UDK 539.3

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ON STRESS INTENSITY FACTORS FOR TRANSIENT THERMAL LOADING IN ORTHOTROPIC THIN PLATE WITH CRACK

The paper deals with the transient thermal stress problem in an orthotropic plate, containing Griffith crack, perpendicular to the surfaces of the plate. It is assumed that transient thermal stress is caused by application of heat flow to the crack faces and the heat flow due to convection from the plate surfaces. The problem is formulated in terms of displacement potentials and the analytical solution is found for the stress intensity factor. Numerical results illustrate the dependence of stress intensity factor on thermal and elastic constants of orthotropic material.

Introduction. To obtain the stress intensity factors for the transient state is a very significant problem. When the crack surfaces are subjected to a transient thermal loading, the stress intensity factor will increase with passage of time and, sooner or later, it will exceed the critical stress intensity factor of the material. Then, the crack extends, suddenly. Thus the analysis of the stress intensity factor under transient thermal loading estimates when the crack extends and, further, the life of the materials. In this study, we assume that the Griffith crack in a thin plate is subjected to known heat flux, which is dependent on time and position.

N. Sumi [11, 12], A. Y. Aköz and T. R. Tauchert [1], C. Atkinson and D. L. Clement [2], D. L. Clements [3], Y. M. Tsai [14] and the author [9] have solved various problems for an anisotropic thermoelastic solid with cracks. The radiation boundary conditions have been considered by G. M. L. Gladwell, J. R. Barber and Z. Olesiak [4], T. Koizumi and H. Niwa [6], V. C. Ting and H. R. Jacobs [13], R. Ishida [5], N. Noda and Y. Matsunaga [7].

Analysis. We shall suppose that the crack, situated in an infinite orthotropic thin plate of thickness $2h$ is opened by the application of heat flux depending on time and position to its flat surfaces and the heat exchange by convection on the upper and lower surface of the plate. The crack is assumed to occupy the region $|x| \le a$ in the plate at $y = 0$ in a rectangular co-ordinate system, as shown in Fig. 1.

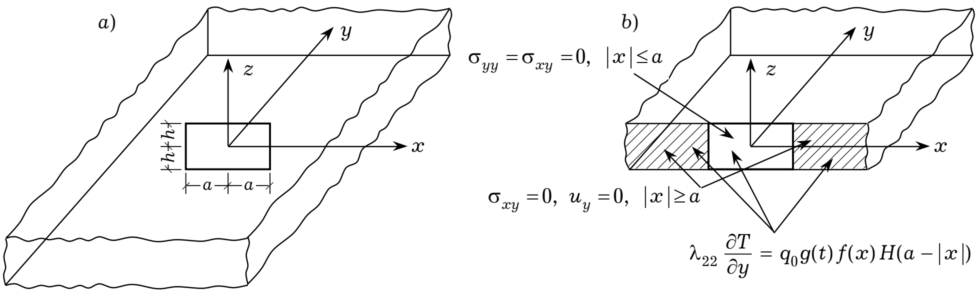

Fig. 1

The heat conduction equation governing an unsteady-state temperature field in orthotropic thin plate with heat dissipation at both plane surfaces is

$$\lambda_{11} \frac{\partial^2 T}{\partial x^2} + \lambda_{22} \frac{\partial^2 T}{\partial y^2} - \frac{\gamma}{h} T = c\rho \frac{\partial T}{\partial t}, \tag{1}$$

where γ is the heat transfer coefficient on the plane surfaces, c is the specific heat, ρ is the density and λ_{11}, λ_{22} are the thermal conductivities in the x- and y-directions, respectively. If the thermal and mechanical conditions on the upper surface of the crack are identical with those on the lower surface, one is able to reduce the problem under consideration to a problem for the semi-infinite thin plate $y \geq 0$. Therefore the initial and boundary conditions for the temperature field are

$$T = 0, \qquad t = 0\,, \tag{2}$$

$$\lambda_{22} \frac{\partial T}{\partial y} = q_0 g(t) f(x) H(a - |\,x\,|), \qquad y = 0\,, \tag{3}$$

where q_0 is the heat flux per unit area per unit time and $H(\cdot)$ denotes Heaviside's step function. Applying both the Laplace transform with respect to the time t and the cosine-Fourier transform with respect to the variable x, and using the convolution theorem for inverse Laplace transform, one can find the solution of (1), which satisfies (2) and (3), in the form

$$T = \int\limits_0^t g(t-\tau)\left[-\frac{4}{\pi^2}\cdot\frac{q_0\chi\lambda^2}{\lambda_{22}}\int\limits_0^\infty \bar{f}(s)\cos(sx)\,ds\int\limits_0^\infty \cos(py)e^{-\chi(m^2+s^2+\lambda^2p^2)\tau}dp\right]d\tau =$$

$$= \int\limits_0^\infty\left[\int\limits_0^\infty \theta(s,p,t)\cos(py)\,dp\right]\cos(sx)\,ds\,, \tag{4}$$

where

$$\theta(s,p,t) = -\frac{4}{\pi^2}\cdot\frac{q_0\chi\lambda^2}{\lambda_{22}}\bar{f}(s)\int\limits_0^t g(t-\tau)e^{-\chi(m^2+s^2+\lambda^2p^2)\tau}\,d\tau\,, \tag{5}$$

$$\bar{f}(s) = \int\limits_0^a f(x)\cos(sx)\,dx, \qquad m^2 = \frac{\gamma}{\lambda_{11}h}, \qquad \chi = \frac{\lambda_{11}}{c\rho}, \qquad \lambda^2 = \frac{\lambda_{22}}{\lambda_{11}}. \tag{6}$$

Next, we consider the stress and displacement fields. The stress-strain equations for an orthotropic medium under plane stress state are

$$\sigma_{xx} = c_{11}e_{xx} + c_{12}e_{yy} - \beta_1 T\,,$$

$$\sigma_{yy} = c_{12}e_{xx} + c_{22}e_{yy} - \beta_2 T\,,$$

$$\sigma_{xy} = 2Ge_{xy}\,, \tag{7}$$

where e_{ij} are the strain components, σ_{ij} are the stress components, c_{ij} are the moduli of elasticity of the material, G is the shear modulus, $\beta_1 = c_{11}\alpha_1 + c_{12}\alpha_2$, $\beta_2 = c_{12}\alpha_1 + c_{22}\alpha_2$ and α_1, α_2 are the thermal expansion coefficients along x- and y-directions, respectively. The strain components are

$$e_{xx} = \frac{\partial u_x}{\partial x}, \qquad e_{yy} = \frac{\partial u_y}{\partial y}, \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \tag{8}$$

where u_x and u_y are the displacement components along the axis. The equations of equilibrium for the plane stressed state in the absence of the body forces are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{9}$$

From (7), (8) and (9), it follows that

$$c_{11}\frac{\partial^2 u_x}{\partial x^2} + G\frac{\partial^2 u_x}{\partial y^2} + (c_{12}+G)\frac{\partial^2 u_y}{\partial x \partial y} = \beta_1\frac{\partial T}{\partial x}\,,$$

$$G\frac{\partial^2 u_y}{\partial x^2} + c_{22}\frac{\partial^2 u_y}{\partial y^2} + (c_{12}+G)\frac{\partial^2 u_x}{\partial x \partial y} = \beta_2\frac{\partial T}{\partial y}\,. \tag{10}$$

The general solution of the equilibrium equations (10) may be obtained as the superposition of two fields. The first one corresponds to the solution of the homogeneous equation (10), for which [8]:

$$u_x = \frac{\partial}{\partial x}(k\varphi_1 + \varphi_2), \qquad u_y = \frac{\partial}{\partial y}(\varphi_1 + k\varphi_2), \tag{11}$$

$$\sigma_{xx} = -G(k+1)\frac{\partial^2}{\partial y^2}(\varphi_1 + \varphi_2),$$

$$\sigma_{yy} = -G(k+1)\frac{\partial^2}{\partial x^2}(\varphi_1 + \varphi_2),$$

$$\sigma_{xy} = G(k+1)\frac{\partial^2}{\partial x \partial y}(\varphi_1 + \varphi_2), \tag{12}$$

$$\frac{\partial^2 \varphi_i}{\partial x^2} + \frac{1}{s_i^2}\cdot\frac{\partial^2 \varphi_i}{\partial y^2} = 0, \qquad i = 1,2, \tag{13}$$

where

$$Gc_{22}s_i^4 - \left(c_{11}c_{22} - c_{12}^2 - 2c_{12}G\right)s_i^2 + Gc_{11} = 0, \qquad i = 1,2,$$

$$k = \frac{c_{22}s_1^2 - G}{c_{12} + G}. \tag{14}$$

The second may be obtained in terms of the thermoelastic displacement potential function $\psi(x, y, t)$ defined as follows:

$$u_x = \frac{\partial \psi}{\partial x}, \qquad u_y = \ell\frac{\partial \psi}{\partial y}. \tag{15}$$

Equations (10) are satisfied if

$$c_{11}\frac{\partial^2 \psi}{\partial x^2} + G\frac{\partial^2 \psi}{\partial y^2} + \ell(c_{12} + G)\frac{\partial^2 \psi}{\partial y^2} = \beta_1 T,$$

$$G\ell\frac{\partial^2 \psi}{\partial x^2} + c_{22}\ell\frac{\partial^2 \psi}{\partial y^2} + (c_{12} + G)\frac{\partial^2 \psi}{\partial x^2} = \beta_2 T. \tag{16}$$

The suitable expression for ψ defined by (16) for the temperature distribution in (4) is

$$\psi = \int_0^\infty \left[\int_0^\infty C(s,p,t)\cos(py)\,dp\right]\cos(sx)\,ds. \tag{17}$$

This satisfies both the equations (16) providing

$$C(s,p,t)\left[c_{11}s^2 + Gp^2 + \ell p^2(c_{12} + G)\right] = -\beta_1\theta(s,p,t),$$

$$C(s,p,t)\left[\ell\left(c_{22}p^2 + Gs^2\right) + s^2(c_{12} + G)\right] = -\beta_2\theta(s,p,t), \tag{18}$$

i.e.

$$\ell(s,p) = \frac{\beta_1 s^2(c_{12} + G) - \beta_2\left(c_{11}s^2 + p^2 G\right)}{\beta_2 p^2(c_{12} + G) - \beta_1\left(c_{22}p^2 + s^2 G\right)},$$

$$C(s,p,t) = \theta(s,p,t)\frac{\beta_2 p^2(c_{12} + G) - \beta_1\left(c_{22}p^2 + s^2 G\right)}{\left(c_{11}s^2 + Gp^2\right)\left(c_{22}p^2 + Gs^2\right) - (c_{12} + G)^2 p^2 s^2}. \tag{19}$$

The solutions of Eqs (13) appropriate to our problem are

$$\varphi_1(x,y) = -\frac{s_2}{G(k+1)(s_1-s_2)}\int_0^\infty s^{-1}A(s)e^{-s_1sy}\cos(sx)\,ds\,,$$

$$\varphi_2(x,y) = \frac{s_1}{G(k+1)(s_1-s_2)}\int_0^\infty s^{-1}B(s)\,e^{-s_2sy}\cos(sx)\,ds\,. \tag{20}$$

Using the obtained potentials, we find:

$$u_x(x,y,t) = \frac{1}{G(k+1)(s_1-s_2)}\int_0^\infty \left[ks_2A(s)e^{-s_1sy} - s_1B(s)e^{-s_2sy}\right]\sin(sx)\,ds -$$

$$-\int_0^\infty\left[\int_0^\infty sC(s,p,t)\cos(py)\,dp\right]\sin(sx)\,ds\,,$$

$$u_y(x,y,t) = \frac{s_1s_2}{G(k+1)(s_1-s_2)}\int_0^\infty \left[A(s)e^{-s_1sy} - kB(s)e^{-s_2sy}\right]\cos(sx)\,ds -$$

$$-\int_0^\infty\left[\int_0^\infty pC(s,p,t)\ell(s,p)\sin(py)\,dp\right]\cos(sx)\,ds\,, \tag{21}$$

$$\sigma_{xx}(x,y,t) = \frac{s_1s_2}{s_1-s_2}\int_0^\infty s\left[s_1A(s)e^{-s_1sy} - s_2B(s)e^{-s_2sy}\right]\cos(sx)\,ds -$$

$$+G\int_0^\infty\left[\int_0^\infty p^2C(s,p,t)\left[\ell(s,p)+1\right]\cos(py)\,dp\right]\cos(sx)\,ds\,,$$

$$\sigma_{yy}(x,y,t) = \frac{1}{s_1-s_2}\int_0^\infty s\left[s_2A(s)e^{-s_1sy} - s_1B(s)\right]e^{-s_2sy}\cos(sx)\,ds +$$

$$+G\int_0^\infty\left[\int_0^\infty s^2C(s,p,t)\left[\ell(s,p)+1\right]\cos(py)\,dp\right]\cos(sx)\,ds\,,$$

$$\sigma_{xy}(x,y,t) = -\frac{s_1s_2}{s_1-s_2}\int_0^\infty s\left[A(s)e^{-s_1sy} - B(s)e^{-s_2sy}\right]\sin(sx)\,ds +$$

$$+G\int_0^\infty\left[\int_0^\infty psC(s,p,t)\left[\ell(s,p)+1\right]\sin(py)\,dp\right]\sin(sx)\,ds\,. \tag{22}$$

The mechanical boundary conditions on the plane $y=0$ are

$$\sigma_{xy} = 0\,, \tag{23}$$

$$\sigma_{yy} = 0, \qquad |\,x\,| \le a, \qquad u_y = 0, \qquad |\,x\,| > a\,. \tag{24}$$

Applying $(22)_3$ to the boundary condition (23) we obtain $A = B$. Substituting $(21)_2$ and $(22)_2$ into the boundary conditions (24), we obtain the following dual integral equations for $A(s)$:

$$\int_0^\infty sA(s)\cos(sx)\,ds = F(x), \qquad |\,x\,| \le a\,,$$

$$\frac{1}{GC_1}\int_0^\infty A(s)\cos(sx)\,ds = 0, \qquad |\,x\,| > a\,, \tag{25}$$

where

$$F(x) = G\int_0^\infty\int_0^\infty s^2 C(s,p,t)\left[\ell(s,p)+1\right]\cos(sx)\,ds\,dp\,,$$

$$C_1 = (k+1)(k-1)^{-1}\left(s_2^{-1} - s_1^{-1}\right). \tag{26}$$

The solution to the dual integral equations of the form (25) is [6]

$$A(s) = \frac{2}{\pi}\int_0^a \beta J_0(s\beta)\left[\int_0^\infty J_0(q\beta)G(q)\,dq\right]d\beta\,, \tag{27}$$

where $J_0(\cdot)$ is a Bessel function of the first kind and zeroth order and

$$G(q) = G\int_0^\infty q^2 C(q,p,t)\left[\ell(q,p)+1\right]dp\,. \tag{28}$$

We obtain the complete solution of the problem by substituting (27), (26) into (21), (22).

Note that

$$GC(s,p,t)\left[\ell(s,p)+1\right] =$$

$$= -\frac{G\left(c_{11}c_{22}-c_{12}^2\right)\left(\alpha_1 p^2+\alpha_2 s^2\right)}{Gc_{11}s^4 + s^2p^2\left[c_{11}c_{22}-c_{12}^2-2Gc_{12}\right]+Gc_{22}p^4}\,\theta(s,p,t) =$$

$$= -\frac{\alpha_1 p^2+\alpha_2 s^2}{s_{22}s^4+\left(2s_{12}+s_{66}\right)s^2p^2+s_{11}p^4}\,\theta(s,p,t)\,, \tag{29}$$

where $s_{11} = \dfrac{c_{22}}{c_{11}c_{22}-c_{12}^2} = \dfrac{1}{E_1}$, $s_{22} = \dfrac{c_{11}}{c_{11}c_{22}-c_{12}^2} = \dfrac{1}{E_2}$,

$$s_{12} = -\frac{c_{12}}{c_{11}c_{22}-c_{12}^2} = -\frac{\nu_{12}}{E_2}, \quad s_{66} = \frac{1}{G}$$

are the elastic compliances of the material in the x- and y-directions, respectively, and E_1, E_2 are the longitudinal elastic moduli, and ν_{12} is Poisson's ratio. The transient thermal stress σ_{yy} has a singularity of order $1/\sqrt{r}$ at the crack tips, where r is the distance from the tip.

Therefore, the stress intensity factor K_I of the mode I is defined as

$$K_I = \lim_{x\to a}\sqrt{2(x-a)}\,\sigma_{yy}\Big|_{y=0}\,. \tag{30}$$

The stress intensity factor K_I is calculated from (30) and we get the formula

$$K_I = \frac{2\sqrt{a}}{\sqrt{\pi}}\int_0^\infty J_0(as)G(s)\,ds =$$

$$= -\frac{8\sqrt{a}}{\pi^2\sqrt{\pi}}\cdot\frac{q_0\chi\lambda^2}{\lambda_{22}}\int_0^\infty s^2\bar{f}(s)J_0(as)\int_0^\infty\frac{\alpha_1p^2+\alpha_2s^2}{s_{22}s^4+\left(2s_{12}+s_{66}\right)s^2p^2+s_{11}p^4}\times$$

$$\times\int_0^t g(t-\tau)e^{-\chi(m^2+s^2+\lambda^2p^2)\tau}\,d\tau\,dp\,ds\,. \tag{31}$$

For isotropic material the stress intensity factor (31) assumes the form

$$K_I^{\text{isot}} =$$

$$= -\frac{8\sqrt{a}}{\pi^2\sqrt{\pi}} \cdot \frac{q_0 \chi E \alpha}{\lambda_{\text{isot}}} \int_0^\infty s^2 \bar{f}(s) J_0(as) \int_0^\infty \frac{1}{p^2 + s^2} \int_0^t g(t-\tau) e^{-\chi(m^2+s^2+p^2)\tau} \, dp \, d\tau \, ds . \qquad (32)$$

Numerical results. In calculating the temperature and transient stress intensity factors the following dimensionless quantities are introduce

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{a}, \qquad t' = \frac{\chi t}{a^2}, \qquad M^2 = a^2 m^2 = \frac{\gamma a^2}{\lambda_{11} h}, \qquad \lambda^2 = \frac{\lambda_{22}}{\lambda_{11}},$$

$$\alpha = \frac{\alpha_2}{\alpha_1}, \qquad E = \frac{E_1}{E_2}, \qquad \mu = \frac{2s_{12} + s_{66}}{2s_{11}}, \qquad \bar{T} = \frac{T\lambda_{22}}{q_0 a},$$

$$\bar{K}_I = \frac{K_I \lambda_{22}}{\alpha_1 E_1 a\, q_0 \sqrt{a}}, \qquad \bar{\sigma}_{yy} = \frac{\sigma_{yy}\, \lambda_{22}}{\alpha_1 E_1\, a\, q_0}.$$

Numerical calculations were carried out for two types of a heat supply q,

*case **1***: $\quad q = q_0 g(t) f(x) = q_0$;

*case **2***: $\quad q = q_0 g(t) f(x) = q_0 e^{-t'}$,

where t' is the Fourier number.

Fig. 2 shows the effects of λ^2 ($\lambda^2 = 0.25, 1, 2, 4$) on the temperature at $\eta = 0$ for the case ***1*** (at $q = q_0$, $M^2 = 1$, $t' = 1$). The temperature at $\eta = 0$ increases with rise of the ratio of thermal conductivity.

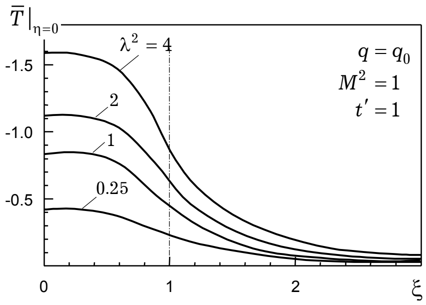

Fig. 2

Figs 3 and 4 show variation of the normal stress $\bar{\sigma}_{yy}$ at $\eta = 0$ with ξ for various values of the effects of λ^2 (fig. 3) and α (fig. 4) for the case ***1*** (at $q = q_0$, $M^2 = 1$, $t' = 1$) of thermal loading.

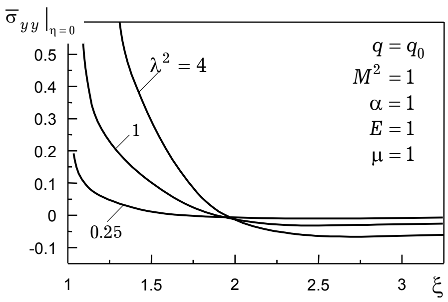

Fig. 3

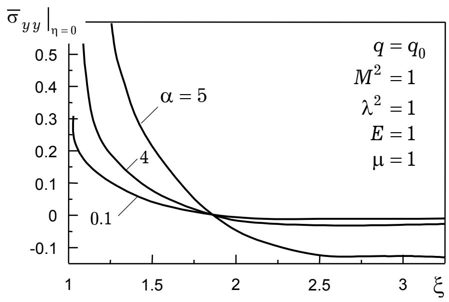

Fig. 4

Figs 5–8 show the effects of the anisotropy of the material constants on the stress intensity factor for case 1 and case 2, when only one among the material constants such as $\lambda^2, \alpha, E, \mu$ indicates various anisotropies and the other material constants are kept equal to those of isotropic conditions.

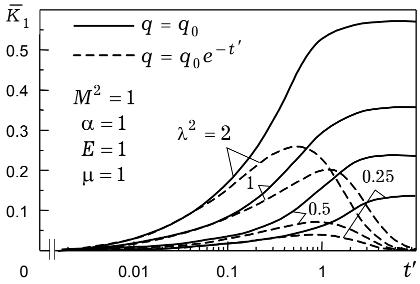

Fig. 5

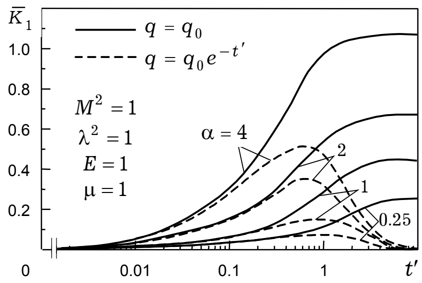

Fig. 6

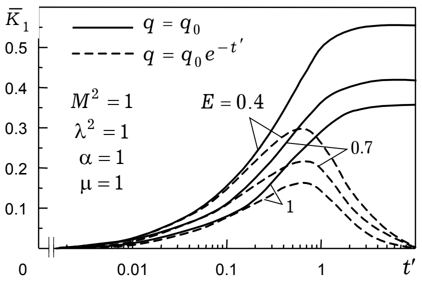

Fig. 7

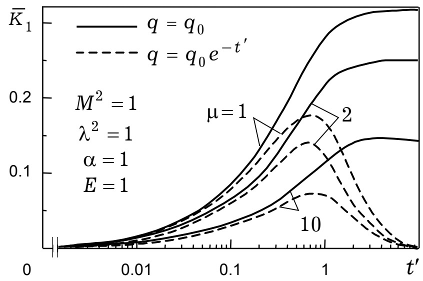

Fig. 8

These figures show that effects of the anisotropy of λ^2, α, E and μ on the stress intensity factor are large.

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ПРО КОЕФІЦІЄНТИ ІНТЕНСИВНОСТІ НАПРУЖЕНЬ ПРИ НЕУСТАЛЕНОМУ ТЕПЛОВОМУ НАВАНТАЖЕННІ В ТОНКІЙ ОРТОТРОПНІЙ ПЛАСТИНЦІ З ТРІЩИНОЮ

Розглядається задача про неусталені теплові напруження в ортотропній пластинці з тріщиною Гріффітса, перпендикулярною до поверхонь пластинки. Припускається, що теплові напруження зумовлені потоком тепла на берегах тріщини та потоком тепла внаслідок конвекції на поверхнях пластинки. Задача формулюється у термінах потенціалів переміщення, для коефіцієнтів інтенсивності напружень отримано аналітичний розв'язок. Числові результати ілюструють залежність коефіцієнта інтенсивності напружень від теплових і пружних параметрів ортотропного матеріалу.

О КОЭФФИЦИЕНТАХ ИНТЕНСИВНОСТИ НАПРЯЖЕНИЙ ПРИ НЕУСТАНОВИВШЕЙСЯ ТЕПЛОВОЙ НАГРУЗКЕ В ТОНКОЙ ОРТОТРОПНОЙ ПЛАСТИНКЕ С ТРЕЩИНОЙ

Рассматривается задача о неустановившихся тепловых напряжениях в ортотропной пластинке с трещиной Гриффитса, перпендикулярной к поверхности пластинки. Предполагается, что тепловые напряжения обусловлены потоком тепла на берегах трещины и потоком тепла вследствие конвекции на поверхностях пластинки. Задача формулируется в терминах потенциалов перемещения, для коэффициента интенсивности напряжений получено аналитическое решение. Численные результаты иллюстрируют зависимость коэффициента интенсивности напряжений от тепловых и упругих параметров ортотропного материала.

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