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REMARKS ON n-EQUIVALENCE OF KNOTS AND LINKS

In this paper, we announce some new results concerning the relationship between Vassiliev knot invariants and canonical and classical genera of knots in a 3-space and study the behavior of finite order knot invariants under the special satellite operations. We also study n-equivalence of links in the sense of Kirk–Livingston in the context of satellite operations.

1. Introduction. The relations of Vassiliev knot invariants, where knots are considered in a 3-space, to classical geometric knot invariants such as genera, signature, the Arf invariant, the 4-ball genus, cobordism have been studied by K. Y. Ng, T. Stanford, A. Stoimenow and the other authors (see [18, 24–27]).

Let $\mathcal{K}$ be the set of oriented knots up to ambient isotopy (knot types) in S^3 and let V be the free abelian group freely generated by the set $\mathcal{K}$. Denote by $\mathcal{K}_n$ the subgroup of V generated by all n-singular knots, $n \geq 1$. The following decreasing sequence of abelian subgroups

$$V \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \ldots$$

is called the Vassiliev–Goussarov filtration of V.

Let H be any abelian group and let $v : \mathcal{K} \to H$ be a map. We use the same notation v for the map $v : \mathcal{K} \to H$ and its natural extension to the homomorphism $v : V \to H$. A Vassiliev invariant of type $n \geq 0$, with values in an abelian group H, is a map $v : \mathcal{K} \to H$ such that its linear extension $v : V \to H$ vanishes on the subgroup $\mathcal{K}_{n+1}$. The smallest number m for which v vanishes on $\mathcal{K}_{m+1}$ is called the order of the Vassiliev invariant v. Two knots are called n-equivalent if they cannot be distinguished by Vassiliev invariants of order $\leq n$, the invariants taking values in any abelian group. A knot K is called n-trivial if it is n-equivalent to a trivial knot O. We refer the reader to [1, 5, 7, 18, 23] for further definitions and the basic properties of Vassiliev knot invariants.

In Section 2 of the present paper, we study relationship between the canonical and classical genera of knots, and Vassiliev invariants. We only outline the proof of the main results concerning this point. The complete proof of the assertions will be given in [20]. The results presented here are analogous to the ones of A. Stoimenow [24, 26] on the relationship between Vassiliev invariants and the 4-ball genus, the Tristram–Levin signature and the unknotting number of knots.

The role of essential tori and Seifert surfaces in the study of Vassiliev knot invariants have been discussed in the papers [8–10]. In [9], E. Kalfagianni and X.-S. Lin studied n-adjacency of knots for any $n \in \mathbb{N}$. It is known [8, 9] that n-adjacency of knots implies their n-equivalence. Kalfagianni and Lin also showed that the (classical) genus of knots can be considered as an obstruction for two knots to be n-adjacent. This is in some contrast with the results presented in the present paper concerning the relationship between the classical genus of knot and Vassiliev invariants.

Let $s_L : \mathcal{K} \to \mathcal{K}$ be a satellite operation defined by a two-component link $L = k \cup t$ in S^3 [12]. G. Kuperberg has shown [12] that for each $n \geq 0$ if the knots K and K' are n-equivalent, then the satellite knots $s_L(K)$ and $s_L(K')$ are also n-equivalent. In Section 3, we show that if the winding number of the knot k in $S^3 \setminus t$ is equal to zero, then the corresponding satellite operation $s_L : \mathcal{K} \to \mathcal{K}$ sends the n-equivalent knots into $(n+1)$-equivalent ones (see also [21]). The proof of this assertion presented here is different from the one given in [21]. In particular, if the two-component link $L = k \bigcup t$ that defines a pattern L for a satellite operation s_L possesses a certain kind of symmetry, then the satellite map $s_L : \mathcal{K} \to \mathcal{K}$ passes the n-equivalent knots into $(n+1)$-equivalent ones (compare with [22]). In Section 3, we also discuss some questions concerning the NIC (non-invertible conjecture) for knots.

In Section 4 we discuss the properties of the Kirk−Livingston invariants of links with respect to satellite operations.

2. Genera of knots and Vassiliev invariants. The well known procedure given originally by Seifert [2] allows to construct for each link diagram L' of a link L in a canonical way a Seifert surface S that is spanned by L. If L' is a knot diagram, the algorithm yields always a connected surface of a knot and its genus is called the genus of the diagram. The canonical genus $\tilde{g}(K)$ of a knot is the minimal genus of all its diagrams. It is known that for a large class of knots K the canonical genus $\tilde{g}(K)$ is equal to its classical genus $g(K)$ (for example, for the class of homogeneous knots [3]). However, in general, the difference $\tilde{g}(K) - g(K)$ can be arbitrarily large [17].

Let K be a knot embedded in the interior of a solid torus $V \subset S^3$. We use the term "K is n-trivial inside V" if there is an embedding of K in $\operatorname{int}(V)$ and a collection of $n+1$ disjoint sets $C_1, \dots, C_{n+1}$ of crossing discs, all containing inside V, such that for each $0 < m \le n+1$, changing all crossings in any m of these sets simultaneously along the corresponding discs, yields an unknotted curve that can be isotoped inside a 3-ball in $\operatorname{int}(V)$.

In [20], we proved the following proposition.

Proposition 2.1. *For each $n \in \mathbb{N}$ there is a non-trivial prime knot K_n of genus one which is essentially embedded in a standard solid torus V and is n-trivial inside V.*

To show this, we construct first for each $n \in \mathbb{N}$ a non-trivial knot K_n' which is essential and n-trivial inside the standard solid torus V, and then take its (untwisted) Whitehead double K_n. Our approach uses the diagrammatic technique and the insertions of geometric pure braid commutators in knot diagrams, as in [18, 19, 23]. The knot K_n can be considered as embedded inside the solid torus V. It follows from [12] that the knot K_n is also n-trivial inside V. Moreover, K_n has genus equal to 1 and is prime. This follows directly from the results of [28]. ◊

For $0 \le i < j \le k-1$, let $p_{i,j} \in P_k$ be the pure braid that links i-th and j-th strands behind the others. Set

$$\hat{p}_n = p_{n-1,n} \cdot p_{n-2,n-1} \cdot \ldots \cdot p_{1,2} \cdot p_{0,1} \cdot p_{1,2}^{-1} \cdot \ldots \cdot p_{n-2,n-1}^{-1} \cdot p_{n-1,n}^{-1} \in P_{n+1}.$$

Theorem 2.2 (Theorem 3.2 of [20]). *For each composite knot K with genus $g(K) = g$ and any $n \in \mathbb{N}$ there is a prime knot R which is n-equivalent to K and has the genus equal to g.*

Here we outline briefly the proof of the theorem. For the detailed proof of it see [20].

Sketch of the p r o o f. We can restrict ourselves to the case $K = K_1 \# K_2$ where K_1 and K_2 are the prime knots. Consider now a minimal Seifert surface S for K. We may assume that $S = S_1 \#_d S_2$, where S_1 and S_2 are disjoint minimal Seifert surfaces for K_1 and K_2, respectively, and d is a trivial band in the band sum $S = S_1 \#_d S_2$. Our purpose now is to replace the band d with the other one, say u, which is also trivial in the band sum $S' = S_1 \#_u S_2$, has the same ends as d (i.e. $d \cap S_1 = u \cap S_1 = a_1$ and $d \cap S_2 = u \cap S_2 = b_2$), and is specified by the condition that allows an application of a doubled one-branch C_{n+1}-move to $\partial S'$ in some ball B. As a result, the surfaces S_1 and S_2 are joined now by a "long" trivial band u which intersects S_1 (S_2) along the side a_1 (b_2, respectively), and such that $u \cap B$ consists of $n+2$ parallel ribbons $u_1, u_2, \ldots, u_{n+2}$ that realize the double of the pure braid $1_{n+2} \subset P_{n+2}$.

By isotoping u in the ball B, we may achieve that the intersection of the resulting band v and B consists of $n+2$ parallel ribbons $v_1, v_2, \ldots, v_{n+2}$ that realize the double p' of the pure braid $\hat{p}_n \cdot \hat{p}_n^{-1} \in P_{n+1} \subset P_{n+2}$. Let $\tilde{p}_n$ denote the double of the pure braid $\hat{p}_n \in P_{n+1} \subset P_{n+2}$ and let $\tilde{p}_{n+1}$ denote the double of the pure braid $\hat{p}_{n+1} \in P_{n+2}$. To continue, we replace the geometric braid $\tilde{p}_n \in P_{2n+4}$ in the box B with a new geometric braid $\tilde{p}_{n+1} \in P_{2n+4}$. This move gives a new band w joining the Seifert surfaces S_1 and S_2. As can be easily seen, this replacement is actually a doubled C_{n+1}-move on the knot K, thus the resulting knot R is n-equivalent to K. Moreover, by our construction, we have $\tilde{S} = S_1 \#_w S_2$, where $\tilde{S}$ is a Seifert surface for the knot $R = \partial \tilde{S}$.

It turns out that the knot R is a non-trivial band sum of prime knots K_1 and K_2. This follows from our construction and Theorem 1 of [6]. Now, by applying the result of D. Gabai [4] on the genus of band sum of two knots which are bounded by disjoint minimal Seifert surfaces, we obtain $g(R) = g(K_1) + g(K_2)$. Therefore $g(R) = g(K)$. ◊

In [20] we also established the following fact.

Theorem 2.3. *For every knot K with genus $g(K)$ and any $n \in \mathbb{N}$ and $m \geq g(K)$ there exists a prime knot R which is n-equivalent to K and has the genus $g(R)$ equal to m.*

Here we present a shorter proof of this assertion.

P r o o f . By Theorem 2.2, we may replace the knot K with a knot L which is prime, n-equivalent to K and has the same genus as the knot K does. Let S_n be a minimal (of minimal genus) Seifert surface for a knot K_n that is positioned in a standard solid torus V, as before, and let S' be a minimal Seifert surface for the knot L. Moreover, let B be a ball inside the solid torus V such that an application of a doubled C_{n+1}-move to the knot K_n inside this ball gives a trivial knot inside V. We may assume that the knot L intersects ∂V along a small arc c only, which is positioned outside the ball B. We may assume also that V and S' are positioned in disjoint balls D and D', respectively. Finally we take a non-trivial band sum of the surfaces S' and S_n with a band b, $\tilde{S} = S' \#_b S_n$, in such a way that $b \cap S_n = b \cap \partial V = c$. The resulting knot L_1 that bounds the surface $\tilde{S}$, will be a prime knot which is n-equivalent to K and such that $g(L_1) = g(K) + 1$. Finally, we use the induction on m to construct the desired knot R. ◊

For the canonical genus of a knot we also established the following fact.

Proposition 2.4. *For each* $n > 1$ *there is a non-trivial and* $(n-1)$*-trivial knot* L_n *the canonical genus of which satisfies the inequality* $\tilde{g}(L_n) \leq 4n - 4$.

The p r o o f of this assertion is omitted here (see [20] for details). ◊

3. Vassiliev knot invariants and satellite operations. First recall the definition of satellite operations for unframed knots in S^3 (see [12] and [21] for details). Fix an orientation on the sphere S^3. Let $\mathcal{K}^f$ be the set of framed, oriented knots in S^3 considered up to (regular) isotopy. Let L be a non-split oriented two-component link with the two unframed components k and t. Suppose t is unknotted. Then L defines a pattern for the satellite operation $s_L : \mathcal{K}^f \to \mathcal{K}^f$ in the following way. For a knot $K \in \mathcal{K}^f$ define the satellite knot $s_L(K)$ as the result of removing a tube around K and a tube around t and gluing the two knot complements together to retain only k so that the given longitude of K (see [2]) is glued to the meridian of t and *vice versa*.

An equivalent definition of a satellite operation on (unframed) knots is the following. First take a diagram of a knot k represented as the closure of a tangle T with m strands and positioned in a flat annulus C. Let B be the rectangle in C in which the tangle T inhabits, $B \simeq I \times I$. Deleting the tangle T in C, we obtain a decomposition of k into m parallel arcs t_i, each of which lies in $C \setminus B$ and has the ends in the set $\operatorname{dom} T \bigcup \operatorname{codom} T$, i.e. on ∂B. Now, given a knot K and any its diagram K', a diagram of the knot $s_L(K)$ can be obtained by cutting a diagram K' to make a box B', putting the tangle T in the place of this box, replacing the remaining part of the diagram K' by m parallel copies of it and adding a number of twists of parallel arcs near the box, if needed, to compensate the framing of the knot diagram K'. Here and below we shall assume that a knot diagram K' has the blackboard framing. By definition, the winding number $w(k)$ of the knot k with respect to L is the number $\operatorname{lk}(k,t)$.

Let $\mathcal{V}_n$ be the quotient of the free abelian group V by the subgroup $\mathcal{K}_{n+1}$. The universal Vassiliev invariant of order n is the induced map $\tilde{V}_n : \mathcal{K} \to \mathcal{V}_n$. By Theorem 4 of [12], for each satellite operation s_L and for every n there exists an endomorphism $s_{L,\mathcal{V}_n} : \mathcal{V}_n \to \mathcal{V}_n$ so that $\tilde{V}_n \circ s_L = s_{L,\mathcal{V}_n} \circ \tilde{V}_n$. It follows that for each n we have the inclusion $s_L(\mathcal{K}_n) \subset \mathcal{K}_n$, and for any satellite operation s_L and any Vassiliev invariant v_n of order n the composition $v_n \circ s_L$ is also an invariant of order $\leq n$. The homomorphism $s_{L,\mathcal{V}_n}$ descends to a homomorphism $\hat{s}_L : \mathcal{K}_n / \mathcal{K}_{n+1} \to \mathcal{K}_n / \mathcal{K}_{n+1}$.

Suppose that a knot diagram R' is obtained from a knot diagram K' by replacing the trivial braid 1_{n+1} with a pure braid $p \in LCS_k(P_{n+1})$ via a tangle map T. In the sequel, we shall assume that the knot diagrams R' and K' have then the same (blackboard) framing.

The following assertion was proved in [21]. Here we outline another proof of it.

Theorem 3.1. *Let* $L = k \bigcup t$ *be a two-component link with components* k *and* t *such that the component* t *is unknotted and has zero framing, and* $w(k) = 0$. *Let* K_1 *and* K_2 *be any two knots in* S^3. *If* K_1 *and* K_2 *are* n*-equivalent, then the satellite knots* $s_L(K_1)$ *and* $s_L(K_2)$ *are* $(n+1)$*-equivalent.*

Sketch of the p r o o f. We may assume that a knot k is represented in a standard solid torus $Q \subset \mathbb{R}^3$ as the closure of some tangle T positioned in a ball $B \subset Q$.

For the rational n-equivalence (i.e. n-equivalence defined by rational Vassiliev invariants) the assertion is a consequence of the more general fact that if the satellite operation s_L satisfies the condition $w(k) = 0$, then the descended endomorphism $\hat{s}_L : \mathcal{K}_{n+1}/\mathcal{K}_{n+2} \to \mathcal{K}_{n+1}/\mathcal{K}_{n+2}$ is trivial, the fact that follows from the proof of Theorem 2.2 of [21] (see also Remark 2.1 of [21]).

In the case of invariants with arbitrary coefficients we argue as follows. Let W and W' be the small regular neighborhoods of the knots K_1 and K_2, respectively, in S^3. The knot $s_L(K_1)$ is obtained by removing the solid torus W and gluing the solid torus Q to $\mathrm{cl}(S^3 \setminus W)$ along ∂Q via a map f. Similarly, the knot $s_L(K_2)$ is obtained by removing the solid torus W' and gluing the solid torus Q to $\mathrm{cl}(S^3 \setminus W')$ along ∂Q via a map f'. Let D be a ball inside the solid torus W where the tangle T for a knot k is inserted under the satellite operation s_L. Moreover, let D' be a ball in the solid torus W' where the tangle T for the knot k is inserted under a satellite operation s_L. Since K_1 and K_2 are n-equivalent, they are related by a sequence of $(n+1)$-singular knots $K^1, K^2, \ldots, K^s$. In other words, the knot K_2 can be obtained from the knot K_1 via generic homotopy $h_t : S^1 \to \mathbb{R}^3$, where $h_0(S^1) = K_1$ and $h_1(S^1) = K_2$, which passes through the $(n+1)$-singular knots $K^1, K^2, \ldots, K^s$ (and may be also through a finite number of knots with more than $n+1$ singularities), so that $K_2 - K_1 = \sum_{i=1}^{s} K^i$ modulo $\mathcal{K}_{n+2}$. We may assume without loss of generality that $s = 2$.

Now, we can replace the homotopy h_t with a homotopy $H_t : Q \to \mathbb{R}^3$ by thickening the knot $h_t(S^1)$ for each parameter t. Then the knot $s_L(K_2)$ can be obtained from the satellite knot $s_L(K_1)$ by a generic homotopy $H_t' = j \circ H_t$, where $j : k \subset Q$ is the inclusion. We may adopt the homotopy H_t' in such a way that it passes the singularities (double points of the intermediate knots) out of the ball $D \subset W$. Note also that the generic homotopy H_t' may be thought of a sequence of isotopies and Habiro's C_{n+1}-moves.

Since the winding number $w(f(k))$ of the knot $f(k) = s_L(K_1)$ in W is equal to zero, the knot $s_L(K_1)$ can be deformed via an appropriate generic homotopy g_t inside the solid torus W (more precisely, by a finite number of crossing changes in the ball D) to a knot $R = g_1(S^1)$ which is contained in some other ball $U \subset W$. Similarly, the knot $s_L(K_2)$ can be homotoped inside the solid torus W' by a finite number of crossing changes in the ball D' so that he resulting knot $g_1'(S^1)$ is contained in a ball $U' \subset W'$. It follows that the knot R has the same type as the one of k. Combining the homotopy H_t' with the homotopy g_t and with the ambient isotopy of $\mathbb{R}^3$, we obtain a generic homotopy e_t which deforms the knot $e_0(S^1) = s_L(K_1)$ to the knot $e_1(S^1) =$

$= s_L(K_2)$ and passes through a finite number of $(n+2)$-singular knots $R_1, \dots,$ $\dots, R_p$ so that $s_L(K_2) - s_L(K_1) = \sum_{i=1}^{p} R_i$ modulo $\mathcal{K}_{n+3}$. It follows that the knots $s_L(K)$ and $s_L(K')$ have the same invariants of order $\le n+1$ with an arbitrary group of coefficients. ◊

For an oriented knot K denote by $\bar{K}$ the knot that is reverse of K. Moreover, for a two-component link $L = k \cup t$ that defines a pattern for a satellite operation, put $L' = k \cup \bar{t}$ and $\bar{L} = \bar{k} \cup t$.

Corollary 3.1. *Let $L = k \cup t$ be a two-component link so that the component t is unknotted. Suppose that $L = L'$ or $L = \bar{L}$. Let K_1 and K_2 be any two knots. If K_1 and K_2 are n-equivalent, then the satellite knots $s_L(K_1)$ and $s_L(K_2)$ are $(n+1)$-equivalent.*

P r o o f. Under the assumption, we have $w(k) = 0$ in both the cases and the assertion follows. ◊

***Remark* 3.1.** *The condition that L possesses a kind of symmetry, as in Corollary* 3.1, *imposes some restrictions on the satellite operation s_L (see* [21]).

An example of the link that possesses the symmetries described in Corollary 3.1, is the well-known Whitehead link. The corresponding satellite operation is the Whitehead doubling of a knot (twisted or untwisted). On the other hand, it is known that the link 9^2_{34} does not possess any such a symmetry (see [12]).

Recall that for each integer $n > 0$ to the operation of inverting the knots in $\mathcal{K}$ there corresponds the involution s on the graded group $\mathcal{A}$ of trivalent diagrams modulo STU-relations (see [1]). If D is any trivalent diagram with m external vertices, then we put $S(D) = (-1)^m \bar{D}$, where $\bar{D}$ is the same trivalent diagram D only with the orientation of the external circle reversed. The map $S : \mathcal{D} \to \mathcal{D}$ can be extended linearly to the graded map $S : \mathbb{Z}\mathcal{D} \to$ $\to \mathbb{Z}\mathcal{D}$. The graded map $S : \mathbb{Z}\mathcal{D} \to \mathbb{Z}\mathcal{D}$ descends obviously to the isomorphism $s : \mathcal{A} \to \mathcal{A}$ of graded abelian groups. For each n denote by $s_n : \mathcal{A}_n \to$ $\to \mathcal{A}_n$ the restriction of the map s to the subgroup $\mathcal{A}_n \subset \mathcal{A}$. Note also that for any trivalent diagram D of degree n with one external vertex we have $s_n(D) = D$ in $\mathcal{A}_n$, since each such a diagram is equal to zero in $\mathcal{A}_n$.

Let $\xi'_n : \mathcal{D}_n \to \mathcal{K}_n/\mathcal{K}_{n+1}$, $n \ge 1$, be a map that "replaces the chords by double points" (see [7]). This map can be extended to a linear map $\xi'_n : \mathbb{Z}\mathcal{D}_n \to$ $\to \mathcal{K}_n/\mathcal{K}_{n+1}$. The map ξ'_n descends to a monomorphism $\xi_n : \mathcal{A}_n \to \mathcal{K}_n/\mathcal{K}_{n+1}$. Note that because the existence of Kontsevich integral over $\mathbb{Q}$, when tensored with $\mathbb{Q}$, the monomorphism ξ_n, $n \ge 1$, becomes an isomorphism of the corresponding vector spaces.

***Remark* 3.2.** *Let s_L be a satellite operation defined by a pattern $L = k \cup t$ with $w(k) = q$. Then, on the level of trivalent diagrams, for each n to the satellite operation s_L there corresponds the operation of q-cabling $\varphi_q : \mathcal{A}_n \to$ $\to \mathcal{A}_n$ which satisfies the following identity: $\xi_n(\varphi_q(D)) = \hat{s}_L \circ \xi_q(D)$ for each trivalent diagram D of degree n (see* [1] *for the definition of the operation of q-cabling on $\mathcal{A}_n$).*

We see from the above discussion and [21] that if $L = k \cup t$ is a two-component link defining the satellite operation $s_L : \mathcal{K} \to \mathcal{K}$ with $w(k) = 0$, where t is the unknotted component, then for any n-trivial knot K the satellite knots $s_L(K)$ and $s_L(\bar{K})$ as well as the knots $s_L(K)$ and $\overline{s_L(K)}$ cannot be distinguished by the invariants of order $\le n+1$. It is natural to ask the converse. More precisely, for a given n-trivial but not (rationally) $(n+1)$-trivial non-invertible knot K (if any exists), what conditions does the equality $s_L(K) = s_L(\bar{K})$ impose on the knot k? The similar question can be asked in the situation when $s_L(K) = \overline{s_L(K)}$. In the following we discuss the above two questions.

Suppose the NIC is true for rational Vassiliev invariants. Let n be the minimal order of rational Vassiliev invariant that can distinguish between some knot K and its inverse $\bar{K}$. Then, by the discussion in [19], there exists a trivalent diagram D of order n with the connected internal graph such that $D \ne s_n(D)$ in $\mathcal{A}'_n = \mathcal{A}_n \otimes \mathbb{Q}$. Conversely, if there exists a non-trivial, non-split trivalent diagram $D \in \mathcal{A}'_n$ such that $D \ne s_n(D)$ then there is a non-invertible knot K and a rational Vassiliev invariant v of order n so that v can distinguish between the knot K and its inverse [19]. Let m be the minimal number for which there exists a non-split trivalent diagram D' with m external vertices so that $D \ne s_n(D)$.

Theorem 3.2. *Let K be a $(n-1)$-trivial knot, $n \ge 2$, such that $K - O = \xi_n(D)$ in $V/\mathcal{K}_{n+1}$, where D is a trivalent diagram, the internal graph of which has m external vertices, $K \ne \bar{K}$ in $V/\mathcal{K}_{n+1}$. Let s_L be a satellite operation with the pattern $L = k \cup t$ so that $s_L(K) = s_L(\bar{K})$ in $V/\mathcal{K}_{n+1}$. Then $w(k) = 0$.*

P r o o f. First note that the knots $K, \bar{K}, s_L(K)$ and $s_L(\bar{K})$ are $(n-1)$-trivial. It follows that the formal difference $\bar{K} - K$ belongs to $\mathcal{K}_n$. Let $w(k) = q$. By the above discussion, modulo $\mathcal{K}_{n+1}$, we have the following equalities: $s_L(\bar{K}) - s_L(K) = s_L(\bar{K}) - s_L(K) - k + k = s_L(\bar{K} - \bar{O}) - s_L(K - O)$. Passing to the map $\hat{s}_L$, we obtain the following $\hat{s}_L(\bar{K} - K) = \hat{s}_L(\bar{K} - \bar{O}) - \hat{s}_L(K - O) = \xi_n(\varphi_q(s_n(D))) - \xi_n(\varphi_q(D)) = \xi_n[q^m \cdot (s_n(D) - D)] + \xi_n[s_n(\varepsilon) - \varepsilon]$, where ε denotes the integral linear combination of trivalent diagrams with the number of external vertices $\le m-1$. By the definition of m, we have $\varepsilon = s_n(\varepsilon)$ in $\mathcal{A}'_n$, so $\hat{s}_L(\bar{K} - \bar{O}) - \hat{s}_L(K - O) = q^m \xi_n[s_n(D) - D] = 0$. Since ξ_n is monomorphism, then $s_n(D) - D = 0$ or $w(k) = 0$. However, by the assumption, $K - \bar{K} \ne 0$ and $\hat{s}_L(\bar{K} - K) = 0$ in $\mathcal{K}_n/\mathcal{K}_{n+1}$. It follows $D \ne s_n(D)$, so we have $w(k) = 0$, completing the proof. ◊

Theorem 3.3. *Let K be an $(n-1)$-trivial knot, $n \ge 2$, such that $K - O = \xi_n(D)$ in $V/\mathcal{K}_{n+1}$, where D is a trivalent diagram, the internal graph of which has m external vertices, and $K \ne \bar{K}$ in $V/\mathcal{K}_{n+1}$. Let s_L be a satellite operation with the pattern $L = k \cup t$ such that $s_L(K) = \overline{s_L(K)}$ in $V/\mathcal{K}_{n+1}$ and $w(k) = q$. Then the following equality in $\mathcal{K}_n/\mathcal{K}_{n+1}$: $\bar{k} - k = \pm q^m \xi_n[s_n(D) - D]$ holds.*

P r o o f. First note that $s_L(O) = k$ and $\overline{s_L(O)} = \bar{k}$ in $V/\mathcal{K}_{n+1}$. Moreover, the knots $K, \bar{K}, s_L(K)$ and $\overline{s_L(K)}$ are $(n-1)$-trivial. Now the proof of the assertion runs in the same way as the one of Theorem 2.2 of [21]. We have the following equalities in $\mathcal{K}_n$ modulo $\mathcal{K}_{n+1}$: $0 = \overline{s_L(K)} - s_L(K) = (\overline{s_L(K-O)} - - s_L(K-O)) + (\overline{s_L(O)} - s_L(O))$. Again, passing to the quotient map $\hat{s}_L$ we obtain the following $0 = q^m \xi_n[s_n(D) - D] + (\bar{k} - k) + \xi_n[s_n(\varepsilon) - \varepsilon]$, where ε denotes the integral linear combination of trivalent diagrams with the number of external vertices $\le m-1$ and the term $(\bar{k} - k)$ is considered modulo $\mathcal{K}_{n+1}$. By the definition of m, we have $\varepsilon = s_n(\varepsilon)$ in $\mathcal{A}'_n$, so $\bar{k} - k = \pm q^m \xi_n[D - s_n(D)]$. This completes the proof. ◊

Now we indicate how relate the results stated in this section to the NIC.

***Remark* 3.3.** *If for each n-trivial non-invertible knot K there exists the pattern $L = k \cup t$ for a satellite operation s_L such that $w(k) \neq 0$ and $\hat{s}_L(K) = = \hat{s}_L(\bar{K})$ in $V/\mathcal{K}_{n+2}$, $n \ge 0$, then the NIC is false for rational Vassiliev invariants.*

Moreover, if for each non-invertible n-trivial knot K there exists a pattern $L = k \cup t$ for a satellite operation s_L such that $w(k) \neq 0$ and $\bar{k} = k$ and $\hat{s}_L(K) = \overline{\hat{s}_L(K)}$ in $V/\mathcal{K}_{n+2}$, $n \ge 0$, then the NIC is false for rational Vassiliev invariants.

This is a direct consequence of Theorems 2.2 and 2.3 of [21] and the above discussion in this section.

4. Kirk–Livingston invariants of links and satellite operations. In this section, we consider the oriented links in a 3-space up to the following two natural equivalence relations:

1) ambient PL-isotopy of links in S^3 or $\mathbb{R}^3$;

2) PL-isotopy, i. e. ambient isotopy plus birth and death of local knots.

There are also at least two ways how to define the finite type invariants of links in S^3, which follow from the different notions of n-singular links. In the first (classical) case, the singularities of transverse type are considered between any two components of a link, including the self-intersections within the same component. The link invariants of finite type in this case have been studied by many authors (see, for example, [13]). In the second case, the singularities are allowed only within the same component of a link. This approach is due to P. Kirk and C. Livingston [11].

Here we study invariants of finite type of in the sense of Kirk–Livingston. Let us recall the definition of them following [11] and [14]. Let $\mathcal{LM}$ (respectively, $\mathcal{LM}^m$) denote the subspace of the space of all link maps $f : S^1_1 \sqcup \ldots \sqcup S^1_m \to \mathbb{R}^3$ where m is arbitrary (respectively, all link maps $f : S^1_1 \sqcup \ldots \sqcup S^1_m \to \mathbb{R}^3$ with m fixed), with only singularities being transversal double points. Note that the only singularities of the same component are allowed here. Let $\mathcal{LM}^m_n$ (respectively, $\mathcal{LM}^m_{\ge n}$) denote its subspace consisting of link maps with precisely (respectively, at least) n singularities.

Given an (ambient isotopy) invariant $v : \mathcal{LM}^m_0 \to G$ on embedding links, taking values in an abelian group G, it can be extended to $\mathcal{LM}^m$ inductively by the formula

$$v(L_s) = v(L_+) - v(L_-),$$

where $L_+, L_- \in \mathcal{LM}_n^m$ differ by a single crossing change, and $L_s \in \mathcal{LM}_{n+1}^m$ is the intermediate link map with one more singular point, defined as in the case of knots (see [14]). If v vanishes on $\mathcal{LM}_{k+1}^m$ for some k, then v is called of finite type k invariant in $\mathcal{LM}$ or the Kirk–Livingston invariant of links. The minimal number k for which v vanishes on $\mathcal{LM}_{k+1}^m$ is called the order of the Kirk – Livingston invariant v. Two m-component links L_1 and L_2 are called then n-equivalent in the sense of Kirk–Livingston (or self-n-equivalent) if they cannot be distinguished by Kirk–Livingston invariants of order $\le n$.

For $m = 1$ the definitions of Kirk–Livingston invariants and the invariants of finite type (Vassiliev invariants) for unframed knots coincide. For $m \ge 1$ any type k invariant in the classical setting (i.e. in the space $\mathcal{L}$ of all singular link maps) is a type k invariant in $\mathcal{LM}$ but not *vice versa*. For example, the linking number lk and the generalized Sato–Levine invariant $\tilde{\beta}$ are the invariants of types 0 and 1 in $\mathcal{LM}$ but of types 1 and 3, respectively, in $\mathcal{L}$ [11]. Moreover, all higher Milnor $\bar{\mu}$-invariants (except for lk) are not of finite type in the classical setting, since they are not well-defined on all links. Note that the type k invariants $v : \mathcal{LM}_0^m \to G$ taking values in an abelian group G form an abelian group, denoted by G_k^m. It is known [11] that $G_1^2 \simeq \mathbb{Z}$ and is conjectured that G_1^2 is not finitely generated for $m > 1$, in contrast to the situation with the Vassiliev link invariants in classical setting. For further definitions and the related discussion see also [11] and [14–16].

Let L and L' be two oriented m-component links with the components $K_1, \ldots, K_m$ ($K_1', \ldots, K_m'$, respectively) and let (Q, k), where k is a knot in a standard solid torus Q, be a pattern for a satellite operation (see Section 3 and [21]). We shall denote this satellite operation by s_Q. We shall represent the knot k as a closure of some tangle T inside the solid torus Q. Let B denote a ball in Q where T inhabits. For a fixed i, let us consider a small regular neighborhood W (W', respectively) of the component K_i (K_i', respectively), which does not intersect the other components of the link L (L', respectively). Replace in W the component K_i with the knot $s_Q(K_i)$. Similarly, replace in W' the component K_i' with the knot $s_Q(K_i')$. Denote by $L_{i,Q}$ and $L_{i,Q}'$ the resulting m-component links. As before, we shall use the notation f for the embedding that glues the solid torus Q to $\mathrm{cl}(\mathbb{R}^3 \setminus W)$ under the satellite operation s_Q. We have the following

Theorem 4.1. *If the links L and L' are self-n-equivalent, then the links $L_{i,Q}$ and $L_{i,Q}'$ are also n-equivalent in the sense of Kirk–Livingston. Moreover, if the winding number of the knot k in Q is zero, then the links $L_{i,Q}$ and $L_{i,Q}'$ are self-$(n+1)$-equivalent.*

P r o o f. Consider the link L as the image of an embedding $g : S_1 \sqcup \ldots \sqcup S_m \to \mathbb{R}^3$, where each S_j is a circle and $f(S_i) = K_i$. Since the links L and L' are self-n-equivalent, they are related by a finite number of singular links, say $L_1, \ldots, L_s \in \mathcal{LM}_{\ge n+1}^m$, where all singularities are within the same components. This means that the link L' is obtained from the link L via a generic homotopy $h_t : S_1 \sqcup \ldots \sqcup S_m \to \mathbb{R}^3$ with the singularities (double points) on each stage t as described above. We thus have $h_1(S_1 \sqcup \ldots \sqcup S_m) = L'$. Now consider instead of $h_t : S_1 \sqcup \ldots \sqcup S_m \to \mathbb{R}^3$ the homotopy $H_t : S_1 \sqcup \ldots \sqcup S_{i-1} \sqcup Q \sqcup S_{i+1} \sqcup \ldots \sqcup S_m \to \mathbb{R}^3$ where the i-th component of L is replaced with a thin solid torus W around this component. Then the link $L'_{i,Q}$ can be obtained from $L_{i,Q}$ via the generic homotopy H_t so that $H_0(S_1 \sqcup \ldots \sqcup S_{i-1} \sqcup S_i \sqcup S_{i+1} \sqcup \ldots \sqcup S_m) = L_{i,Q}$ and $H_1(S_1 \sqcup \ldots \sqcup S_{i-1} \sqcup S_i \sqcup S_{i+1} \sqcup \ldots \sqcup S_m) = L'_{i,Q}$. We may assume that there are a finite number of singularities in each intermediate link $H_t(S_1 \sqcup \ldots \sqcup S_i \sqcup \ldots \sqcup S_m)$ and every such a singularity is within the same component. More precisely, under this homotopy, there are intermediate singular links $R_1, \ldots, R_t \in \mathcal{LM}^m_{n+1}$ and, may be, the links with more than $n+1$ singularities, so that modulo $\mathcal{LM}^m_{n+2}$ we have in $\mathbb{Z}(\mathcal{LM}^m)$ the following: $L'_{i,Q} - L_{i,Q} = \sum_{j=1}^{t} R_j$. Note that the expression $\sum_{j=1}^{t} R_j$ for the formal difference $L'_{i,Q} - L_{i,Q}$ gives in return a bigger number of intermediate $(n+1)$-singular links than the one for the formal difference $L' - L$ considered before. It follows that the links $L_{i,Q}$ and $L'_{i,Q}$ are related by a finite number of k-singular links, where $k \geq n+1$, with the type of singularities described above. This proves the first part of the assertion.

To prove the second part of the assertion, we argue in the same way as before, and in the proof of Theorem 3.1. The condition $w(k) = 0$ in Q allows us, as in the proof of Theorem 3.1, to deform the component $s_Q(K_1)$ via an appropriate generic homotopy g_t inside the solid torus W to a knot $R = g_1(S_i)$ which is contained in a ball $B \subset W$. Actually we perform a finite number of crossing changes in the ball $f(B)$ to achieve this. Combining the homotopy H_t with the homotopy g_t and with the ambient isotopy, we obtain a generic homotopy $e_t : S_1 \sqcup \ldots \sqcup S_m \to \mathbb{R}^3$ which connects the links $L'_{i,Q}$ and $L_{i,Q}$ and for each fixed value of t the immersion e_t has only a finite number of double points. We can adopt the homotopy e_t in such a way that it passes through a finite number of intermediate $(n+2)$-singular links $L_1, \ldots, L_s$ and, may be, the links with more than $n+2$ singularities, where each such singularity is within the same component. It follows that the links $L'_{i,Q}$ and $L_{i,Q}$ differ in $\mathbb{Z}(\mathcal{LM}^m)$ by a finite number of k-singular links with $k \geq n+2$, $L'_{i,Q} - L_{i,Q} \in \mathcal{LM}^m_{\geq n+2}$, thus they have the same Kirk–Livingston invariants of order $\leq n+1$. ◊

To the best of our knowledge, up to now, there is no combinatorial or algebraic description of self-n-equivalence for links in S^3 in any of the two cases (that is, for equivalence relations on links defined by an ambient or PL-isotopy of links).

Recently S. Melikhov and D. Repovš [15, 16] have defined for each integer $k \geq 0$ an equivalence relation, called k-quasi-isotopy, on the set of oriented links in $\mathbb{R}^3$, as follows:

Definition 4.1 (**Melikhov and Repovš [16]**). Let k be any nonnegative integer. A PL-map $f : S_1^1 \sqcup \ldots \sqcup S_m^1 \to \mathbb{R}^3$ with precisely one double point $f(p) = f(q)$, $p, q \in S_i^1$, is called a *strong* k*-quasi-embedding*, if in addition to the singleton $B_0 = \{f(p)\}$ there is a sequence of closed PL 3-balls $B_1 \subset \ldots \subset$ $\subset B_k$ in the complement C to all other components S_j^1, $j \neq i$, such that each B_{n+1}, where $0 \le n \le k$, contains the f-image of an arc $J_n \subset S_i^1$ such that $J_n \supset f^{-1}(B_n)$. Also, all PL embeddings $f : S_1^1 \sqcup \ldots \sqcup S_m^1 \to \mathbb{R}^3$ are to be thought of as contained in the class of strong k-quasi-embeddings.

If the above balls B_n are replaced with arbitrary compact polyhedra $P_n \subset \mathbb{R}^3$, where $P_0 = \{f(p)\}$, such that each inclusion $P_n \bigcup f(J_n) \subset P_{n+1}$ induces trivial homomorphism of fundamental groups, then f is called a k*-quasi-embedding*. Replacing in the definition of k-quasi-embedding the induced homomorphism of fundamental groups by the induced homomorphism of the first homology groups, one obtains the definition of a *week* k*-quasi-embedding*.

Definition 4.2 (**Melikhov and Repovš [16]**). Let $f_0, f_1 : S_1^1 \sqcup \ldots \sqcup S_m^1 \to \mathbb{R}^3$ be two links. We say that they are (*weekly, strongly*) k*-quasi-isotopic*, if they are PL-homotopic through maps f_t with at most single transversal self-intersections of the components, all of which are (weak, strong) k-quasi-embeddings.

In this setting, 0-quasi-isotopy coincides with the certain link homotopy, whereas 1-quasi-isotopy does not follow from the link concordance. Note also that the notion of n-quasi-isotopy for links was expected to give a geometric characterization of the classes of links that are indistinguishable by Kirk – Livingston invariants of order $\le n$, well defined up to PL-isotopy.

For each $n > 0$, let $\mathcal{LM}_{n,0}^m$ denote the subspace of $\mathcal{LM}_n^m$ consisting of the link maps ℓ such that all singularities of ℓ are contained in a ball B such that $\ell^{-1}(B)$ is an arc. The link maps $\ell, \ell' \in \mathcal{LM}_n^m$ are called *geometrically* k*-equivalent* [16] if they are homotopic in the space $\mathcal{LM}_n^m \bigcup \mathcal{LM}_{n+1,k}^m$, where $\mathcal{LM}_{i,k}^m$ for $k > 0$, $i > 0$, is the space of all links maps with i singularities which are geometrically $(k-1)$-equivalent to a link map in $\mathcal{LM}_{i,0}^m$.

Note that geometric n-equivalence of two links implies that they have the same Kirk – Livingston invariants of order $\le n$, well defined for PL-isotopy. Moreover, if two links are n-quasi-isotopic, then they are geometric n-equivalent (see [14], Theorem 2.2 and the discussion below the proof of this theorem).

We do not know whether there exist counterparts of Theorem 4.1 for n-quasi-isotopy and geometric n-equivalence (for the discussion on a weak n-quasi-isotopy concerning this subject see [16]).

1. *Bar-Natan D.* On the Vassiliev knot invariants // Topology. – 1995. – **34.** – P. 423–472.
2. *Burde G., Zieschang H.* Knots. – New York: Walter de Gruyter, 2003. – 559 p.
3. *Cromwell P. M.* Homogeneous links // J. London Math. Soc. Ser. 2. – 1989. – **39.** – P. 535–552.
4. *Gabai D.* Genus is superadditive under band connected sum // Topology. – 1987. – **26.** – P. 209–210.
5. *Gusarov M. N.* On n-equivalence of knots and invariants of finite degree // Advances in Soviet Math.: Topology of manifolds and varieties / Ed. O. Viro. – 1994. – P. 173–192.
6. *Eudave-Muños M.* Prime knots obtained by band sums // Pacif. J. Math. – 1988. – **139**. – P. 53–57.

7. *Habiro K.* Claspers and finite type invariants of links // Geom. and Topol. – 2000. – **4.** – P. 1–83.
8. *Kalfagianni E.* Surgery n-triviality and companion tori // J. Knot Theory and its Ramificat. – 2004. – **13.** – P. 441–456.
9. *Kalfagianni E., Lin X.-S.* Knot adjacency, genus and essential tori (with an Appendix by Darryl McCullough). – Preprint. – 2002.
10. *Kalfagianni E., Lin X.-S.* Regular Seifert surfaces and Vassiliev knot invariants. – Preprint: math. GT/9804032S. – 1998.
11. *Kirk P., Livingston C.* Vassiliev invariants of two component links and the Casson-Walker invariant // Topology. – 1997. – **36.** – P. 1333–1353.
12. *Kuperberg G.* Detecting knot invertibility // J. Knot Theory and its Ramificat. – 1996. – **5.** – P. 173–181.
13. *Le T. T. Q., Murakami J.* The universal Vassiliev-Kontsevich invariant for framed oriented links // Compositio Math. – 1996. – **102.** – P. 41–64.
14. *Melikhov S. A.* Colored finite type invariants and a multi-variable analogue of the Conway polynomial. – Preprint: Steklov Math. Inst., Dec. 2003.
15. *Melikhov S. A., Repovš D.* A geometric filtration of links modulo knots: I. Question of nilpotence // J. Knot Theory and its Ramificat. – 2005. – **14.** – P. 571–602.
16. *Melikhov S. A., Repovš D.* A geometric filtration of links modulo knots: II. Comparison // J. Knot Theory and its Ramificat. – 2005. – **14.** – P. 603–626.
17. *Nakamura T.* On canonical genus of fibered knots // J. Knot Theory and its Ramificat. – 2002. – **11.** – P. 341–352.
18. *Ng K. Y., Stanford T.* On Gusarov's groups of knots // Math. Proc. Camb. Phil. Soc. – 1999. – **126**. – P. 63–76.
19. *Plachta L.* C_n-moves, braid commutators and Vassiliev knot invariants // J. Knot Theory and its Ramificat. – 2004. – **13**. – P. 809–828.
20. *Plachta L.* Genera, band sum of knots and Vassiliev invariants // Pacif. J. Math. – (submitted).
21. *Plachta L.* Knots, satellite operations and invariants of finite order // J. Knot Theory and its Ramificat. – 2006. – **15**. – (in press).
22. *Przytycki J.* Vassiliev-Goussarov skein modules of 3-manifolds and criteria for periodicity of knots // Conf. Proc. Topology (Knoxville, TN, 1992): Lecture Notes Geom. Topology, III. – Cambridge, MA: Int. Press, 1994. – P. 143–162.
23. *Stanford T.* Vassiliev invariants and knots modulo pure braid subgroups. – Preprint: math. GT/9805092. – 1998.
24. *Stoimenow A.* Knots of genus one // Proc. Amer. Math. Soc. – 2001. – **129,** No. 7. – P. 2141–2156.
25. *Stoimenow A.* Knots of genus two. – Preprint (available on the home page: http: // www.ms.u-tokyo.ac.jp/ stoimeno/papers.
26. *Stoimenow A.* Some applications of Tristram-Levine signatures // Advances in Math. – 2005. – **194,** No. 2. – P. 463–484.
27. *Stoimenow A.* Vassiliev invariants and rational knots of unknotting number one // Topology. – 2003. – **42,** No. 1. – P. 227–241.
28. *Scharlemann M. G., Thompson A.* Unknotting number, genus, and companion tori // Math. Ann. – 1988. – **336**. – P. 191–205.

ДО n-ЕКВІВАЛЕНТНОСТІ ВУЗЛІВ І ЛІНКІВ

Описано нові співвідношення між інваріантами Васильєва та канонічним і класичним родами вузлів у тривимірному просторі. Досліджується також поведінка інваріантів скінченного типу вузлів при дії на вузлах спеціальних сателітних операцій. Крім того, вивчається n-еквівалентність лінків у сенсі Кірка – Лівінґстона в контексті сателітних операцій.

К n-ЭКВИВАЛЕНТНОСТИ УЗЛОВ И ЗАЦЕПЛЕНИЙ

Описаны новые соотношения между инвариантами Васильева и каноничным и классическим родами узлов в трехмерном пространстве. Исследуется также поведение инвариантов конечного типа узлов при действии на узлах специальных сателитных операций. Кроме того, изучается n-эквивалентность зацеплений в смысле Кирка – Ливингстона в контексте сателитных операций.

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