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ON SEMITOPOLOGICAL BICYCLIC EXTENSIONS OF LINEARLY ORDERED GROUPS

For a linearly ordered group G let us define a subset $A \subseteq G$ to be a shift-set if for any $x, y, z \in A$ with $y < x$ we get $x \cdot y^{-1} \cdot z \in A$. We describe the natural partial order and solutions of equations on the semigroup $B(A)$ of shifts of positive cones of A. We study topologizations of the semigroup $B(A)$. In particular, we show that, for an arbitrary countable linearly ordered group G and a non-empty shift-set A of G, every Baire shift-continuous T_1-topology τ on $B(A)$ is discrete. Also we prove that, for an arbitrary linearly non-densely ordered group G and a non-empty shift-set A of G, every shift-continuous Hausdorff topology τ on the semigroup $B(A)$ is discrete.

Introduction and preliminaries. We shall follow the terminology of [17, 21, 23, 27, 36, 43, 44].

A *semigroup* is a non-empty set with a binary associative operation. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element y in S is called the *inverse* of x and denoted by x^{-1}. The map defined on an inverse semigroup S which maps every element x of S to its inverse x^{-1} is called the *inversion*.

For a semigroup S by $E(S)$ we denote the set of idempotents in S. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as the *band* of S. A *semilattice* is a commutative semigroup of idempotents.

Let $\mathfrak{I}_X$ denote the set of all partial one-to-one transformations of an infinite set X together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathfrak{I}_X$. The semigroup $\mathfrak{I}_X$ is called the *symmetric inverse semigroup* over the set X (see [21].). The symmetric inverse semigroup was introduced by Wagner [1] and it plays a major role in the theory of semigroups.

The bicyclic monoid $C(p, q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The bicyclic monoid is a combinatorial bisimple F-inverse semigroup and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known Andersen's result [9] states that a $(0-)$simple semigroup is completely $(0-)$ simple if and only if it does not contain the bicyclic monoid. The bicyclic monoid does not embed into stable semigroups [38].

Recall from [27] that a *partially-ordered group* is a group $(G, \cdot)$ equipped with a translation-invariant partial order $\le$; in other words, the binary relation $\le$ has the property that, for all $a, b, g \in G$, if $a \le b$ then $a \cdot g \le b \cdot g$ and $g \cdot a \le g \cdot b$.

By e we denote the identity of a group G. The set $G^+ = \{x \in G : e \le x\}$ in a partially ordered group G is called the *positive cone* of G and satisfies the properties:

1°) $G^+ \cdot G^+ \subseteq G^+$;

2°) $G^+ \bigcap (G^+)^{-1} = \{e\}$;

3°) $x^{-1} \cdot G^+ \cdot x \subseteq G^+$ for each $x \in G$.

Any subset P of a group G that satisfies the conditions ***1°–3°*** induces a partial order on G ($x \le y$ if and only if $x^{-1} \cdot y \in P$) for which P is the positive cone. An elements of the set $G^+ \setminus \{e\}$ is called *positive*.

A *linearly ordered* or *totally ordered group* is an ordered group G whose order relation «$\le$» is total (see [16] and [20]).

From now on we shall assume that G is a non-trivial linearly ordered group.

For every $g \in G$ the set

$$G^+(g) = \{x \in G : g \le x\}.$$

is called the *positive cone on element* g in G.

For arbitrary elements $g, h \in G$ we consider a partial map $\alpha_h^g : G \to G$ defined by the formula

$$(x)\alpha_h^g = x \cdot g^{-1} \cdot h, \qquad \text{for} \qquad x \in G^+(g).$$

We observe that Lemma XIII.1 from [16] implies that for such partial map $\alpha_h^g : G \to G$ the restriction $\alpha_h^g : G^+(g) \to G^+(h)$ is a bijective map.

We consider the semigroups

$$B(G) = \{\alpha_h^g : G \to G : g, h \in G\},$$

$$B^+(G) = \{\alpha_h^g : G \to G : g, h \in G^+\},$$

endowed with the operation of the composition of partial maps. Simple verifications show that

$$\alpha_h^g \cdot \alpha_\ell^k = \alpha_b^a, \quad \text{where} \quad a = (h \vee k) \cdot h^{-1} \cdot g \quad \text{and} \quad b = (h \vee k) \cdot k^{-1} \cdot \ell, \qquad (1)$$

for $g, h, k, \ell \in G$, where by $h \vee k$ we denote the join of h and k in the linearly ordered set $(G, \le)$. Therefore, property ***1°*** of the positive cone and condition (1) imply that $B(G)$ and $B^+(G)$ are subsemigroups of $\Im_G$.

By Proposition 1.2 from [31] for a linearly ordered group G the following assertions hold:

(***i***) elements α_h^g and α_g^h are inverse of each other in $B(G)$ for all $g, h \in G$ (respectively, $B^+(G)$ for all $g, h \in G^+$);

(***ii***) an element α_h^g of the semigroup $B(G)$ (respectively, $B^+(G)$) is an idempotent if and only if $g = h$;

(***iii***) $B(G)$ and $B^+(G)$ are inverse subsemigroups of $\Im_G$;

(***iv***) the semigroup $B(G)$ (respectively, $B^+(G)$) is isomorphic to the set $S_G = G \times G$ (respectively, $S_G^+ = G^+ \times G^+$) with the following semigroup operation:

$$(a,b)(c,d) = \begin{cases} (c \cdot b^{-1} \cdot a, d), & b < c, \\ (a, d), & b = c, \\ (a, b \cdot c^{-1} \cdot d), & b > c, \end{cases} \qquad (2)$$

where $a, b, c, d \in G$ (respectively, $a, b, c, d \in G^+$).

It is obvious that:

1°) if G is isomorphic to the additive group of integers $(\mathbb{Z}, +)$ with usual linear order $\leq$, then the semigroup $B^+(G)$ is isomorphic to the bicyclic monoid $C(p, q)$ and the semigroup $B^+(G)$ is isomorphic to the extended bicyclic semigroup $C_{\mathbb{Z}}$ (see [24]);

2°) if G is the additive group of real numbers $(\mathbb{R}, +)$ with usual linear order $\leq$, then the semigroup $B(G)$ is isomorphic to $B^2_{(-\infty,\infty)}$ (see [40, 39]) and the semigroup $B^+(G)$ is isomorphic to $B^1_{[0,\infty)}$ (see [4–8]);

3°) the semigroup $B^+(G)$ is isomorphic to the semigroup $S(G)$ which is defined in [25, 26].

In the paper [31] semigroups $B(G)$ and $B^+(G)$ are studied for a linearly ordered group G. That paper describes Green's relations on $B(G)$ and $B^+(G)$ and their bands and shows that these semigroups are bisimple. Also in [31] it is proved that, for a commutative linearly ordered group G, all non-trivial congruences on the semigroups $B(G)$ and $B^+(G)$ are group congruences if and only if the group G is Archimedean; and the structure of group congruences on the semigroups $B(G)$ and $B^+(G)$ is described.

In this paper we present a more general construction than the semigroups $B(G)$ and $B^+(G)$. Namely, for a linearly ordered group G let us define a subset $A \subseteq G$ to be a *shift-set* if for any $x, y, z \in A$ with $y < x$ we get $x \cdot y^{-1} \cdot z \in A$. For any shift-set $A \subseteq G$ let

$$B(A) = \{\alpha^a_b : G^+(a) \to G^+(b) : a, b \in A\}$$

be the semigroup of partial bijections defined by the formula

$$(x)\alpha^a_b = x \cdot a^{-1} \cdot b \quad \text{for } x \in G^+(a).$$

The semigroup $B(A)$ is isomorphic to the semigroup $S_A = A \times A$ endowed with the binary operation defined by formula (2). For $A = G$ the semigroup $B(A)$ coincides with $B(G)$ and for $A = G^+$ it coincides with the semigroup $B^+(G)$.

Later in this paper for a non-empty shift-set $A \subseteq G$ we identify the semigroup $B(A)$ with the semigroup S_A endowed with the multiplication defined by formula (2). We observe that $B(A)$ is an inverse subsemigroup of $B(G)$ for any non-empty shift-set A of a linearly ordered group G. Moreover, the results of [31] imply that an element (a, b) of $B(A)$ is an idempotent iff $a = b$, and (b, a) is inverse of (a, b) in $B(G)$.

We recall that a topological space X is said to be

- *locally compact*, if every point $x \in X$ has an open neighbourhood with the compact closure;
- *Čech-complete*, if X is Tychonoff and X is a G_δ-set in its Čech – Stone compactification βX;
- *Baire*, if, for each sequence $(U_i)_{i=1}^{\infty}$ of open dense subsets of X, the intersection $\bigcap_{i=1}^{\infty} U_i$ is a dense subset in X.

Every Hausdorff locally compact space is Čech-complete, and every Čech-complete space is Baire (see [23]).

A *semitopological* (*topological*) *semigroup* is a topological space with a separately continuous (jointly continuous) semigroup operation.

A topology τ on a semigroup S is called:

- *semigroup* if (S, τ) is a topological semigroup;
- *shift-continuous* if (S, τ) is a semitopological semigroup.

The bicyclic monoid admits only the discrete semigroup Hausdorff topology and if a topological semigroup S contains it as a dense subsemigroup then $C(p,q)$ is an open subset of S [22]. We observe that the openness of $C(p,q)$ in its closure easily follows from the non-topologizability of the bicyclic monoid, because the discrete subspace D is open in its closure $\bar{D}$ in any T_1-space containing D. Bertman and West in [15] extend this result for the case of Hausdorff semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic monoid [10, 37]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups studied in [11, 12, 33]. Independently Taimanov in [3] constructed a semigroup $\mathfrak{A}_{æ}$ of cardinality æ which admits only the discrete semigroup topology. Also, Taimanov [2] gave sufficient conditions on a commutative semigroup to have a non-discrete semigroup topology. In the paper [29] it was shown that for the Taimanov semigroup $\mathfrak{A}_{æ}$ from [3] the following conditions hold: every T_1-topology τ on the semigroup $\mathfrak{A}_{æ}$ such that $(\mathfrak{A}_{æ}, \tau)$ is a topological semigroup is discrete; for every T_1-topological semigroup which contains $\mathfrak{A}_{æ}$ as a subsemigroup, $\mathfrak{A}_{æ}$ is a closed subsemigroup of S; and every homomorphic non-isomorphic image of $\mathfrak{A}_{æ}$ is a zero-semigroup. Also in the paper [24] it is proved that the discrete topology is the unique shift-continuous Hausdorff topology on the extended bicyclic semigroup $C_{\mathbb{Z}}$. Also, for many $(0-)$bisimple semigroups of transformations S the following statement holds: *every shift-continuous Hausdorff Baire* (*in particular locally compact*) *topology S is discrete* (see [18, 19, 32, 34, 35]). In the paper [42] Mesyan, Mitchell, Morayne and Péresse showed that if E is a finite graph, then the only locally compact Hausdorff semigroup topology on the graph inverse semigroup $G(E)$ is the discrete topology. In [14] it was proved that the conclusion of this statement also holds for graphs E consisting of one vertex and infinitely many loops (i.e., infinitely-generated polycyclic monoids). A surprising dichotomy for the bicyclic monoid with adjoined zero $C^0 = C(p,q) \amalg \{0\}$ was proved in [28]: every Hausdorff locally compact semitopological bicyclic monoid C^0 with adjoined zero is either compact or discrete. The above dichotomy was extended by Bardyla in [13] to locally compact λ-polycyclic semitopological monoids and to locally compact semitopological interassociates of the bicyclic monoid [30].

For a linearly ordered group G and a non-empty shift-set A of G, the natural partial order and solutions of equations on the semigroup $B(A)$ are described. We study topologizations of the semigroups $B(A)$. In particular, we show that for an arbitrary countable linearly ordered group G and a non-empty shift-set A of G, every Baire shift-continuous T_1-topology τ on $B(A)$ is discrete. We also prove that for an arbitrary linearly non-densely ordered group G and a non-empty shift-set A of G, every shift-continuous Hausdorff topology τ on the semigroup $B(A)$ is discrete, and hence $(B(A), \tau)$ is a discrete subspace of any Hausdorff semitopological semigroup which contains $B(A)$ as a subsemigroup.

1. Solutions of some equations and the natural partial order on the semigroup $B(A)$**.** It is well known that every inverse semigroup S admits the *natural partial order*:

$$s \preceq t \quad \text{if and only if} \quad s = et \quad \text{for some} \quad e \in E(S).$$

This order induces the natural partial order on the semilattice $E(S)$, and for arbitrary $s, t \in S$ the following conditions are equivalent:

$$(\alpha): \quad s \preceq t; \qquad (\beta): \quad s = ss^{-1}t; \qquad (\gamma): \quad s = ts^{-1}s \tag{3}$$

(see [41, Chap. 3]).

Proposition 1. *Let* G *be a linearly ordered group and* A *be a non-empty shift-set in* G. *Then the following assertions hold*:

(***i***) *if* $(g,g),(h,h) \in E(B(A))$ *then* $(g,g) \preceq (h,h)$ *if and only if* $g \geq h$ *in* A;

(***ii***) *the semilattice* $E(B(A))$ *is isomorphic to* A *considered as* $\vee$*-semilattice under the isomorphism* $i : E(B(A)) \to A,\ i : (g,g) \to g$;

(***iii***) $(g,h)\mathcal{R}(k,\ell)$ *in* $B(A)$ *if and only if* $g = k$ *in* A;

(***iv***) $(g,h)\mathcal{L}(k,\ell)$ *in* $B(A)$ *if and only if* $h = \ell$ *in* A;

(***v***) $(g,h)\mathcal{H}(k,\ell)$ *in* $B(A)$ *if and only if* $g = k$ *and* $h = \ell$ *in* A, *and hence every* $\mathcal{H}$*-class in* $B(A)$ *is a singleton*;

(***vi***) $B(A)$ *is a bisimple semigroup and hence it is simple*;

P r o o f. Assertions (***i***) and (***ii***) are trivial, (***iii***)−(***v***) follow from Proposition 2.1 from [31] and Proposition 3.2.11 from [41], and (***vi***) follows from Proposition 3.2.5 from [41]. ♦

Later we need the following lemma, which describes the natural partial order on the semigroup $B(A)$.

Lemma 1. *Let* G *be a linearly ordered group and* A *be a non-empty shift-set in* G. *Then for arbitrary elements* $(a,b),(c,d) \in B(A)$ *the following conditions are equivalent:*

(***i***) $(a,b) \preceq (c,d)$ *in* $B(A)$;

(***ii***) $a^{-1} \cdot b = c^{-1} \cdot d$ *and* $a \geq c$ *in* A;

(***iii***) $b^{-1} \cdot a = d^{-1} \cdot b$ *and* $b \geq d$ *in* A.

P r o o f. (***i***) $\Rightarrow$ (***ii***). The equivalence of conditions (α) and (β) in (3) implies that $(a,b) \preceq (c,d)$ in $B(A)$ if and only if $(a,b) = (a,b)(a,b)^{-1}(c,d)$. Therefore we have that

$$(a,b) = (a,b)(a,b)^{-1}(c,d) = (a,b)(b,a)(c,d) = (a,a)(c,d) =$$

$$= \begin{cases} (c \cdot a^{-1} \cdot a, d), & a < c, \\ (c,d), & a = c, \\ (a, a \cdot c^{-1} \cdot d), & a > c. \end{cases}$$

This implies that

$$(a,b) = \begin{cases} (c,d), & a < c, \\ (c,d), & a = c, \\ (a, a \cdot c^{-1} \cdot d), & a > c, \end{cases}$$

and hence the condition $(a,b) \preceq (c,d)$ in $B(A)$ implies that $a^{-1} \cdot b = c^{-1} \cdot d$ and $a \geq c$ in A.

(***ii***) $\Rightarrow$ (***i***). Fix arbitrary $(a,b),(c,d) \in B(A)$ such that $a^{-1} \cdot b = c^{-1} \cdot d$ and $a \geq c$ in A. Then we have that

$$(a,b) = (a,b)(a,b)^{-1}(c,d) = (a,b)(b,a)(c,d) =$$

$$= (a,a)(c,d) = (a, a \cdot c^{-1} \cdot d) = (a,b)$$

and hence $(a,b) \preceq (c,d)$ in $B(A)$.

The proof of the equivalence (***ii***) $\Leftrightarrow$ (***iii***) is trivial. ♦

The definition the semigroup operation in $B(A)$ implies that $(a,b) = (a,c)(c,d)(d,b)$ for arbitrary elements a, b, c, d of the group A. The following two propositions give descriptions of solutions of some equations in the semigroup $B(A)$.

Proposition 2. *Let G be a linearly ordered group, A be a non-empty shift-set in G, and a, b, c, d be arbitrary elements of A. Then the following conditions hold:*

(***i***) *$(a,b) = (a,c)(x,y)$ for $(x,y) \in B(A)$ if and only if $(c,b) \preceq (x,y)$ in $B(A)$;*

(***ii***) *$(a,b) = (x,y)(d,b)$ for $(x,y) \in B(A)$ if and only if $(a,d) \preceq (x,y)$ in $B(A)$;*

(***iii***) *$a,b = (a,c)(x,y)(d,b)$ for $(x,y) \in B(A)$ if and only if $(c,d) \preceq (x,y)$ in $B(A)$.*

P r o o f. (***i***) ($\Rightarrow$). Suppose that $(a,b) = (a,c)(x,y)$ for some $(x,y) \in B(A)$. Then we have that

$$(a,c)(x,y) = \begin{cases} (a\cdot, c \cdot x^{-1} \cdot y), & c > x, \\ (a,y), & c = x, \\ (x \cdot c^{-1} \cdot a, y), & c < x. \end{cases}$$

Then in the case when $c > x$ we get that $b = c \cdot x^{-1} \cdot y$ and hence Lemma 1 implies that $(c,b) \leq (x,y)$ in $B(A)$. Also, in the case when $c = x$ we have that $b = y$, which implies the inequality $(c,b) \leq (x,y)$ in $B(A)$. The case $c < x$ does not hold because the group operation on G implies that $x \cdot c^{-1} \cdot a < a$.

($\Leftarrow$). Suppose that the relation $(c,b) \leq (x,y)$ holds in $B(A)$. Then by Lemma 1 we have that $c^{-1} \cdot b = x^{-1} \cdot y$ and $c \geq x$ in A, and hence the semigroup operation of $B(A)$ implies that

$$(a,c)(x,y) = (a, c \cdot x^{-1} \cdot y) = (a, c \cdot c^{-1} \cdot b) = (a,b).$$

The proof of statement (***ii***) is similar to statement (***i***).

(***iii***) ($\Rightarrow$). Suppose that $(a,b) = (a,c)(x,y)(d,b)$ for some $(x,y) \in B(A)$.

Then we have that

$$(a,c)(x,y) = \begin{cases} (a\cdot, c \cdot x^{-1} \cdot y), & c > x, \\ (a,y), & c = x, \\ (x \cdot c^{-1} \cdot a, y), & c < x. \end{cases}$$

Therefore,

($\boldsymbol{a}$) if $c > x$, then

$$(a,c)(x,y)(d,b) = (a, c\cdot x^{-1}\cdot y)(d,b) =$$

$$= \begin{cases} (a\cdot, c\cdot x^{-1}\cdot y\cdot d^{-1}\cdot b), & c\cdot x^{-1}\cdot y > d, \\ (a,b), & c\cdot x^{-1}\cdot y = d, \\ (d\cdot y^{-1}\cdot x\cdot c^{-1}\cdot a, b), & c\cdot x^{-1}\cdot y < d; \end{cases}$$

($\boldsymbol{b}$) if $c = x$, then

$$(a,c)(x,y)(d,b) = (a,y)(d,b) = \begin{cases} (a\cdot, y\cdot d^{-1}\cdot b), & y > d, \\ (a,b), & y = d, \\ (d\cdot y^{-1}\cdot a, b), & y < d; \end{cases}$$

($\boldsymbol{c}$) if $c < x$, then

$$(a,c)(x,y)(d,b) = (x\cdot c^{-1}\cdot a, y)(d,b) = \begin{cases} (x\cdot c^{-1}\cdot a\cdot, y\cdot d^{-1}\cdot b), & y > d, \\ (x\cdot c^{-1}\cdot a, b), & y = d, \\ (d\cdot y^{-1}\cdot x\cdot c^{-1}\cdot a, b), & y < d. \end{cases}$$

Then the equality $(a,b) = (a,c)(x,y)(d,b)$ implies that

in case ($\boldsymbol{a}$): if $c > x$, then $c\cdot x^{-1}\cdot y\cdot d^{-1} = e$ in G,

in case ($\boldsymbol{b}$): if $c = x$, then $y = d$,

and the case ($\boldsymbol{c}$) does not hold. Hence, by Lemma 1 we get that $(c,d) \preceq (x,y)$ in $B(A)$.

($\Leftarrow$). Suppose that the relation $(c,d) \le (x,y)$ holds in $B(A)$. Then by Lemma 1 we have that $c^{-1}\cdot d = x^{-1}\cdot y$ and $c \ge x$ in A, and hence the semigroup operation of $B(A)$ implies that

$$(a,c)(x,y)(d,b) = (a,c)(x,y)(c\cdot x^{-1}\cdot y, b) = (a,c)(c\cdot x^{-1}\cdot y\cdot y^{-1}\cdot x, b) =$$

$$= (a,c)(c\cdot x^{-1}\cdot x, b) = (a,c)(c,b) = (a,b),$$

because $c\cdot x^{-1}\cdot y \ge y$ in A.

Proposition 3. *Let G be a linearly ordered group, A be a non-empty shift-set in G, and a, b, c, d be arbitrary elements of A. Then the following conditions hold:*

($\boldsymbol{i}$) *if $a < c$ in A, then the equation $(a,b) = (c,d)(x,y)$ has no solutions in $B(A)$;*

($\boldsymbol{ii}$) *if $a > c$ in A, then the equation $(a,b) = (c,d)(x,y)$ has the unique solution $(x,y) = (a\cdot c^{-1}\cdot d, b)$ in $B(A)$;*

($\boldsymbol{iii}$) *the equation $(a,b) = (a,d)(x,y)$ has the unique solution $(x,y) = (d,b)$ in $B(A)$;*

($\boldsymbol{iv}$) *if $b < d$ in A then the equation $(a,b) = (x,y)(c,d)$ has no solutions in $B(A)$;*

($\boldsymbol{v}$) *if $b > d$ in A, then the equation $(a,b) = (x,y)(c,d)$ has the unique solution $(x,y) = (a, b\cdot d^{-1}\cdot c)$ in $B(A)$;*

($\boldsymbol{vi}$) *the equation $(a,b) = (x,y)(c,b)$ has the unique solution $(x,y) = (a,c)$ in $B(A)$.*

P r o o f. (***i***). Assume that $a < c$. Then formula (2) implies that $d < x$ in A and hence $(a,b) = (x \cdot d^{-1} \cdot c, y)$. This implies that $a = x \cdot d^{-1} \cdot c$ and $b = y$. Since $d < x$, the equality $a = x \cdot d^{-1} \cdot c$ implies that $a > c$, which contradicts the assumption of statement (***i***).

(***ii***). Assume that $a > c$. Then formula (2) implies that $d < x$ in A and hence we have that $(a,b) = (x \cdot d^{-1} \cdot c, y)$. This implies the equalities $x = a \cdot c^{-1} \cdot d$ and $y = b$.

(***iii***) follows from formula (2).

The proofs of statements (***iv***), (***v***) and (***vi***) are dual to the proofs of (***i***), (***ii***), and (***iii***), respectively. ♦

Later we need the following proposition which follows from formula (2) and describes right and left principal ideals in the semigroup $B(A)$ for a non-empty shift-set A in G.

Proposition 4. *Let G be a linearly ordered group and A be a non-empty shift-set in G. Then the following conditions hold:*

(***i***) $(a,a)B(A) = \{(x,y) \in B(A) : x \geq a \text{ in } A\}$;

(***ii***) $B(A)(a,a) = \{(x,y) \in B(A) : y \geq a \text{ in } A\}$.

2. On topologizations of the semigroup $B(A)$. It is obvious that every left (right) topological group G with an isolated point is discrete. This implies that every countable T_1-Baire left (right) topological group is a discrete space, too. Later we shall show that the similar statement holds for Baire semitopological semigroup $B(A)$ over a non-empty shift-set A of a countable linearly ordered group G.

For an arbitrary element (a,b) of the semigroup $B(A)$ we denote

$$\uparrow_{\leq} (a,b) = \{(x,y) \in B(A) : (a,b) \preceq (x,y)\} .$$

Lemma 2. *Let G be a linearly ordered group, A be a non-empty shift-set in G, and τ be a shift-continuous topology on $B(A)$ such that $(B(A),\tau)$ contains an isolated point. Then the space $(B(A),\tau)$ is discrete.*

P r o o f. Suppose that (a,b) is an isolated point of the topological space $(B(A),\tau)$. Assume that for an arbitrary $u \in A$ there exists $c \in A$ such that $u > c$, which implies $d = c \cdot u^{-1} \cdot b < b$. By Proposition 3(***v***) the equation $(a,b) = (x,y)(c,d)$ has the unique solution

$$(x,y) = (a, b \cdot d^{-1} \cdot c) = \left(a, b \cdot (c \cdot u^{-1} \cdot b)^{-1} \cdot c\right) =$$

$$= (a, b \cdot b^{-1} \cdot u \cdot c^{-1} \cdot c) = (a,u)$$

in $B(A)$. If u is the smallest element of A, then by Proposition 3(***vi***), the equation $(a,b) = (x,y)(u,b)$ has the unique solution $(x,y) = (a,u)$. In both cases the continuity of right translations in $(B(A),\tau)$ implies that for arbitrary $u \in A$ the pair (a,u) is an isolated point of the topological space $(B(A),\tau)$.

Fix an arbitrary element v of A. Assume that there exists $d \in A$ such that $d < v$, which implies $c = d \cdot v^{-1} \cdot a < a$. Then by Proposition 3(***ii***), the equation $(a,u) = (c,d)(x,y)$ has the unique solution

$$(x,y) = (a \cdot c^{-1} \cdot d, u) = \left(a \cdot (d \cdot v^{-1} \cdot a)^{-1} \cdot d, u\right) =$$

$$= (a \cdot a^{-1} \cdot v \cdot d^{-1} \cdot d, u) = (v,u)$$

in $B(A)$. If v is the smallest element of A, then by Proposition 3(***iii***), the equation $(a,u)=(a,v)(x,y)$ has the unique solution $(x,y)=(v,u)$. Since (a,u) is an isolated point of $(B(A),\tau)$, in both cases the continuity of left translations in $(B(A),\tau)$ implies that for arbitrary $u\in A$ the pair (v,u) is an isolated point of the topological space $(B(A),\tau)$. This completes the proof of the lemma. ♦

Theorem 1. *Let A be a countable non-empty shift-set in a linearly ordered group G and τ be a T_1-Baire shift-continuous topology on $B(A)$. Then the topological space $(B(A),\tau)$ is discrete.*

P r o o f. By Proposition 1.30 from [36] every countable Baire T_1-space contains a dense subspace of isolated points, and hence the space $(B(A),\tau)$ contains an isolated point. Then we apply Lemma 2. ♦

Theorem 1 implies the following

Corollary 1. *Let A be a countable non-empty shift-set in a linearly ordered group G, and τ be a shift-continuous Čech complete (locally compact) T_1-topology on $B(A)$. Then the topological space $(B(A),\tau)$ is discrete.*

Remark 1. Let $\mathbb{R}$ be the set of reals with usual topology. It is obvious that $S_{\mathbb{R}}=\mathbb{R}\times\mathbb{R}$ with the semigroup operation

$$(a,b)(c,d)=\begin{cases}(a-b+c,d), & b<c,\\ (a,d), & b=c,\\ (a,b-c+d), & b>c,\end{cases}$$

is isomorphic to the semigroup $B(G)$, where G is the additive group of reals $(\mathbb{R},+)$ with usual linear order $\le$. Then simple verifications show that S with the product topology τ_p is a topological inverse semigroup (also, see [39, 40]). Then the subspace $S_{\mathbb{Q}}=\{(x,y)\in S_{\mathbb{R}}: x \text{ and } y \text{ are rational}\}$ with the induced semigroup operation from S is a countable non-discrete non-Baire topological inverse subsemigroup of (S,τ_p). Also, the same we get in the case of subsemigroup $S_{\mathbb{Q}}^{+}=\{(x,y)\in S_{\mathbb{Q}}: x\ge 0 \text{ and } y\ge 0\}$ of (S,τ_p) (see [4–8]). The above arguments show that the condition in Theorem 1 that τ is a T_1-*Baire* topology is essential.

Recall that a linearly ordered group G is said to be *densely ordered* if for every positive element $g\in G$ there exists a positive element $h\in G$ such that $h<g$.

Remark 2. It is obviously that for a linearly ordered group G the following conditions are equivalent:

(***i***) G is not densely ordered;

(***ii***) for every $g\in G$ there exists a unique $g^{+}\in G$ such that $G^{+}(g)\setminus G^{+}(g^{+})=\{g\}$;

(***iii***) for every $g\in G$ there exists a unique $g^{-}\in G$ such that $G^{+}(g)\setminus G^{+}(g^{+})=\{g\}$, where $G^{-}(g)$ is the *negative cone* on the element g, i.e., $G^{-}(g)=\{x\in G: x\le g\}$.

In what follows, for a linearly ordered group G which is not densely ordered and an arbitrary element g of a non-empty shift-set A in G by g^{+} (respectively, g^{-}) we denote the minimum (respectively, maximum) element of the set $G^{+}(g)\setminus\{g\}\cap A$ (respectively, $G^{-}(g)\setminus\{g\}\cap A$).

Theorem 2. *Let G be a linearly ordered group which is not densely ordered and A be a non-empty shift-set in G. Then every shift-continuous Hausdorff topology τ on the semigroup $B(A)$ is discrete, and hence $B(A)$ is a discrete subspace of any semitopological semigroup which contains $B(A)$ as a subsemigroup.*

P r o o f. We fix an arbitrary idempotent (a,a) of the semigroup $B(A)$ and suppose that (a,a) is a non-isolated point of the topological space $(B(A),\tau)$. Since the maps $\lambda_{(a,a)}: B(A)\to B(A)$ and $\rho_{(a,a)}: B(A)\to B(A)$ defined by the formula $(x,y)\lambda_{(a,a)}=(a,a)(x,y)$ and $(x,y)\rho_{(a,a)}=(x,y)(a,a)$ are continuous retractions, we conclude that $(a,a)B(A)$ and $B(A)(a,a)$ are closed subsets in the topological space $(B(A),\tau)$ (see [23, Exercise 1.5.C]). For an arbitrary element b of the shift-set A in the linearly ordered group G we put

$$DL_{(b,b)}[(b,b)]=\{(x,y)\in B(A):(x,y)(b,b)=(b,b)\}.$$

Lemma 1 and Proposition 2 imply that

$$DL_{(b,b)}[(b,b)]={\uparrow_{\leq}}(b,b)=\{(x,x)\in B(A): x\leq b \text{ in } A\}$$

and since right translations are continuous maps in $(B(A),\tau)$ we get that $DL_{(b,b)}[(b,b)]$ is a closed subset of the topological space $(B(A),\tau)$ for every $b\in A$. Then there exists an open neighbourhood $W_{(a,a)}$ of the point (a,a) in the topological space $(B(A),\tau)$ such that

$$W_{(a,a)}\subseteq B(A)\setminus\left((a^+,a^+)B(A)\cup B(A)(a^+,a^+)\cup DL(a^-,a^-)\right).$$

Since $(B(A),\tau)$ is a semitopological semigroup we conclude that there exists an open neighbourhood $V_{(a,a)}$ of the idempotent (a,a) in the topological space $(B(A),\tau)$ such that the following conditions hold:

$$V_{(a,a)}\subseteq W_{(a,a)},\qquad (a,a)\cdot V_{(a,a)}\subseteq W_{(a,a)},\qquad V_{(a,a)}\cdot(a,a)\subseteq W_{(a,a)}.$$

Hence at least one of the following conditions holds:

(***a***) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x,y)\in B(A)$ such that $x<y\leq a$ in the group A;

or

(***b***) the neighbourhood $V_{(a,a)}$ contains infinitely many points $(x,y)\in B(A)$ such that $y<x\leq a$ in the group A.

In the case (***a***) by Proposition 2 we have that

$$(a,a)(x,y)=(a,a\cdot x^{-1}\cdot y)\notin W_{(a,a)},$$

because $x^{-1}\cdot y\geq e$ in G, and in the case (***b***) by Proposition 2 we have that

$$(x,y)(a,a)=(a\cdot y^{-1}\cdot x,a)\notin W_{(a,a)}$$

because $y^{-1}\cdot x\geq e$ in G, which contradicts the separate continuity of the semigroup operation in $(B(A),\tau)$. The obtained contradiction implies that the set $V_{(a,a)}$ is a singleton, and hence the idempotent (a,a) is an isolated point of the topological space $(B(A),\tau)$.

Now, we apply Lemma 2 and get that the topological space $(B(A),\tau)$ is discrete. ♦

Theorem 2 implies the following three corollaries.

Corollary 2. *Let* G *be a linearly ordered group which is not densely ordered and* A *be a non-empty shift-set in* G. *Then every semigroup Hausdorff topology* τ *on the semigroup* $B(A)$ *is discrete.*

Corollary 3 [24]. *Every shift-continuous Hausdorff topology* τ *on the bicyclic extended semigroup* $C_{\mathbb{Z}}$ *is discrete.*

Corollary 4 [15, 22]. *Every shift-continuous Hausdorff topology* τ *on the bicyclic monoid* $C(p,q)$ *is discrete.*

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НАПІВТОПОЛОГІЧНІ БІЦИКЛІЧНІ РОЗШИРЕННЯ ЛІНІЙНО ВПОРЯДКОВАНИХ ГРУП

Підмножину $A \subseteq G$ *лінійно впорядкованої групи* G *називають трансляційною, якщо для довільних* $x, y, z \in A$, $y < x$, *елемент* $x \cdot y^{-1} \cdot z \in A$. *Описано природний частковий порядок і розв'язки рівнянь на півгрупі* $B(A)$ *зсувів додатних конусів множини* A. *Вивчається топологізація півгрупи* $B(A)$. *Зокрема, показано, що для довільної зліченної лінійно впорядкованої групи* G *і непорожньої трансляційної множини* A, $A \subseteq G$, *кожна берівська трансляційно неперервна* T_1*-топологія* τ *на* $B(A)$ *є дискретною. Також доведено, що для довільної лінійно нещільно впорядкованої групи* G *і непорожньої трансляційної множини* A *кожна трансляційно неперервна гаусдорфова топологія* τ *на півгрупі* $B(A)$ *є дискретною.*

ПОЛУТОПОЛОГИЧЕСКИЕ БИЦИКЛИЧЕСКИЕ РАСШИРЕНИЯ ЛИНЕЙНО УПОРЯДОЧЕННЫХ ГРУПП

Подмножество $A \subseteq G$ *линейно упорядоченной группы* G *называют трансляционным, если для произвольных* $x, y, z \in A$, $y < x$, *элемент* $x \cdot y^{-1} \cdot z \in A$. *Описан естественный частичный порядок и решения уравнений на полугруппе* $B(A)$ *сдвигов положительных конусов множества* A. *Изучается топологизация полугруппы* $B(A)$. *В частности, показано, что для произвольной счётной линейно упорядоченной группы* G *и непустого трансляционного множества* A, $A \subseteq G$, *каждая бэровская трансляционно непрерывная* T_1*-топология* τ *на* $B(A)$ *является дискретной. Также доказано, что для произвольной линейно неплотно упорядоченной группы* G *и непустого трансляционного множества* A *каждая трансляционно непрерывная гаусдорфова топология* τ *на полугруппе* $B(A)$ *является дискретной.*

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