UDK 539.3

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RECIPROCITY THEOREM FOR MECHANICAL PROBLEM IN BRITTLE DAMAGED BODY WITH THERMAL DISTORTION

The initial-boundary problem of mechanics is formulated in the paper in an incremental version for a viscoelastic-brittle damaged medium with thermal distortion. Next, the reciprocity theorem is derived for the stated problem. A way of calculation of the global damage parameter for the body is formulated on the basis of a special case of the theorem. The problem is also illustrated by the numerical example.

1. Introduction. The reciprocity theorem for a brittle damaged body with thermal distortion is formulated in the paper. The considerations are made under the following assumptions:

– damage in the body is considered as a continuous field described by the damage tensor [9] of rank two;
– a field of temperature in the body is treated as a known function;
– the undamaged body is treated as a linear viscoelastic one.

Further, the special case of the theorem is presented, which allows for a definition of the global scalar damage parameter of the body. This approach makes possible to connect a local damage evolution in engineering building structure with an averaging description of its global stiffness change. This is a very important problem in a diagnostics of building structures. Finally, a numerical example is discussed, which shows a relation between a scalar-global description of damage in concrete and a local-tensor one.

2. Equations of the problem – general form of the reciprocity theorem. Let us consider a body, which is isotropic in the initial moment and occupies the area V restricted by the surface F (Fig. 1). The body is subjected to an action of mass forces ρF_i, a known increment of temperature θ in the area V and static external mechanical load P_i. The body has brittle properties so the microcracks evolution in structure of the body is taken under consideration. The damage evolution causes an anisotropic stiffness change of the body. The unknown of the problem – displacement, strain and stress fields – u_i, ε_{ij} and σ_{ij} – must be determined from the following system of equations: equation of equilibrium, geometrical equation and physical equation (they are given in an incremental form because of the physical non-linearity of the problem):

$$\Delta\sigma_{ij,j} + \rho\Delta F_i = 0\,, \tag{1}$$

$$\Delta\varepsilon_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i})\,, \tag{2}$$

$$\Delta\sigma_{ij} = {}^{t}C_{ijkl}(t) * d(\Delta\varepsilon_{kl} - \Delta\varepsilon^{\theta}_{kl}), \qquad C_{ijkl}(t) = E_{ijkl}(t) - C^{*}_{ijkl}(t)\,,$$

$$E_{ijkl}(t) = 2\mu(t)\delta_{ik}\delta_{jl} + \lambda(t)\delta_{ij}\delta_{kl}\,, \qquad \varepsilon^{\theta}_{ij} = \alpha^{\theta}\delta_{ij}\theta, \qquad \theta = T - T_0\,. \tag{3}$$

In the equations above the symbols $C_{ijkl}(t)$, $E_{ijkl}(t)$, $\mu(t)$, $\lambda(t)$, δ_{ij}, T, T_0, α^{θ}, $\varepsilon^{\theta}_{ij}$, t, $\Delta\ldots$, ${}^{t}\ldots$ denote respectively: anisotropic and isotropic relaxation functions tensor, relaxation functions, Kronecker delta, current and initial temperature, coefficient of thermal expansion, thermal strain tensor, time, increment and tangent of a function. The damage evolution in material is taken into account here by an introduction of the anisotropic relaxation functions tensor C^*_{ijkl} [4]. Components of this tensor are equal to zero in the initial moment and depend on the damage measure – the damage effect tensor D_{ij} [6]

$$C^*_{ijkl}(x_i, t=0) = 0, \qquad C^*_{ijkl} = C^*_{ijkl}(D_{mn}). \tag{4}$$

The damage measure must be determined form the damage evolution equation [1–3, 6–9, 11] formulated so as to be a time non-decreasing function because of the thermomechanical limitations [1, 3, 8, 11]

$$D_{ij} = D_{ij}(\sigma_{ij}) \quad \text{or} \quad \dot{D}_{ij} = \dot{D}_{ij}(\sigma_{ij}) \quad \text{and} \quad \dot{D}_{ij} \geq 0. \tag{5}$$

The presented system of equations must be complemented by the following initial-boundary conditions:

$$u_i(x_i, t) = u_i^0(x_i, t), \qquad x_i \in V, \qquad t = 0, \tag{6}$$

$$\Delta\sigma_{ij} n_j = \Delta P_i(x_i, t), \qquad x_i \in F_\sigma, \qquad t \geq 0, \tag{7}$$

$$\Delta u_i(x_i, t) = \Delta\tilde{u}_i(x_i, t), \qquad x_i \in F_u, \qquad t \geq 0, \tag{8}$$

where $F_u \cup F_\sigma = F$, $F_u \cap F_\sigma = \varnothing$, n_j – normal vector to the surface F.

Now, we can give a derivation of the reciprocity theorem for the stated problem. Let us consider two sets of increments of the following fields satisfying the equations (1)–(3) and the initial-boundary conditions (4), (6)–(8):

$$\text{Set 1:} \quad \Delta u_i,\ \Delta\varepsilon_{ij},\ \Delta\varepsilon^\theta_{ij},\ \Delta\sigma_{ij},\ \Delta\rho F_i,\ \Delta P_i \to {}^t C_{ijkl}(t), \tag{9}$$

$$\text{Set 2:} \quad \Delta\hat{u}_i,\ \Delta\hat{\varepsilon}_{ij},\ \Delta\hat{\varepsilon}^\theta_{ij},\ \Delta\hat{\sigma}_{ij},\ \Delta\rho\hat{F}_i,\ \Delta\hat{P}_i \to {}^t\hat{C}_{ijkl}(t). \tag{10}$$

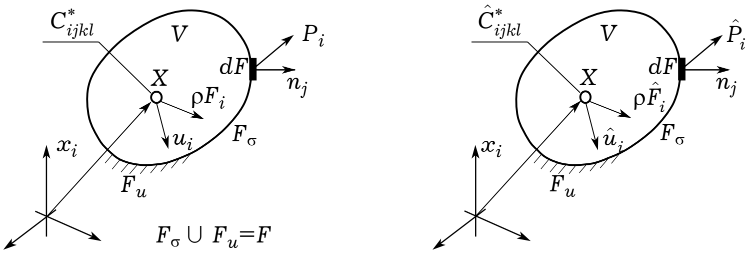

Fig. 1

Then, different tangent tensors ${}^tC_{ijkl}$ and ${}^t\hat{C}_{ijkl}$ are present in the physical equations for each of the sets. Finally, an analysis of a reciprocal symmetry of the physical equations

$$\Delta\sigma_{ij} * d\Delta\hat{\varepsilon}_{ij} = E_{ijkl}(t) * d(\Delta\varepsilon_{kl} - \Delta\varepsilon^\theta_{kl}) * d\Delta\hat{\varepsilon}_{ij} - {}^tC^*_{ijkl}(t) * d(\Delta\varepsilon_{kl} - \Delta\varepsilon^\theta_{kl}) * d\Delta\hat{\varepsilon}_{ij}, \tag{11}$$

$$\Delta\hat{\sigma}_{ij} * d\Delta\varepsilon_{ij} = E_{ijkl}(t) * d(\Delta\hat{\varepsilon}_{kl} - \Delta\hat{\varepsilon}^\theta_{kl}) * d\Delta\varepsilon_{ij} - {}^t\hat{C}^*_{ijkl}(t) * d(\Delta\hat{\varepsilon}_{kl} - \Delta\hat{\varepsilon}^\theta_{kl}) * d\Delta\varepsilon_{ij} \tag{12}$$

leads to the following identity

$$E_{ijkl}(t) * d\Delta\varepsilon_{kl} * d\Delta\hat{\varepsilon}_{ij} \equiv E_{ijkl}(t) * d\Delta\hat{\varepsilon}_{kl} * d\Delta\varepsilon_{ij}. \tag{13}$$

The expression (13) is a basis for a formulation of the reciprocity theorem in a local form

$$\Delta\sigma_{ij} * d\Delta\hat{\varepsilon}_{ij} - \Delta\hat{\sigma}_{ij} * d\Delta\varepsilon_{ij} + E_{ijkl}(t) * d\Delta\varepsilon^{\theta}_{kl} * d\Delta\hat{\varepsilon}_{kl} -$$
$$- E_{ijkl}(t) * d\Delta\hat{\varepsilon}^{\theta}_{kl} * d\Delta\varepsilon_{kl} + {}^{t}C^{*}_{ijkl}(t) * d(\Delta\varepsilon_{kl} - \Delta\varepsilon^{\theta}_{kl}) * d\Delta\hat{\varepsilon}_{ij} -$$
$$- {}^{t}\hat{C}^{*}_{ijkl}(t) * d(\Delta\hat{\varepsilon}_{kl} - \Delta\hat{\varepsilon}^{\theta}_{kl}) * d\Delta\varepsilon_{ij} = 0 \tag{14}$$

and a global form

$$\int_F (\Delta P_i * d\Delta\hat{u}_i - \Delta\hat{P}_i * d\Delta u_i) dF + \int_V (\Delta\rho F_i * d\Delta\hat{u}_i - \Delta\rho\hat{F}_i * d\Delta u_i) dV +$$
$$+ \int_V (E_{ijkl}(t) * d\Delta\varepsilon^{\theta}_{kl} * d\Delta\hat{\varepsilon}_{ij} - E_{ijkl}(t) * d\Delta\hat{\varepsilon}^{\theta}_{kl} * d\Delta\varepsilon_{ij}) dV +$$
$$+ \int_V ({}^{t}C^{*}_{ijkl} * d(\Delta\varepsilon_{kl} - \Delta\varepsilon^{\theta}_{kl}) * d\Delta\hat{\varepsilon}_{ij} -$$
$$- {}^{t}\hat{C}^{*}_{ijkl} * d(\Delta\hat{\varepsilon}_{kl} - \Delta\hat{\varepsilon}^{\theta}_{kl}) * d\Delta\varepsilon_{ij}) dV = 0 . \tag{15}$$

3. A particular case of the theorem – definition of the global damage parameter. A particular case of the theorem makes possible to get a formula describing a global stiffness change of the body. Let us consider a situation, in which the damage filed takes place in the first set of the fields (9)–(10) and doesn't take place in the second one. In order to facilitate the problem we can analyse here an elastic-brittle case, in which increments of external mechanical load and mass forces are neglected and increments of thermal strains are the same in both of the sets

$$\hat{C}^{*}_{ijkl} = 0, \qquad \rho F_i = \rho\hat{F}_i = P_i = \hat{P}_i = 0, \quad \Delta\varepsilon^{\theta}_{ij} = \Delta\hat{\varepsilon}^{\theta}_{ij},$$
$$C^{*}_{ijkl}(t) = C^{*}_{ijkl}H(t), \qquad E_{ijkl}(t) = E_{ijkl}H(t), \tag{16}$$

where $H(t)$ – Hevyside's function.

In this case, we are able to obtain an expression

$$\int_V (E_{ijkl}(\Delta\hat{\varepsilon}_{ij} - \Delta\varepsilon_{ij})\Delta\varepsilon^{\theta}_{kl}) dV + \int_V ({}^{t}C^{*}_{ijkl}(\Delta\varepsilon_{kl} - \Delta\varepsilon^{\theta}_{kl})\Delta\hat{\varepsilon}_{ij}) dV = 0 . \tag{17}$$

Then, it is possible to introduce into considerations the global damage parameter Ω on the basis of the definition

$$\Omega = \frac{\int_V (E_{ijkl}(\Delta\varepsilon_{ij} - \Delta\hat{\varepsilon}_{ij})\Delta\varepsilon^{\theta}_{kl}) dV}{\int_V (E_{ijkl}(\Delta\varepsilon_{kl} - \Delta\varepsilon^{\theta}_{kl})\Delta\hat{\varepsilon}_{ij}) dV} . \tag{18}$$

4. Numerical example and conclusions. Simulations of the damage evolution in concrete based on the local and global approaches were compared in the example. The global approach was based on the formula (18) defining the damage parameter Ω. The local approach used a definition of the damage tensor Ω_{ij} formulated by Litewka [7] with taking into account the limitation (5)

$$\Omega_{ij} = C s^{+}_{kl} s^{+}_{kl} \delta_{ij} + D\sqrt{\sigma^{+}_{kl}\sigma^{+}_{kl}}\, \sigma^{+}_{ij}, \qquad \Omega_{ij} \in [0,1), \tag{19a}$$

$$x^{+} = \frac{1}{2}(x + |x|) - \frac{1}{2}\zeta(-x + |-x|), \qquad \zeta \in [0,1), \tag{19b}$$

$$\Delta\Omega_{ij} > 0, \text{ when } \Delta\sigma_{ij} \geq 0, \ \sigma_{ij} > 0 \text{ or } \Delta\sigma_{ij} \leq 0, \ \sigma_{ij} < 0, \tag{19c}$$

$$\Delta\Omega_{ij} = 0, \text{ when } \Delta\sigma_{ij} \leq 0, \ \sigma_{ij} > 0 \text{ or } \Delta\sigma_{ij} \geq 0, \ \sigma_{ij} < 0. \tag{19d}$$

Performance of the operation $(...)^{+}$ according to the notation (19*b*) means here a transformation of the stress tensor σ_{ij} and the stress deviator s_{ij} to its principal directions and a reduction of their principal negative components proportionally to the constant ζ [8]. This operation is introduced because of the fact the microcracks evolution takes place mainly in the planes which are perpendicular to directions of the principal tensile stresses. To simplify the considerations, one analysed a case of plane stress in a rectangular area with a gap according to the Fig. 2.

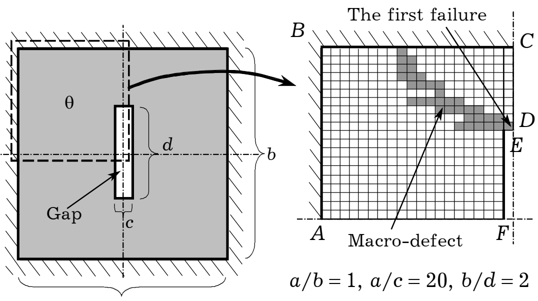

Fig. 2

The system was subjected to cyclic heating and cooling uniformly in a whole volume with a run shown on the Fig. 3. One used in the problem the following physical relation formulated by Litewka [6]

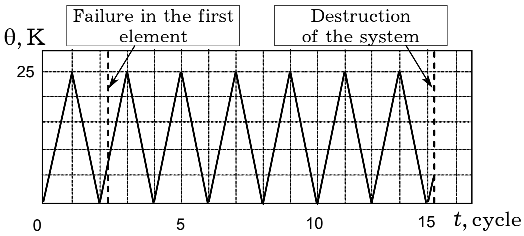

Fig. 3

$$\varepsilon_{ij} = A_{ijkl}(D_{mn})\sigma_{kl} + \varepsilon^{\theta}_{ij},$$

$$A_{ijkl} = -\frac{\nu_0}{E_0}\delta_{ij}\delta_{kl} + \frac{1+\nu_0}{2E_0}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) +$$

$$+ \alpha(\delta_{ij}D_{kl} + D_{ij}\delta_{kl}) + \gamma(\delta_{ik}D_{jl} + \delta_{il}D_{jk} + \delta_{jk}D_{il} + \delta_{jl}D_{ik}),$$

$$\mathbf{C} = \mathbf{A}^{-1}, \tag{20}$$

$$D_p = \frac{\Omega_p}{1-\Omega_p}, \qquad p = 1,2,3. \tag{21}$$

The components of the damage effect tensor D_{ij} in the equation (20) were determined here on the basis of the relations (21) between principal values of the tensors Ω_{ij} and D_{ij}. A rest of conditions necessary for a definition of the example was formulated according to the notations (4) and (16).

So defined system is double symmetric for which the stress, strain and damage tensors have a form

$$\sigma_{ij} = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad \varepsilon_{ij} = \begin{Vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{Vmatrix}, \quad \Omega_{ij} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 \\ \Omega_{21} & \Omega_{22} & 0 \\ 0 & 0 & \Omega_{33} \end{bmatrix}. \tag{22}$$

In the equations above the symbols $\mathbf{C}$, $\mathbf{A}$, E_0, ν_0, α, γ, C, D, ζ denote respectively: stiffness and compliance matrixes, Young's modulus, Poisson's ratio, material parameters expressing an influence of stresses on the damage evolution in concrete.

The presented example was solved with the help of own computer-program written in the Matlab environment. Computations were based on the incremental formulation of FEM. An analysis of the damage evolution required in this case taking into consideration the failure criterions for concrete. Two criterions complementing themselves mutually were used here:

- Kupfer's failure criterion [5];
- criterion of positively definite tangent stiffness matrix $\mathbf{C}$ [11].

Satisfying the mentioned criterions meant a formation of macro-defect in material. This fact was simulated in the numerical procedure by a reduction of stiffness in an adequate finite element to zero [11]. Computations were ended in the moment, when the system became a mechanism or it had to be represented by the other initial-boundary problem.

During solving the problem the following material parameters for concrete C30 were used [7, 8, 10]:

$$E_0 = 30800\,\text{MPa}, \quad \nu_0 = 0.19, \quad \zeta = 0.1, \quad \alpha = -1.457\cdot 10^{-6}\,\text{MPa}^{-1},$$

$$\gamma = 6.205\cdot 10^{-6}\,\text{MPa}^{-1}, \qquad C/\zeta^2 = 1.845\cdot 10^{-3}\,\text{MPa}^{-2},$$

$$D/\zeta^2 = 2.979\cdot 10^{-4}\,\text{MPa}^{-2}, \qquad \alpha^{\theta} = 10^{-5}\,\text{K}^{-1},$$

$$f_{\text{ctm}} = 2.22\ \text{MPa}, \qquad f_{\text{cm}} = 28.14\ \text{MPa}, \tag{23}$$

where f_{ctm} – tensile strength, f_{cm} – compression strength.

On the basis of numerical simulations it was found that during the following cycles of heating and cooling the failure criterion was satisfied in the elements starting form the top of the gap $D{-}E{-}F$ (see Fig. 2). A macro-defect was made this way. The defect grew stable and slantly in comparison with a direction of the gap. This situation was caused by a concentration of stresses (see Fig. 4) and microdamage (see Fig. 5) in that area of the system. The cumulation of microdamage at the top of the gap caused that tangent stiffness matrix in the elements placed there stopped satisfying the criterion of positive definition.

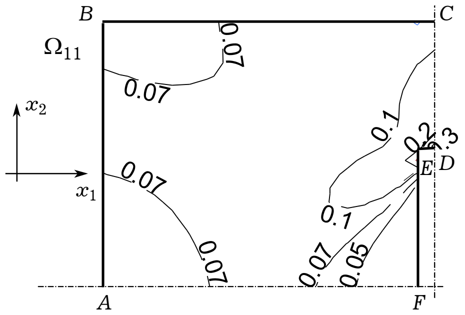

Fig. 4

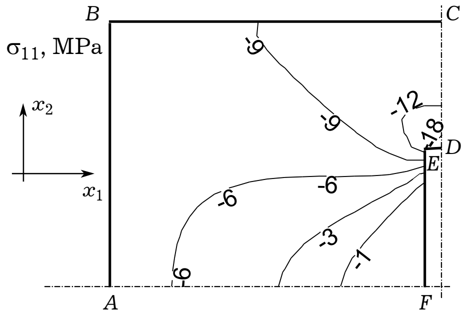

Fig. 5

After a solution of the brittle-elastic problem presented above one solved an equivalent elastic problem – (also with help of FEM). The comparison of these two solutions made possible to compute a run of the global damage parameter Ω according to the formula (18). It was found that just before the first failure detected in the element at the top of the gap, the parameter had been equal about to 0.1–0.2. A full run of this parameter till first failure is shown on the Fig. 6.

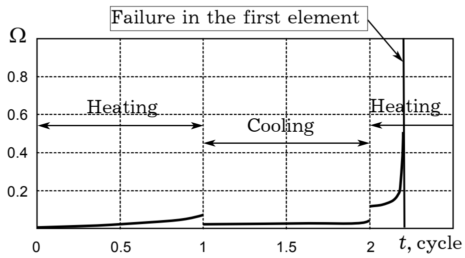

Fig. 6

The obtained results give a conclusion that a prediction of damage of engineering structure with help of global measures should be used with some limitations. On the other hand, it is very useful and it simplifies the considerations but the maximal allowable value of the global damage parameter should be restricted. The presented approach give also a chance for detecting damaged areas in structures on the basis of thermal strain measurements.

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ТЕОРЕМА ВЗАЄМНОСТІ ЗАДАЧІ МЕХАНІКИ ДЛЯ КРИХКО ПОШКОДЖЕНОГО ТІЛА З ТЕРМІЧНИМИ ДИСТОРСІЯМИ

У термінах приростів розглянуто крайову задачу механіки для в'язкопружного тіла з крихкими пошкодженнями й термічними дисторсіями. Для таких тіл сформульовано теорему взаємності робіт. На основі часткового випадку цього твердження отримано формулу, яка дозволяє обчислювати глобальний параметр пошкодження. Наведено числовий приклад розвитку мікропошкоджень в прямокутнику з центральним макродефектом за циклічного нагрівання.

ТЕОРЕМА ВЗАИМНОСТИ ЗАДАЧИ МЕХАНИКИ ДЛЯ ХРУПКО ПОВРЕЖДЕННОГО ТЕЛА С ТЕРМИЧЕСКИМИ ДИСТОРСИЯМИ

В терминах приращений рассмотрена краевая задача механики для вязкоупругого тела с хрупкими повреждениями и термическими дисторсиями. Для таких тел сформулирована теорема взаимности работ. На основе частного случая этого утверждения получена формула, позволяющая определить глобальный параметр повреждения. Приведен числовой пример развития микроповреждений в прямоугольнике с центральным макродефектом при циклическом нагревании.

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