЙ СРЕДЕ

В нелокальной теории упругости напряжения в отсчетной точке тела зависят не только от деформации в этой точке, но также от деформации во всех остальных точках твердого тела. Используя фундаментальное решение уравнения Гельмгольца в качестве соответствующего модуля нелокальности (весовой функции в интегральном соотношении между напряжениями и деформациями), определено распределение напряжений в двумерной нелокально упругой среде под действием сосредоточенной нагрузки. Нелокальные напряжения не содержат нефизических сингулярностей, в отличие от решений, полученных в рамках классической теории упругости.

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