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ON BRANDT λ^0-EXTENSIONS OF SEMIGROUPS WITH ZERO

We introduce the Brandt λ^0-extension $B_\lambda^0(S)$ of a semigroup S with zero and establish some algebraic properties of the semigroup $B_\lambda^0(S)$ with respect to the semigroup S. Also we introduce the topological Brandt λ^0-extension of a topological semigroup S with zero and study its topological properties with respect to the topological semigroup S. In particular we show that any topological Brandt λ^0-extension of an (absolutely) H-closed topological inverse semigroup S is (absolutely) H-closed in the class of topological inverse semigroups. Also we construct topologies on $B_\lambda^0(S)$ which preserve the absolute H-closedness and H-closedness. Using the construction of topological Brandt λ^0-extensions of topological semigroups we give an example of absolutely H-closed metrizable inverse topological semigroup S with an absolutely H-closed ideal I such that S/I is not a topological semigroup.

Introductions and preliminaries. In this paper all spaces are Hausdorff. A *topological* (*inverse*) *semigroup* is a topological space together with a continuous multiplication (and an inversion, respectively). Further we follow the terminology of [1, 2, 4]. If S is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of S, and by S^1 [S^0] we denote the semigroup S with the adjoined unit [zero] (see [2]). By ω we denote the first infinite ordinal. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\mathrm{cl}_Y(A)$ we denote the topological closure of A in Y.

Let S be a semigroup with zero and I_λ be a set of cardinality $\lambda \geq 2$. On the set $B_\lambda(S) = I_\lambda \times S \times I_\lambda \cup \{0\}$ we define the semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma, \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$ and $a, b \in S$. If $S = S^1$ then the semigroup $B_\lambda(S)$ is called *the Brandt λ-extension of the semigroup S* [7]. Obviously, $\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \mid 0_S \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$ and we shall call $B_\lambda^0(S)$ *the Brandt λ^0-extension of the semigroup S with zero*. Further, if $A \subseteq S$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if A does not contain zero, and $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in I_\lambda$. If $\mathcal{I}$ is a trivial semigroup (i.e. $\mathcal{I}$ contains only one element), then by $\mathcal{I}^0$ we denote the semigroup $\mathcal{I}$ with the adjoined zero. Obviously, for any $\lambda \geq 2$ the Brandt λ^0-extension of the semigroup $\mathcal{I}^0$ is isomorphic to the semigroup of $I_\lambda \times I_\lambda$-matrix units and any Brandt λ^0-extension of a semigroup with zero contains the semigroup of $I_\lambda \times I_\lambda$-matrix units. Further by B_λ we shall denote the semigroup of $I_\lambda \times I_\lambda$-matrix units and by $B_\lambda^0(1)$ the subsemigroup of $I_\lambda \times I_\lambda$-matrix units of the Brandt λ^0-extension of a monoid S with zero.

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In our paper we establish some algebraic properties of the semigroup $B_\lambda^0(S)$ with respect to a semigroup S. Also we introduce a topological Brandt λ^0-extension of a topological semigroup S with zero and study its topological properties with respect to the topological semigroup S. In particular, we show that any topological Brandt λ^0-extension of an (absolutely) H-closed topological inverse semigroup S is (absolutely) H-closed in the class of topological inverse semigroups. Also we construct topologies on $B_\lambda^0(S)$ which preserve the absolute H-closedness and H-closedness. Using the construction of topological Brandt λ^0-extensions of topological semigroups we give an example of an absolutely H-closed metrizable inverse topological semigroup S with an absolutely H-closed ideal I such that S/I is not a topological semigroup.

1. Algebraic properties of $B_\lambda^0(S)$. This section contains algebraic properties of the semigroup $B_\lambda^0(S)$ with respect to the semigroup S. We remark that a non-zero element (α, e, β) of the semigroup $B_\lambda^0(S)$ is idempotent if and only if e is an idempotent in S and $\alpha = \beta$. Obviously, a non-zero idempotent (α, e, α) of the semigroup $B_\lambda^0(S)$ is primitive if and only if e is a primitive idempotent of S.

Proposition 1. *Let S be a semigroup with zero. Then the following conclusions hold:*

(***i***) *S is regular if and only if $B_\lambda^0(S)$ is regular;*

(***ii***) *S is orthodox if and only if $B_\lambda^0(S)$ is orthodox;*

(***iii***) *S is inverse if and only if $B_\lambda^0(S)$ is inverse;*

(***iv***) *S is 0-simple if and only if $B_\lambda^0(S)$ is 0-simple;*

(***v***) *S is completely 0-simple if and only if $B_\lambda^0(S)$ is completely 0-simple.*

P r o o f. Statement (***i***) follows from the fact that an element (α, x, β) of $B_\lambda^0(S)$ is regular if and only if x is regular in S.

(***ii***) If T is a subsemigroup of S, then $B_\lambda^0(T)$ as a subset of $B_\lambda^0(S)$ is a subsemigroup of $B_\lambda^0(S)$. Therefore if S is an orthodox semigroup then so is $B_\lambda^0(S)$. Conversely, suppose that $B_\lambda^0(S)$ be an orthodox semigroup and e and f are idempotents of S. Then the element $(\alpha, e, \alpha)(\alpha, f, \alpha) = (\alpha, ef, \alpha)$ is a non-zero idempotent of $B_\lambda^0(S)$ if $ef \neq 0$. Therefore ef is an idempotent of S and the semigroup S is orthodox.

Statement (***iii***) follows from the fact that the idempotents of the semigroup S commute if and only if the idempotents of the semigroup $B_\lambda^0(S)$ commute.

(***iv***) Suppose the contrary, i.e. there exists a 0-simple semigroup such that the semigroup $B_\lambda^0(S)$ contains a non-zero proper ideal I. Then there exist $\alpha, \beta \in I_\lambda$ and a non-empty subset $A \neq \{0\}$ of S such that $A_{\alpha\beta} \subseteq I$. Since $S_{\alpha\alpha} I \subseteq I$ and $S_{\alpha\alpha} A_{\alpha\beta} \subseteq S_{\alpha\beta}$, we have $S_{\alpha\alpha} A_{\alpha\beta} \subseteq A_{\alpha\beta}$. Therefore, A is a non-zero proper ideal of S and we obtain a contradiction. Since $B_\lambda^0(J)$ is a non-zero proper ideal in $B_\lambda^0(S)$ where J is a non-zero proper ideal of S, we get that if the semigroup $B_\lambda^0(S)$ is 0-simple then so is S.

Since every completely 0-simple semigroup contains a primitive idempotent, statement (***iv***) implies (***v***). ◊

A semigroup homomorphism $h : S \to T$ is called *annihilating* if there exists $c \in T$ such that $h(a) = c$ for all $a \in S$.

A semigroup S is called *congruence-free* if it has only two congruences: identical and universal [11]. Obviously, a semigroup S is congruence-free if and only if any homomorphism h of S into an arbitrary semigroup T is an isomorphism «into» or is annihilating.

Theorem 1. *A semigroup S with zero is congruence-free if and only if $B_\lambda^0(S)$ is congruence-free for all $\lambda \geq 2$.*

P r o o f. ($\Rightarrow$) Suppose the contrary, i.e. let there exists a congruence-free semigroup S with zero such that the semigroup $B_\lambda^0(S)$ is not congruence-free for some $\lambda \geq 2$. Then there exists a semigroup homomorphism $g : B_\lambda^0(S) \to T$ into a semigroup T which is neither an isomorphism nor annihilating. Therefore there exist $x, y \in B_\lambda^0(S)$ such that $x \neq y$ and $g(x) = g(y)$. We consider the following cases.

1°. Let $x = 0$, $y = (\alpha, s, \beta)$ for some $s \in S \setminus \{0\}$ and $\alpha, \beta \in I_\lambda$. Let (γ, t, δ) be any nonzero element of $B_\lambda^0(S)$. Since the semigroup S is congruence-free and hence is 0-simple, there exist $a, b \in S \setminus \{0\}$ such that $t = asb$ and therefore we get $g((\gamma, t, \delta)) = g((\gamma, a, \alpha) \cdot (\alpha, s, \beta) \cdot (\beta, b, \delta)) = g((\gamma, a, \alpha)) \cdot g((\alpha, s, \beta)) \cdot$ $\cdot\, g((\beta, b, \delta)) = g((\gamma, a, \alpha)) \cdot g(0) \cdot g((\beta, b, \delta)) = g((\gamma, a, \alpha) \cdot 0 \cdot (\beta, b, \delta)) = g(0)$ for any nonzero element (γ, t, δ) of $B_\lambda^0(S)$.

2°. Let $x = (\alpha, s, \beta)$, $y = (\alpha, t, \beta)$ for some $\alpha, \beta \in I_\lambda$ and $s, t \in S \setminus \{0\}$ such that $s \neq t$. Since the semigroup S is congruence-free, the restriction homomorphism $g|_S : S \to T$ is annihilating and therefore $g(x) = g(y) = g(0)$. Then case ***1°*** implies that g is an annihilating homomorphism.

3°. Let $x = (\alpha, s, \beta)$, $y = (\gamma, t, \delta)$ for some $s, t \in S \setminus \{0\}$ and $\alpha, \beta \in I_\lambda$ such that $\alpha \neq \gamma$ or $\beta \neq \delta$. Since the semigroup S is 0-simple, there exist $a, b \in S \setminus \{0\}$ such that $s = atb$ and we have $g((\alpha, s, \beta)) = g((\alpha, atb, \beta)) =$ $= g((\alpha, a, \gamma) \cdot (\gamma, t, \delta) \cdot (\delta, b, \beta)) = g((\alpha, a, \gamma)) \cdot g((\gamma, t, \delta)) \cdot g((\delta, b, \beta)) = g((\alpha, a, \gamma)) \cdot$ $\cdot\, g((\alpha, s, \beta)) \cdot g((\delta, b, \beta)) = g((\alpha, a, \gamma) \cdot (\alpha, s, \beta) \cdot (\delta, b, \beta)) = g(0)$. Therefore by case ***1°*** the homomorphism g is annihilating. We thus showed that in all three cases ***1°–3°*** the homomorphism g is annihilating. The derived contradiction shows, that $B_\lambda^0(S)$ is congruence-free and justifies the implication.

($\Leftarrow$) Suppose there exists a non-congruence-free semigroup S with zero such that $B_\lambda^0(S)$ is congruence-free semigroup for some $\lambda \geq 2$. Then there exists a semigroup T with zero and a surjective homomorphism $h : S \to T$ such that $h(s) = h(t)$ for some different $s, t \in S$. We extend the homomorphism h up to homomorphism $\tilde{h} : B_\lambda^0(S) \to B_\lambda^0(T)$ by the formulae $\tilde{h}((\alpha, s, \beta)) =$ $= (\alpha, h(s), \beta)$ and $\tilde{h}(0_S) = 0_T$, where 0_S and 0_T are the zeros of the semigroups $B_\lambda^0(S)$ and $B_\lambda^0(T)$, respectively. Therefore, we get $\tilde{h}((\alpha, s, \beta)) = \tilde{h}((\alpha, t, \beta))$, which contradicts the assumption that the semigroup $B_\lambda^0(S)$ is congruence-free. The obtained contradiction implies that S is a congruence-free semigroup. ◊

Proposition 2. *Let* S *be a semigroup with zero. Let* $h : B_\lambda^0(S) \to T$ *be a homomorphism such that* $h((\alpha, x, \beta)) = h(0)$ *for some* $x \in S$, $\alpha, \beta \in I_\lambda$. *Then* $h((\gamma, y, \delta)) = h(0)$ *for all* $y \in SxS$, $\gamma, \delta \in I_\lambda$.

P r o o f. Assume that $y \in SxS$. Then $y = axb$ for some $a, b \in S$. Therefore $h((\gamma, y, \delta)) = h((\gamma, a, \alpha) \cdot (\alpha, x, \beta) \cdot (\beta, b, \delta)) = h((\gamma, a, \alpha)) \cdot h((\alpha, x, \beta)) \cdot h((\beta, b, \delta)) = h((\gamma, a, \alpha)) \cdot h(0) \cdot h((\beta, b, \delta)) = h((\gamma, a, \alpha) \cdot 0 \cdot (\beta, b, \delta)) = h(0)$. ◊

Corollary 1. *Let* S *be a monoid with zero. A homomorphism* $h : B_\lambda^0(S) \to T$ *is annihilating if and only if the homomorphism* $h|_{B_\lambda} : B_\lambda = B_\lambda^0(1) \to T$ *is annihilating.*

Proposition 3. *Let* S *be a monoid with zero. Let* $h : B_\lambda^0(S) \to T$ *be a homomorphism and* $h((\alpha_1, a, \beta_1)) = h((\alpha_2, b, \beta_2))$ *for some* $a, b \in S$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_\lambda$. *If* $\alpha_1 \neq \alpha_2$ *or* $\beta_1 \neq \beta_2$ *then* $h((\alpha_1, a, \beta_1)) = h(0)$.

P r o o f. Assume that $\alpha_1 \neq \alpha_2$. Then $h((\alpha_1, a, \beta_1)) = h((\alpha_1, 1, \alpha_1)(\alpha_1, a, \beta_1)) = h((\alpha_1, 1, \alpha_1)) \cdot h((\alpha_1, a, \beta_1)) = h((\alpha_1, 1, \alpha_1)) \cdot h((\alpha_2, b, \beta_2)) = h((\alpha_1, 1, \alpha_1) \cdot (\alpha_2, b, \beta_2)) = h(0)$. The proof of the case $\beta_1 \neq \beta_2$ is similar. ◊

Proposition 4. *Let* $\lambda \geq 2$, S *be a monoid with zero and* T *be a semigroup. Let* $h : B_\lambda^0(S) \to T$ *be a homomorphism,* A *and* B *be disjoint subsets of* $h(B_\lambda^0(S))$. *If the sets* A *and* B *intersect at least two different subsets of the type* $h(S_{\alpha\beta})$, $\alpha, \beta \in I_\lambda$, *then* $h(0) \in A \cdot B$ *or* $h(0) \in B \cdot A$.

P r o o f. The cases $h(0) \in A$, or $h(0) \in B$ are trivial. Otherwise, for $i = 1, 2, 3, 4$ we fix $\alpha_i, \beta_i \in I_\lambda$ such that $A \cap h(S_{\alpha_1\beta_1}) \neq \varnothing$, $A \cap h(S_{\alpha_2\beta_2}) \neq \varnothing$, $B \cap h(S_{\alpha_3\beta_3}) \neq \varnothing$ and $B \cap h(S_{\alpha_4\beta_4}) \neq \varnothing$. By Proposition 3 the sets $h(S_{\alpha_1\beta_1}) \setminus h(0)$ and $h(S_{\alpha_2\beta_2}) \setminus h(0)$ are disjoint in $h(B_\lambda^0(S))$, hence $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Let x_1, x_2, x_3, x_4 be elements of the semigroup S such that $h((\alpha_1, x_1, \beta_1))$, $h((\alpha_2, x_2, \beta_2)) \in A$ and $h((\alpha_3, x_3, \beta_3))$, $h((\alpha_4, x_4, \beta_4)) \in B$. If $\alpha_1 \neq \alpha_2$, then $\alpha_1 \neq \beta_3$ or $\alpha_2 \neq \beta_3$, and hence $h(0) = h((\alpha_3, x_3, \beta_3) \cdot (\alpha_1, x_1, \beta_1)) = h((\alpha_3, x_3, \beta_3)) \cdot h((\alpha_1, x_1, \beta_1)) \in B \cdot A$, or $h(0) = h((\alpha_3, x_3, \beta_3) \cdot (\alpha_2, x_2, \beta_2)) = h((\alpha_3, x_3, \beta_3)) \cdot h((\alpha_2, x_2, \beta_2)) \in B \cdot A$. If $\beta_1 \neq \beta_2$ then $\beta_1 \neq \alpha_3$ or $\beta_2 \neq \alpha_3$, and hence $h(0) = h((\alpha_1, x_1, \beta_1) \cdot (\alpha_3, x_3, \beta_3)) = h((\alpha_1, x_1, \beta_1)) \cdot h((\alpha_3, x_3, \beta_3)) \in A \cdot B$, or $h(0) = h((\alpha_2, x_2, \beta_2) \cdot (\alpha_3, x_3, \beta_3)) = h((\alpha_2, x_2, \beta_2)) \cdot h((\alpha_3, x_3, \beta_3)) \in A \cdot B$. ◊

2. Topological Brandt λ^0-extensions of topological semigroups with zero. Further, by $\mathcal{S}$ we denote some class of topological semigroups with zero.

***Definition* 1.** Let λ be a cardinal ≥ 2, and $(S, \boldsymbol{\tau}) \in \mathcal{S}$. Let $\boldsymbol{\tau}_B$ be a topology on $B_\lambda^0(S)$ such that

a) $(B_\lambda^0(S), \boldsymbol{\tau}_B) \in \mathcal{S}$;   ***b***) $\boldsymbol{\tau}_B|_{(\alpha, S, \alpha) \cup \{0\}} = \boldsymbol{\tau}$ for some $\alpha \in I_\lambda$.

Then $(B_\lambda^0(S), \tau_B)$ is called *a topological Brandt* λ^0*-extension of* $(S, \boldsymbol{\tau})$ *in* $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then $(B_\lambda^0(S), \tau_B)$ is called *a topological Brandt* λ^0*-extension of* $(S, \boldsymbol{\tau})$.

Lemma 1. *Let* $\lambda \geq 2$ *and* $B_\lambda^0(S)$ *be a topological* λ^0*-extension of a topological monoid* S *with zero. Let* T *be a topological semigroup and* $h : B_\lambda^0(S) \to T$ *be a continuous homomorphism. Then the sets* $h(A_{\alpha\beta})$ *and* $h(A_{\gamma\delta})$ *are homeomorphic in* T *for all* $\alpha, \beta, \gamma, \delta \in I_\lambda$, *and all* $A \subseteq S$.

P r o o f. If h is an annihilating homomorphism, then the statement of the Lemma is trivial. Otherwise, we fix arbitrary $\alpha, \beta, \gamma, \delta \in I_\lambda$ and define the maps $\varphi_{\alpha\beta}^{\gamma\delta} : T \to T$ and $\varphi_{\gamma\delta}^{\alpha\beta} : T \to T$ by the formulae $\varphi_{\alpha\beta}^{\gamma\delta}(s) = h((\gamma, 1, \alpha)) \cdot s \cdot h((\beta, 1, \delta))$ and $\varphi_{\gamma\delta}^{\alpha\beta}(s) = h((\alpha, 1, \gamma)) \cdot s \cdot h((\delta, 1, \beta))$, $s \in T$. Obviously $\varphi_{\gamma\delta}^{\alpha\beta}(\varphi_{\alpha\beta}^{\gamma\delta}(h((\alpha, x, \beta)))) = h((\alpha, x, \beta))$ and $\varphi_{\alpha\beta}^{\gamma\delta}(\varphi_{\gamma\delta}^{\alpha\beta}(h((\gamma, x, \delta)))) = h((\gamma, x, \delta))$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, $x \in S^1$, and hence $\varphi_{\alpha\beta}^{\gamma\delta}|_{A_{\alpha\beta}} = (\varphi_{\gamma\delta}^{\alpha\beta})^{-1}|_{A_{\alpha\beta}}$. Since the maps $\varphi_{\alpha\beta}^{\gamma\delta}$ and $\varphi_{\gamma\delta}^{\alpha\beta}$ are continuous on T, the map $\varphi_{\gamma\delta}^{\alpha\beta}|_{h(A_{\alpha\beta})} : h(A_{\alpha\beta}) \to h(A_{\gamma\delta})$ is a homeomorphism. ◊

Proposition 5. *Let* $\lambda \geq 2$ *and let* $B_\lambda^0(S)$ *be a topological* λ^0*-extension of a topological monoid* S *with zero. Let* T *be a topological semigroup and* $h : B_\lambda^0(S) \to T$ *be a continuous homomorphism. Assume that a set* $A \subseteq h(B_\lambda^0(S))$ *is such that* A *intersects at least two different subsets of the type* $h(S_{\alpha\beta})$. *Then* $h(0) \in A \cdot A$.

P r o o f. The case $h(0) \in A$ is trivial. Assume that $h(0) \notin A$, $A \cap h(S_{\alpha_1\alpha_2}) \neq \varnothing$ and $A \cap h(S_{\beta_1\beta_2}) \neq \varnothing$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_\lambda$, i.e. there exist $x, y \in S^1$ such that $h((\alpha_1, x, \alpha_2)) \in A$ and $h((\beta_1, y, \beta_2)) \in A$. If $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$, then $h(0) = h((\alpha_1, x, \alpha_2)) \cdot h((\alpha_1, x, \alpha_2)) \in A \cdot A$ or $h(0) = h((\beta_1, y, \beta_2)) \cdot h((\beta_1, y, \beta_2)) \in A \cdot A$. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\alpha_2 \neq \beta_1$, and hence $h(0) = h((\alpha_1, x, \alpha_2)) \cdot h((\beta_1, y, \beta_2)) \in A \cdot A$. ◊

Lemma 2. *Let* $\lambda \geq 2$, $B_\lambda^0(S)$ *and* T *be topological semigroups and* $h : B_\lambda^0(S) \to T$ *be a continuous homomorphism. Let* $h(B_\lambda^0(S))$ *be a dense subsemigroup of* T *and* $h(S_{\alpha\beta})$ *be a closed subset in* T *for some* $\alpha, \beta \in I_\lambda$. *Then* $a \cdot a = h(0)$ *for all* $a \in T \setminus h(B_\lambda^0(S))$, *and* $h(0)$ *is the zero of* T.

P r o o f. Since $h(B_\lambda^0(S))$ is a dense subsemigroup of T, by Proposition 2 [7], $h(0)$ is the zero of T. Assume that $a \cdot a = b \neq h(0)$ for some $a \in T \setminus h(B_\lambda^0(S))$. Then for any open neighborhood $U(b) \not\ni h(0)$ there exists an open neighborhood $V(a) \not\ni h(0)$ such that $V(a) \cdot V(a) \subseteq U(b)$. By Lemma 1 the set $h(S_{\gamma\delta})$ is closed for each $\gamma, \delta \in I_\lambda$. Therefore the neighborhood $V(a)$ intersects infinitely many sets of the type $h(S_{\alpha\beta})$, $\alpha, \beta \in I_\lambda$. Then by Proposition 5 we have $h(0) \in V(a) \cdot V(a) \subseteq U(b)$, a contradiction with the choice of $U(b)$. ◊

Theorem 2. *Let* S *be a topological inverse monoid with zero. Let* $\lambda \geq 2$, $B_\lambda^0(S)$ *and* T *be topological inverse semigroups,* $h : B_\lambda^0(S) \to T$ *be a continuous homomorphism such that the set* $h(S_{\alpha\beta})$ *is closed in* T *for some* $\alpha, \beta \in I_\lambda$. *Then* $h(B_\lambda^0(S))$ *is a closed subsemigroup of* T.

P r o o f. In the case $2 \le \lambda < \omega$ the statement of the Theorem follows from Lemma 1.

Let $\lambda \ge \omega$. We denote $G = \mathrm{cl}_T(h(B_\lambda^0(S)))$. By Proposition II.2 [3], G is a topological inverse semigroup. Let $b \in G \setminus h(B_\lambda^0(S))$. Then by Lemma 1, $b, b^{-1} \in G \setminus E(G)$. We remark that $b \cdot b^{-1} \ne h(0)$ and $b^{-1} \cdot b \ne h(0)$. Indeed, if we assume that $b \cdot b^{-1} = h(0)$ or $b^{-1} \cdot b = h(0)$, then since $h(0)$ is the zero of G, we would get $b = b \cdot b^{-1} \cdot b = h(0) \cdot b = h(0)$ or $= h(0)$, which would contradict the inclusion $b \in G \setminus h(B_\lambda^0(S))$.

Therefore there exist $e, f \in E(G) = E(h(B_\lambda^0(S)))$, such that $b \cdot b^{-1} = e$ and $b^{-1} \cdot b = f$. We consider first the case $e \ne f$. Let $W(e) \not\ni h(0)$ and $W(f) \not\ni h(0)$ be disjoint open neighborhood s of e and f in T, respectively. Then there exist disjoint open neighborhood s $U(b) \not\ni h(0)$ and $U(b^{-1}) \not\ni h(0)$ in T such that $U(b) \cdot U(b^{-1}) \subseteq W(e)$ and $U(b^{-1}) \cdot U(b) \subseteq W(f)$. By Lemma 1 the set $h(S_{\alpha\beta})$ is closed in T for each $\alpha, \beta \in I_\lambda$, and hence the sets $U(b)$ and $U(b^{-1})$ intersect infinitely many different sets of the type $h(S_{\gamma\delta}) \setminus h(0)$, $\gamma, \delta \in I_\lambda$. Thus by Proposition 5 we get $h(0) \in U(b) \cdot U(b^{-1}) \subseteq W(e)$ or $h(0) \in U(b^{-1}) \cdot$ $\cdot\, U(b) \subseteq W(f)$, a contradiction with the choice of the neighborhoods $W(e)$ and $W(f)$. In the case $e = f$ we similarly derive a contradiction. The obtained contradictions imply the statement of the theorem. ◊

***Definition* 2** [12]. Let $\mathcal{S}$ be a class of topological semigroups. A semigroup $S \in \mathcal{S}$ is called *H-closed in $\mathcal{S}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as subsemigroup. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*.

***Definition* 3** [13]. Let $\mathcal{S}$ be a class of topological semigroups. A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class $\mathcal{S}$* if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

Lemma 1 and Theorem 2 imply

Theorem 3. *For any cardinal $\lambda \ge 2$, every topological Brandt λ^0-extension $B_\lambda^0(S)$ of an absolutely H-closed topological inverse monoid S with zero in the class of topological inverse semigroups, is absolutely H-closed in the class of topological inverse semigroups.*

Corollary 2. *For any cardinal $\lambda \ge 2$, every topological Brandt λ^0-extension $B_\lambda^0(S)$ of a compact topological inverse semigroup S with zero in the class of topological inverse semigroups, is absolutely H-closed in the class of topological inverse semigroups.*

Theorem 4. *Let S be a topological inverse monoid with zero. Then the following conditions are equivalent:*

(***i***) *S is an absolutely H-closed semigroup in the class of topological inverse semigroups;*

(***ii***) *there exists a cardinal $\lambda \ge 2$ such that any topological Brandt λ^0-extension $B_\lambda^0(S)$ of the semigroup S is absolutely H-closed in the class of topological inverse semigroups;*

(***iii***) *for each cardinal* $\lambda \ge 2$, *every topological Brandt* λ^0*-extension* $B_\lambda^0(S)$ *of the semigroup* S *is absolutely* H*-closed in the class of topological inverse semigroups.*

P r o o f. The implication (***iii***) $\Rightarrow$ (***ii***) is trivial, and Theorem 3 claims the implications (***i***) $\Rightarrow$ (***ii***) and (***i***) $\Rightarrow$ (***iii***).

We shall show that the implication (***ii***) $\Rightarrow$ (***i***) holds. Suppose the contrary, i.e. that there exists a non-absolutely H-closed topological inverse monoid S with zero in the class of topological inverse semigroups, and for some cardinal $\lambda^* \ge 2$ every topological Brandt λ_0^*-extension $B_{\lambda^*}^0(S)$ is absolutely H-closed in the class of topological inverse semigroups. Then there exist a topological inverse semigroup T and a continuous homomorphism «into» $h : S \to T$ such that $h(S)$ is not a closed subsemigroup of T.

Let $\boldsymbol{\tau}_S$ and $\boldsymbol{\tau}_T$ be direct sum topologies on $B_{\lambda^*}^0(S)$ and $B_{\lambda^*}^0(T)$, respectively (see [5, p. 129]). Then $(B_{\lambda^*}^0(S), \boldsymbol{\tau}_S)$ and $(B_{\lambda^*}^0(T), \boldsymbol{\tau}_T)$ are topological inverse semigroups, S and T^1 are homeomorphic to $S_{\alpha\beta}$ and $T_{\alpha\beta}$, for all $\alpha, \beta \in I_\lambda$ (see [5, p. 129]). We define the map $\tilde{h} : B_{\lambda^*}^0(S) \to B_{\lambda^*}^0(T)$ as follows: $\tilde{h}(0) = 0$ and $\tilde{h}((\alpha, s, \beta)) = (\alpha, h(s), \beta)$ for all $\alpha, \beta \in I_\lambda$, $s \in S \setminus \{0\}$. Obviously, the homomorphism $\tilde{h} : (B_{\lambda^*}^0(S), \boldsymbol{\tau}_S) \to (B_{\lambda^*}^0(T), \boldsymbol{\tau}_T)$ is continuous and $\tilde{h}(B_{\lambda^*}^0(S))$ is not a closed subsemigroup of $(B_{\lambda^*}^0(T), \boldsymbol{\tau}_T)$. Therefore there exists a topological Brandt λ_0^*-extension $(B_{\lambda^*}^0(S), \tau_S)$, which is not absolutely H-closed in the class of topological inverse semigroups. The obtained contradiction implies the statement of the theorem. ◊

Taking $h : B_\lambda^0(S) \to T$ is a topological isomorphism «into» in Lemma 1 and Theorem 2, we get

Theorem 5. *For any cardinal* $\lambda \ge 2$, *every topological Brandt* λ^0*-extension* $B_\lambda^0(S)$ *of a* H*-closed topological inverse monoid* S *with zero in the class of topological inverse semigroups is* H*-closed in the class of topological inverse semigroups.*

The proof of the next theorem is similar that of Theorem 4.

Theorem 6. *Let* S *be a topological inverse monoid with zero. Then the following conditions are equivalent:*

(***i***) S *is an* H*-closed semigroup in the class of topological inverse semigroups;*

(***ii***) *there exists a cardinal* $\lambda \ge 2$ *such that any topological Brandt* λ^0*-extension* $B_\lambda^0(S)$ *of the semigroup* S *is* H*-closed in the class of topological inverse semigroups;*

(***iii***) *for each cardinal* $\lambda \ge 2$, *every topological Brandt* λ^0*-extension* $B_\lambda^0(S)$ *of the semigroup* S *is* H*-closed in the class of topological inverse semigroups.*

Let $(S, \boldsymbol{\tau})$ be a topological semigroup with zero 0_S and $\lambda \geqslant \omega$. Let $V(0_S)$ be an open neighborhood of the zero of the semigroup $(S, \boldsymbol{\tau})$. For all $\alpha, \beta \in I_\lambda$ we put

$$V_\alpha(V(0_S)) = B^0_\lambda(S) \setminus \{(\alpha, s, \gamma) \mid \gamma \in I_\lambda,\ s \in S \setminus V(0_S)\}$$

and

$$H_\beta(V(0_S)) = B^0_\lambda(S) \setminus \{(\gamma, s, \beta) \mid \gamma \in I_\lambda,\ s \in S \setminus V(0_S)\}.$$

We define

$$U^{\alpha_1,\dots,\alpha_n}(V(0_S)) = \bigcap_{i=1}^{n} V_{\alpha_i}(V(0_S)), \qquad U_{\beta_1,\dots,\beta_m}(V(0_S)) = \bigcap_{j=1}^{m} H_{\beta_j}(V(0_S)),$$

$$U^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m}(V(0_S)) = U^{\alpha_1,\dots,\alpha_n}(V(0_S)) \cap U_{\beta_1,\dots,\beta_m}(V(0_S)),$$

where $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m \in I_\lambda$, $m,n \in \mathbb{N}$.

Let $\Omega(s)$ be a base of the topology $\boldsymbol{\tau}$ at the point $s \in S$. Further, we define the following families

$$\Omega_v = \{U^{\alpha_1,\dots,\alpha_n}(V(0_S)) \mid \alpha_1,\dots,\alpha_n \in I_\lambda, n \in \mathbb{N}, V(0_S) \in \Omega(0_S)\} \cup$$
$$\cup \{(\alpha, V(s), \beta) \mid V(s) \in \Omega(s),\ s \in S \setminus \{0_S\},\ \alpha,\beta \in I_\lambda\},$$
$$\Omega_h = \{U_{\beta_1,\dots,\beta_m}(V(0_S)) \mid \beta_1,\dots,\beta_m \in I_\lambda, m \in \mathbb{N}, V(0_S) \in \Omega(0_S)\} \cup$$
$$\cup \{(\alpha, V(s), \beta) \mid V(s) \in \Omega(s), s \in S \setminus \{0_S\}, \alpha,\beta \in I_\lambda\},$$
$$\Omega_i = \{U^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m}(V(0_S)) \mid \alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m \in I_\lambda,\ n,m \in \mathbb{N},\ V(0_S) \in$$
$$\in \Omega(0_S)\} \cup \{(\alpha, V(s), \beta) \mid V(s) \in \Omega(s), s \in S \setminus \{0_S\}, \alpha,\beta \in I_\lambda\}.$$

Obviously, the conditions (BP1)–(BP3) [4] hold for the families Ω_v, Ω_h and Ω_i, and hence Ω_v, Ω_h and Ω_i are the bases of topologies $\boldsymbol{\tau}_v(S)$, $\boldsymbol{\tau}_h(S)$ and $\boldsymbol{\tau}_i(S)$ on the semigroup $B^0_\lambda(S)$, respectively.

***Definition* 4.** Let S be a topological semigroup with zero 0. Then S is called *a left* [*right*] 0*-bounded semigroup* if for any open neighborhood $U(0)$ of zero there exists an open neighborhood $V(0)$ such that $V(0)\cdot S \subseteq U(0)$ [$S \cdot V(0) \subseteq U(0)$]. A left and right 0-bounded topological semigroup is called 0*-bounded*.

Theorem 7. *Every compact topological semigroup with zero is* 0*-bounded*.

P r o o f. Let S be a compact topological semigroup with zero 0 and $U(0)$ be an open neighborhood of 0. Since the multiplication in S is continuous, for any $s \in S$ there exist open neighborhoods $V(s)$ and $V_s(0)$ of s and 0, respectively, such that $V(s)V_s(0) \subseteq U(0)$ and $V_s(0)V(s) \subseteq U(0)$. The compactness of S implies that the open cover $\gamma = \{V(s) \mid s \in S\}$ contains a finite subcover $\gamma_0 = \{V(s_j) \mid s_j \in S,\ j = 1,\dots,k\}$. Put $V(0) = \bigcap_{j=1}^{k} V_{s_j}(0)$. Therefore, we get

$$SV(0) = (V(s_1) \cup \dots \cup V(s_k))V(0) \subseteq V(s_1)V(0) \cup \dots \cup V(s_k)V(0) \subseteq U(0)$$

and

$$V(0)S = V(0)(V(s_1) \cup \dots \cup V(s_k)) \subseteq V(0)V(s_1) \cup \dots \cup V(0)V(s_k) \subseteq U(0). \ \Diamond$$

Proposition 6. *Let* $\lambda \ge \omega$ *and* (S,τ) *be a topological semigroup with zero. Then the semigroup* $(S,\boldsymbol{\tau})$ *is left* [*right*] 0*-bounded if and only if* $(B^0_\lambda(S), \boldsymbol{\tau}_v(S))$ [$(B^0_\lambda(S), \boldsymbol{\tau}_h(S))$] *is a topological semigroup*.

P r o o f. ($\Rightarrow$) We consider only the case $(B^0_\lambda(S), \boldsymbol{\tau}_v(S))$. The proof of the statement for the semigroup $(B^0_\lambda(S), \boldsymbol{\tau}_h(S))$ is similar.

It is sufficient to consider the following cases.

1°. Let $ab = c \neq 0$ in S and $U(a)U(b) \subseteq U(c)$. If $\beta \neq \gamma$, then

$$(\alpha, U(a), \beta)(\gamma, U(b), \delta) = \{0\} \subseteq U^{\alpha_1,\ldots,\alpha_n}(U(0))$$

for any $\alpha_1, \ldots, \alpha_n \in I_\lambda$ and any open neighborhood $U(0)$ of the zero 0, and

$$(\alpha, U(a), \beta)(\beta, U(b), \delta) \subseteq (\alpha, U(c), \delta).$$

2°. Let $ab = 0$ in S and $U(a)U(b) \subseteq U(0)$. If $\beta \neq \gamma$, then

$$(\alpha, U(a), \beta)(\gamma, U(b), \delta) = \{0\} \subseteq U^{\alpha_1,\ldots,\alpha_n}(U(0))$$

and

$$(\alpha, U(a), \beta)(\beta, U(b), \delta) \subseteq U^{\alpha_1,\ldots,\alpha_n}(U(0))$$

for any $\alpha_1, \ldots, \alpha_n \in I_\lambda$ and any open neighborhood $U(0)$ of the zero 0.

3°. If $V(0)$ and $U(0)$ are open neighborhoods of zero in S such that $V(0)S \subseteq U(0)$, then $U^{\alpha_1,\ldots,\alpha_n}(V(0))U^{\alpha_1,\ldots,\alpha_n}(V(0)) \subseteq U^{\alpha_1,\ldots,\alpha_n}(U(0))$.

4°. If $V(a)$, $V(0)$ and $U(0)$ are open neighborhoods of a and zero in S such that $V(a)V(0) \subseteq U(0)$ and $V(0)V(a) \subseteq U(0)$, then

$$(\alpha, V(a), \beta)U^{\alpha_1,\ldots,\alpha_n,\beta}(V(0)) \subseteq U^{\alpha_1,\ldots,\alpha_n}(U(0))$$

and

$$U^{\alpha_1,\ldots,\alpha_n}(V(0))(\alpha, V(a), \beta) \subseteq U^{\alpha_1,\ldots,\alpha_n}(U(0)).$$

($\Leftarrow$) Suppose the contrary, i.e. that $(B^0_\lambda(S), \boldsymbol{\tau}_v(S))$ is a topological semigroup and $(S, \boldsymbol{\tau})$ is a non-left 0-bounded topological semigroup. Then there exists an open neighborhood $U(0)$ of zero in $(S, \boldsymbol{\tau})$ such that $V(0)S \nsubseteq U(0)$ for any open neighborhood $V(0)$ of the zero 0 in $(S, \boldsymbol{\tau})$. Therefore for every open neighborhood $W(0)$ of zero in $(S, \boldsymbol{\tau})$ and any $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k \in I_\lambda$, $m, k \in \mathbb{N}$, the following condition holds $U^{\beta_1,\ldots,\beta_k}(W(0))U^{\beta_1,\ldots,\beta_k}(W(0)) \nsubseteq U^{\alpha_1,\ldots,\alpha_m}(U(0))$, which contradicts the assumption that $(B^0_\lambda(S), \boldsymbol{\tau}_v(S))$ is a topological semigroup. $\Diamond$

Proposition 7. *Let $\lambda \geq \omega$ and let $(S, \boldsymbol{\tau})$ be a topological (inverse) semigroup with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$ is a topological (inverse) semigroup.*

The p r o o f of Proposition 7 is similar to the one of Proposition 6.

Proposition 2 [7] implies the following

Lemma 3. *Let $\lambda \geq \omega$, $B^0_\lambda(S)$ and T be topological semigroups and $h : B^0_\lambda(S) \to T$ be a continuous homomorphism such that $h(B^0_\lambda(S))$ is a dense subset in T. Then $0_T = h(0)$ is the zero of the semigroup T.*

Theorem 8. *Let $\lambda \geq \omega$ and $(S, \boldsymbol{\tau})$ be an absolutely H-closed topological (inverse) monoid with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$ is an absolutely H-closed topological (inverse) semigroup.*

P r o o f. Suppose the contrary, i.e. that $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$ is not an absolutely H-closed topological semigroup. Then there exists a continuous homomorphism $h : B^0_\lambda(S) \to T$ from $B^0_\lambda(S)$ into a topological semigroup T such that $h(B^0_\lambda(S))$ is not a closed subset in T. Without loss of generality we can suppose that the set $h(B^0_\lambda(S))$ is dense in T and $h(B^0_\lambda(S)) \neq T$. Then there exists $x \in \overline{h(B^0_\lambda(S))} \setminus h(B^0_\lambda(S)) \subseteq T$. By Lemma 3, $h(0) = 0_T$ and hence $x \cdot 0_T =$

$= 0_T \cdot x = 0_T$. Since T is a topological semigroup, for any open neighborhood $W(0_T)$ of 0_T in T there exist open neighborhoods $V(0_T)$ and $U(0_T)$ of 0_T in T and an open neighborhood $V(x)$ of x in T such that $V(0_T) \cap V(x) = \varnothing$, $U(0_T) \cap V(x) = \varnothing$, $V(0_T) \subseteq W(0_T)$, $U(0_T) \subseteq W(0_T)$, $V(0_T) \cdot V(x) \subseteq U(0_T)$, and $V(x) \cdot V(0_T) \subseteq U(0_T)$.

Since $0 \in h^{-1}(U(0_T))$ and $h^{-1}(U(0_T))$ is an open subset in $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$, there exist $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in I_\lambda$ such that $U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}(V(0_S)) \subseteq h^{-1}(U(0_S))$.

By Lemma 1 the sets $h(S_{\alpha\beta})$ and $h(S_{\gamma\delta})$ are homeomorphic in T, and hence are closed subsets of T for all $\alpha, \beta, \gamma, \delta \in I_\lambda$. Therefore at least one of the following conditions holds:

(***i***) for some $i_0 \in \{1, 2, \ldots, n\}$, the set $B_{i_0} = h^{-1}(V(x)) \cap \{(\alpha_{i_0}, s, \gamma) \mid s \in S^1, \gamma \in I_\lambda\}$ intersects infinitely many subsets $S_{\alpha\beta}$;

(***ii***) for some $j_0 \in \{1, 2, \ldots, m\}$, the set $B^{j_0} = h^{-1}(V(x)) \cap \{(\gamma, s, \alpha_{j_0}) \mid s \in S^1, \gamma \in I_\lambda\}$ intersects infinitely many subsets $S_{\alpha\beta}$.

Indeed, suppose that for any $\alpha_{i_0} \in I_\lambda$ the set B_{i_0} intersects finitely many subsets $S_{\alpha\beta}$, i.e. $S_{\alpha\beta} \cap S_{\alpha_i\beta_i} \neq \varnothing$ only for $i = 1, 2, \ldots, n$. By Lemma 1 the set $h(S_{\alpha_i\beta_i})$ is closed in T and hence $h(S_{\alpha_1\beta_1}) \cup \ldots \cup h(S_{\alpha_n\beta_n})$ is a closed subset of T. Therefore x is not a limit point of the set $h(B^0_\lambda(S))$ in the topological space T. This contradicts the choice of α. Therefore the set B_{i_0} intersects infinitely many subsets $S_{\alpha\beta}$ for some $i_0 \in \{1, 2, \ldots, n\}$.

Taking i_0 as in (***i***), we define

$$\Gamma_{i_0} = \{\gamma \in I_\lambda \mid \text{there exists } s \in S \text{ such that } (\alpha_{i_0}, s, \gamma) \in h^{-1}(V(x))\}.$$

For any element $U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S))$ of the base of the topology $\boldsymbol{\tau}_i(S)$ at zero, where $\delta_1, \ldots, \delta_k \in I_\lambda$ and $U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S)) \subseteq h^{-1}(V(x))$ we have that the set $\{(\gamma, s, \gamma) \mid \gamma \in \Gamma_{i_0}, s \in S\} \cap U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S))$ contains infinitely many subsets $S_{\alpha\alpha}$ and hence the set Γ_{i_0} is infinite. Since $(\alpha_{i_0}, s, \gamma) \cdot (\gamma, s, \gamma) \neq (\alpha, s, \beta)$, for $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$, $\beta \notin \{\beta_1, \ldots, \beta_m\}$ and $i_0 \in \{1, 2, \ldots, n\}$ we have

$$B_{i_0} \cdot U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S)) \nsubseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}(V(0_S)),$$

which contradicts the inclusion $V(x) \cdot V(0_T) \subseteq U(0_T)$.

Let $j_0 \in \{1, 2, \ldots, m\}$ be such that the set $B^{j_0} = h^{-1}(V(x)) \cap \{(\gamma, s, \alpha_{j_0}) \mid s \in S^1, \gamma \in I_\lambda\}$ intersects infinitely many subsets $S_{\alpha\beta}$. We define

$$\Gamma^{j_0} = \{\gamma \in I_\lambda \mid \text{there exists } s \in S \text{ such that } (\gamma, s, \alpha_{j_0}) \in h^{-1}(V(x))\}.$$

For any element $U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S))$ of the base of the topology $\boldsymbol{\tau}_i(S)$ at zero, where $\delta_1, \ldots, \delta_k \in I_\lambda$ and $U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S)) \subseteq h^{-1}(V(x))$ we have that the set $\{(\gamma, s, \gamma) \mid \gamma \in \Gamma^{j_0}, s \in S\} \cap U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}(V(0_S))$ contains infinitely many subsets

$S_{\alpha\alpha}$ and hence the set Γ^{j_0} is infinite. Since $(\gamma, s, \gamma)\cdot(\gamma, s, \alpha_{j_0}) \neq (\beta, s, \alpha)$ for $\alpha \notin \{\alpha_1, \dots, \alpha_n\}$, $\beta \notin \{\beta_1, \dots, \beta_m\}$ and $j_0 \in \{1, 2, \dots, m\}$ we have

$$U^{\delta_1,\dots,\delta_k}_{\delta_1,\dots,\delta_k}(V(0_S)) \cdot B^{j_0} \nsubseteq U^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m}(V(0_S)),$$

which contradicts the inclusion $V(0_T)\cdot V(x) \subseteq U(0_T)$.

Therefore, the contradictions derived show that $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$ is an absolutely H-closed topological (inverse) semigroup. ◊

Theorem 8 implies

Corollary 3. *Let $\lambda \geq \omega$ and let $(S, \boldsymbol{\tau})$ be a compact topological (inverse) monoid with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$ is an absolutely H-closed topological (inverse) semigroup.*

The proof of next Theorem is similar that of Theorem 8.

Theorem 9. *Let $\lambda \geq \omega$ and let $(S, \boldsymbol{\tau})$ be a left [right] 0-bounded absolutely H-closed topological monoid with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_h(S))$ $[(B^0_\lambda(S), \boldsymbol{\tau}_v(S))]$ is an absolutely H-closed topological semigroup.*

Theorem 9 implies

Corollary 4. *Let $\lambda \geq \omega$ and let $(S, \boldsymbol{\tau})$ be a compact topological (inverse) monoid with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_h(S))$ and $(B^0_\lambda(S), \boldsymbol{\tau}_v(S))$ are absolutely H-closed topological semigroups.*

If in the proof of Theorem 8 we suppose that the homomorphism $h : B^0_\lambda(S) \to T$ is an embedding, then we get Theorem 10, and similarly Theorem 11.

Theorem 10. *Let $\lambda \geq \omega$ and let $(S, \boldsymbol{\tau})$ be a H-closed topological (inverse) monoid with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_i(S))$ is a H-closed topological (inverse) semigroup.*

Theorem 11. *Let $\lambda \geq \omega$ and let $(S, \boldsymbol{\tau})$ be a left [right] 0-bounded H-closed topological monoid with zero. Then $(B^0_\lambda(S), \boldsymbol{\tau}_h(S))$ $[(B^0_\lambda(S), \boldsymbol{\tau}_v(S))]$ is a H-closed topological semigroup.*

A. D. Wallace in [14] proved that if S is a compact topological semigroup and ρ is a closed congruence on S, then S/ρ is a compact topological semigroup. As a consequence of this result we have that if I is a closed ideal of a compact topological semigroup S, then S/I is a compact topological semigroup. J. D. Lawson and B. L. Madison in [10] generalized this Wallace's result and showed that if S is a locally compact σ-compact topological semigroup and ρ is a closed congruence on S, then S/ρ is a topological semigroup. As an immediate corollary of the Lawson – Madison Theorem, we have a topological version of the Rees quotient semigroup: if S is a locally compact σ-compact topological semigroup and I is a closed ideal of S, then S/I is a topological semigroup.

The next theorem is a generalization of the Wallace Theorem on the Rees quotient semigroup.

Theorem 12. *Let S be a topological semigroup and I be a compact ideal in S. Then S/I is a topological semigroup.*

P r o o f. Let $\pi : S \to S/I$ be a natural homomorphism. By Proposition 2.1 [10] it is sufficient to prove that the map $\pi \times \pi : S \times S \to S/I \times S/I$ is quotient. We shall show that the map $\pi : S \to S/I$ is perfect. Since for any $\tilde{a} \in S/I$ the set $\pi^{-1}(\tilde{a})$ is compact in S, it is sufficient to prove that π is a closed map. Let A be a closed subset in S. We remark that the restriction

$\pi|_{S\setminus I}: S\setminus I \to (S/I)\setminus \pi(I)$ of the map π is a homeomorphism. Hence, if $A\cap I=\varnothing$ then $\pi(A)$ is a closed subset of S/I. Suppose that $A\cap I\neq\varnothing$ and $\pi(A)$ is not a closed subset in S/I. Since the map π is quotient, $\pi^{-1}(\pi(A))$ is a nonclosed subset of S. But the set $\pi^{-1}(\pi(A))$ is closed in S as a union of the closed subset A and the compactum I, a contradiction. The obtained contradiction implies that π is a closed map. Then by Theorem 3.7.7 [4] the map $\pi\times\pi: S\times S\to S/I\times S/I$ is perfect and hence by Corollary 2.4.8 [4] is quotient. Therefore S/I is a topological semigroup. ◊

In [9] O. Hryniv constructed an example of a locally compact metrizable topological semigroup S with a closed ideal I such that S/I is not a topological semigroup. In our paper we construct an example of an absolutely H-closed countable metrizable topological semigroup S with an absolutely H-closed ideal I such that S/I is not a topological semigroup.

***Example* 1** [6]. Let $\mathbb{N}$ be the set of positive integers. Let $\{x_n\}$ be an increasing sequence in $\mathbb{N}$. Put $\mathbb{N}^*=\{0\}\cup\{\frac{1}{n}\mid n\in\mathbb{N}\}$. We define the semilattice operation on $\mathbb{N}^*$ as follows: $ab=\min\{a,b\}$, for $a,b\in\mathbb{N}^*$. Obviously, 0 is the zero of $\mathbb{N}^*$. We put $U_n(0)=\{0\}\cup\left\{\frac{1}{x_k}\mid k\geq n\right\}$, $n\in\mathbb{N}$. A topology $\boldsymbol{\tau}$ on $\mathbb{N}^*$ is defined as follows:

a) all nonzero elements of $\mathbb{N}^*$ are isolated points in $\mathbb{N}^*$;

b) $\mathcal{B}(0)=\{U_n(0)\mid n\in\mathbb{N}\}$ is the base of the topology $\boldsymbol{\tau}$ at the point $0\in\mathbb{N}^*$.

It is easy to see that $(\mathbb{N}^*,\boldsymbol{\tau})$ is a countable linearly ordered σ-compact locally compact metrizable topological semilattice and if $x_{k+1}>x_k+1$ for any $k\in\mathbb{N}$ then $(\mathbb{N}^*,\boldsymbol{\tau})$ is a non-compact semilattice. ⊳

By Proposition 1 [6] $(\mathbb{N}^*,\boldsymbol{\tau})$ is an H-closed topological semilattice and hence by Theorem 1 [6] the semilattice $(\mathbb{N}^*,\boldsymbol{\tau})$ is an absolutely H-closed.

Let $\mathbf{0}\notin\mathbb{N}^*$. We extend the semilattice operation from $\mathbb{N}^*$ to $\widetilde{\mathbb{N}}^*=\mathbb{N}^*\cup\mathbf{0}$ as follows: $\mathbf{0}x=x\mathbf{0}=\mathbf{0}\mathbf{0}=\mathbf{0}$. We define the topological space $\widetilde{\mathbb{N}}^*$ to be a topological sum of the space $(\mathbb{N}^*,\boldsymbol{\tau})$ and the single space $\mathbf{0}$.

Proposition 8. $\widetilde{\mathbb{N}}^*$ *is an absolutely H-closed metrizable topological semilattice.*

Theorem 13. *Let $\lambda=\omega$. Then $(B^0_\lambda(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$ and $(B^0_\lambda(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_v(\widetilde{\mathbb{N}}^*))$ are metrizable topological semigroups.*

P r o o f. We consider only the case $(B^0_\lambda(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$. In the case $(B^0_\lambda(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_v(\widetilde{\mathbb{N}}^*))$ the proof is similar.

Obviously, the topological semilattice $\widetilde{\mathbb{N}}^*$ is a zero-dimensional topological space, i.e. there exists a base of $\widetilde{\mathbb{N}}^*$ which consists from clopen subsets. Hence by the definition of the topology $\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*)$ every non-zero element of the topological semigroup $(B^0_\lambda(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$ has a base which contains clopen subsets. Since $\mathbf{0}$ is an isolated point in $\widetilde{\mathbb{N}}^*$, every element $U_{\beta_1,\ldots,\beta_m}(\mathbf{0})$ of the base Ω_h of the topology $\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*)$ has an open complement in $B^0_\lambda(\widetilde{\mathbb{N}}^*)$ and hence

$U_{\beta_1,\ldots,\beta_m}(\mathbf{0})$ is a closed subset of $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$. Therefore the topological space $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$ is 0-dimensional and hence is regular. Since $\lambda=\omega$, the definition of the base Ω_h implies that $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$ is a second countable space, and hence by Theorem 4.2.9 [4] the topological space $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$ is metrizable. ◊

The set $\mathcal{J}(\widetilde{\mathbb{N}}^*)=\{0\}\cup\{(\alpha,\mathbf{0},\beta)\mid \mathbf{0}\in\widetilde{\mathbb{N}}^*,\alpha,\beta\in I_\lambda\}$ is an ideal of $B_\lambda^0(\widetilde{\mathbb{N}}^*)$. By Theorem 6 [8] the semigroup $\mathcal{J}(\widetilde{\mathbb{N}}^*)$ with the induced topology $\boldsymbol{\tau}_h$ from $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$ is an absolutely H-closed topological semigroup and hence is a closed ideal of $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))$. Similarly we get that $\mathcal{J}(\widetilde{\mathbb{N}}^*)$ with the induced topology $\boldsymbol{\tau}_v$ from $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_v(\widetilde{\mathbb{N}}^*))$ is an absolutely H-closed topological semigroup and hence is a closed ideal of $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_v(\widetilde{\mathbb{N}}^*))$.

Obviously, the Rees quotient semigroup $B_\lambda^0(\widetilde{\mathbb{N}}^*)/\mathcal{J}(\widetilde{\mathbb{N}}^*)$ is algebraically isomorphic to the semigroup $B_\lambda^0(\mathbb{N}^*)$.

Lemma 4. *The topological semilattice* $(\mathbb{N}^*,\boldsymbol{\tau})$ *is* 0*-bounded if and only if it is compact.*

P r o o f. ($\Rightarrow$) Suppose there exists an increasing sequence $\{x_n\}$ in $\mathbb{N}$ such that $(\mathbb{N}^*,\boldsymbol{\tau})$ is a 0-bounded non-compact topological semilattice. Then there exists $k_0\in\mathbb{N}$ such that $x_{k+1}>x_k+1$ for all $k>k_0$, $k\in\mathbb{N}$. Then $U_i(0)\mathbb{N}^*=\mathbb{N}^*U_i(0)\nsubseteq U_j(0)$ for any $i,j\in\mathbb{N}$, a contradiction. The obtained contradiction implies the implication.

The implication ($\Leftarrow$) follows from Theorem 7. ◊

Lemma 4 implies

Theorem 14. *Let* $\lambda\geq\omega$ *and* $\{x_n\}$ *be an increasing sequence in* $\mathbb{N}$ *such that* $x_{k+1}>x_k+1$ *for any* $k\in\mathbb{N}$ *and define the topological semigroup* $(\mathbb{N}^*,\boldsymbol{\tau})$ *as above. Then the semigroup operations in* $(B_\lambda^0(\mathbb{N}^*),\boldsymbol{\tau}_h(\mathbb{N}^*))$ *and* $(B_\lambda^0(\mathbb{N}^*),\boldsymbol{\tau}_v(\mathbb{N}^*))$ *are discontinuous.*

Theorem 15. *Let* $\lambda\geq\omega$ *and* $\{x_n\}$ *be an increasing sequence in* $\mathbb{N}$ *such that* $x_{k+1}>x_k+1$ *for any* $k\in\mathbb{N}$ *and define the topological semigroup* $(\mathbb{N}^*,\boldsymbol{\tau})$ *as above. Then the topological Rees quotient semigroups* $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))/\mathcal{J}(\widetilde{\mathbb{N}}^*)$ *and* $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_v(\widetilde{\mathbb{N}}^*))/\mathcal{J}(\widetilde{\mathbb{N}}^*)$ *are not topological semigroups.*

P r o o f. We consider only the case $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))/\mathcal{J}(\widetilde{\mathbb{N}}^*)$. The proof of the statement for the semigroup $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_v(\widetilde{\mathbb{N}}^*))/\mathcal{J}(\widetilde{\mathbb{N}}^*)$ is similar.

At first we determine a base of the topology of the quotient space $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))/\mathcal{J}(\widetilde{\mathbb{N}}^*)$. Since for any $n\in\mathbb{N}$ the point $\frac{1}{n}$ is isolated in the topological space $(\mathbb{N}^*,\tau)$, Proposition 2.4.3 [4] implies that any non-zero element of the semigroup $(B_\lambda^0(\widetilde{\mathbb{N}}^*),\boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*))/\mathcal{J}(\widetilde{\mathbb{N}}^*)$ is isolated.

By $\Phi_{I_\lambda,\mathbb{N}}$ we denote the set of maps from I_λ into the set of positive integers $\mathbb{N}$. Let $n \in \mathbb{N}$ and let $U_n(0)$ be the set defined in Example 1. We define $M_n = \mathbb{N} \setminus U_n(0)$ for any $n \in \mathbb{N}$. For all $\varphi_1, \dots, \varphi_k \in \Phi_{I_\lambda,\mathbb{N}}$ and all $\alpha_1, \dots, \alpha_k \in I_\lambda$, $k \in \mathbb{N}$ we put $M^{\alpha_1,\dots,\alpha_k}_{\varphi_1,\dots,\varphi_k} = B^0_\lambda(\widetilde{\mathbb{N}}^*) \setminus \left(\bigcup_{i=1}^{k} \bigcup_{\gamma \in I_\lambda} (\gamma, M_{\varphi_i(\gamma)}, \alpha_i) \right)$. Proposition 2.4.3 [4] implies that the family $\widetilde{\mathcal{B}}(0) = \{ M^{\alpha_1,\dots,\alpha_k}_{\varphi_1,\dots,\varphi_k} \mid \alpha_1, \dots, \alpha_k \in I_\lambda,\ \varphi_1, \dots, \varphi_k \in \Phi_{I_\lambda,\mathbb{N}},\ k \in \mathbb{N} \}$ is a base at zero of the topology of the space $(B^0_\lambda(\widetilde{\mathbb{N}}^*), \boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*)) / \mathcal{J}(\widetilde{\mathbb{N}}^*)$.

Since the topological semigroup $(\mathbb{N}^*, \boldsymbol{\tau})$ is not compact, there exists $n_0 \in \mathbb{N}$ such that $\mathbb{N}^* U_k(0) = U_k(0) \mathbb{N}^* \nsubseteq U_{n_0}(0)$ for all $k \in \mathbb{N}$. We consider $\psi_0 \in \Phi_{I_\lambda,\mathbb{N}}$ such that $\psi_0(\gamma) = n_0$ for all $\gamma \in I_\lambda$ and fix $\alpha_0 \in I_\lambda$. Obviously, $M^{\alpha_0}_{\psi_0} \in \widetilde{\mathcal{B}}(0)$. We remark that $(\gamma, \mathbb{N}, \beta)(\beta, U_{\varphi(\beta)}(0) \setminus \{0\}, \alpha) = (\gamma, \mathbb{N}(U_{\varphi(\beta)}(0) \setminus \{0\}), \alpha) \nsubseteq (\gamma, U_{n_0}(0), \alpha)$ for all $\alpha, \beta, \gamma \in I_\lambda$ and all $\varphi \in \Phi_{I_\lambda,\mathbb{N}}$. Therefore, for every $\alpha_1, \dots, \alpha_j \in I_\lambda$ and every $\varphi_1, \dots, \varphi_j \in \Phi_{I_\lambda,\mathbb{N}}$, where $j \in \mathbb{N}$, we have $M^{\alpha_1,\dots,\alpha_j}_{\varphi_1,\dots,\varphi_j} M^{\alpha_1,\dots,\alpha_j}_{\varphi_1,\dots,\varphi_j} \nsubseteq M^{\alpha_0}_{\psi_0}$, and hence the semigroup operation in the semigroup $(B^0_\lambda(\widetilde{\mathbb{N}}^*), \boldsymbol{\tau}_h(\widetilde{\mathbb{N}}^*)) / \mathcal{J}(\widetilde{\mathbb{N}}^*)$ is discontinuous at zero. ◊

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ПРО λ^0-РОЗШИРЕННЯ БРАНДТА НАПІВГРУП З НУЛЕМ

Вводиться λ^0-розширення Брандта $B_\lambda^0(S)$ напівгрупи S з нулем і встановлено деякі алгебраїчні властивості напівгрупи S, які зберігаються напівгрупою $B_\lambda^0(S)$. Також введено топологічне λ^0- розширення Брандта топологічної напівгрупи S з нулем і встановлено його топологічні властивості в залежності від топологічної напівгрупи S. Зокрема, доведено, що топологічне λ^0-розширення Брандта (абсолютно) H-замкненої топологічної інверсної напівгрупи S є (абсолютно) H-замкненою напівгрупою у класі топологічних інверсних напівгруп. Побудовано топології на напівгрупі $B_\lambda^0(S)$, які зберігають абсолютну H-замкненість і H-замкненість. За допомогою топологічного λ^0-розширення Брандта побудовано приклад абсолютно H-замкненої метризовної інверсної топологічної напівгрупи S з абсолютно H-замкненим ідеалом I такої, що фактор-напівгрупа Ріса S/I не є топологічною напівгрупою.

О λ^0-РАСШИРЕНИЯХ БРАНДТА ПОЛУГРУПП С НУЛЁМ

Вводится λ^0-расширение Брандта $B_\lambda^0(S)$ полугруппы S с нулём и установлены некоторые алгебраические свойства полугруппы S, которые сохраняются полугруппой $B_\lambda^0(S)$. Также введено топологическое λ^0-расширение Брандта топологической полугруппы S с нулём и установлено его топологические свойства в зависимости от топологической полугруппы S. В частности, доказано, что топологическое λ^0-расширение Брандта (абсолютно) H-замкнутой топологической инверсной полугруппы S есть (абсолютно) H-замкнутая полугруппа в классе топологических инверсных полугрупп. Построено топологии на $B_\lambda^0(S)$, которые сохраняют абсолютную H-замкнутость и H- замкнутость. С помощью конструкции топологического λ^0-расширения Брандта построен пример абсолютно H-замкнутой метризуемой инверсной топологической полугруппы S с абсолютно H-замкнутым идеалом I такой, что фактор-полугруппа Рисса S/I не является топологической полугруппой.

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