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A PRIORI ESTIMATES FOR PERIODIC LINEAR ELLIPTIC FIRST-ORDER SYSTEMS

We prove the uniqueness and the continuous dependence results for initial-boundary periodic problem for the first-order semi-linear elliptic systems in the whole scale of Sobolev spaces of periodic functions. We derive a priori estimates with respect to the spaces of solutions and right-hand sides ensuring the optimal regularity trade-off for our problem.

Introduction. We investigate a periodic-Dirichlet problem for a first-order elliptic system with discontinuous coefficients. Specifically, in the domain $\{(x,t) \,|\, 0 < x < 1, \ -\infty < t < \infty\}$, we consider system

$$\partial_t u + \lambda \partial_x u + a(x)u + b(x)v = f(x,t),$$

$$\partial_t v - \mu \partial_x v + c(x)u + d(x)v = g(x,t), \tag{1}$$

subjected to periodic conditions

$$u(x, t+T) = u(x,t),$$

$$v(x, t+T) = v(x,t) \tag{2}$$

and reflection boundary conditions

$$u(0,t) = r_0 v(0,t),$$

$$v(1,t) = r_1 u(1,t). \tag{3}$$

The unknown functions u and v and all the data in (1) are complex functions; λ, μ, r_0 and r_1 are complex constants, and $T > 0$. The functions f and g are assumed to be T-periodic in t and $a, b, c, d \in L^\infty(0,1)$.

Our goal is to investigate uniqueness and continuous dependence of solutions, where we deal with spaces of solutions V^γ and spaces of right hand sides W^γ in (1) providing us with an optimal regularity trade-off of the following kind. From one side, for all $(u,v) \in V^\gamma$ the left-hand side of (1) belongs to W^γ and, from the other side, for all $(f,g) \in W^\gamma$ solutions to (1)–(3) belong to V^γ. Similar results for the first-order hyperbolic system of kind (1) with $\lambda = \mu \equiv 1$ subjected to conditions (2) and (3) is obtained in [2, 3]. It contrasts to our elliptic case in the sense that the optimal regularity trade-off for the hyperbolic problem is attained in the pair of spaces $(\tilde{V}^\gamma, W^\gamma)$ where V^γ is a subspace of $\tilde{V}^\gamma$ which does not coincide with $\tilde{V}^\gamma$. This means that elliptic operators are «more regular» than hyperbolic operators, namely, that they improve the regularity of the right hand side (f,g) better than the hyperbolic operators do this. An explanation of this phenomenon is that hyperbolic operators give «non-uniform» regularity in all directions on the real plane: In the so-called characteristic directions singularities of the derivatives of the unknown functions cancel out each other, while in all other directions they do not.

In the present paper we derive a priori estimates in the whole scale of Sobolev spaces of periodic functions and, as a consequence, obtain the uniqueness and the continuous dependence results.

2. Spaces of solutions and right-hand sides. We here introduce two scales of Banach spaces V^γ (for solutions) and W^γ (for right-hand sides in (1)) with a scale parameter $\gamma \in \mathbb{R}$, consisting of complex valued functions.

We will achieve the following properties:

- elements of V^γ satisfy (2) and have traces in x and elements of W^γ satisfy (2);
- elements of W^γ allow discontinuities in x;
- for any $\gamma \in \mathbb{R}$, the pair (V^γ, W^γ) gives an optimal regularity for (1)−(3).

We first introduce Sobolev spaces of T-periodic functions (see, e.g. [1, 4, 5]). Let $S^T = \mathbb{R}/T\mathbb{Z}$. Define the (Banach) space $C^k(S^T)$ of k-times continuously differentiable functions on S^T by

$$C^k(S^T) = \{f : S^T \to C \mid f \circ q \in C^k(\mathbb{R})\},$$

where q is the quotient map $q : \mathbb{R} \to \mathbb{R}/T\mathbb{Z}$. For the (Freshet) space of smooth functions we hence have

$$C^\infty(S^T) = \bigcap_{k\geq 1} C^k(S^T).$$

As a topological vector space this is a projective limit of $C^k(S^T)$ with projections being the natural inclusions. Now the space of distributions on S^T is defined as the ascending union (colimit) of duals of the spaces $C^k(S^T)$:

$$C^\infty(S^T)^* = \bigcup_k C^k(S^T)^* = \operatorname*{colim}_{k\to\infty} C^k(S^T)^*.$$

We are now prepared to define Sobolev spaces of periodic functions: Set $\omega = 2\pi/T$ and $\varphi_k(t) = e^{ik\omega t}$ and define

$$H^\gamma(S^T) = \Big\{u \in C^\infty(S^T)^* \mid \| u \|^2_{H^\gamma(S^T)} =$$

$$= T^{-1} \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \left|[u, \varphi_{-k}]_{C^\infty(S^T)}\right|^2 < \infty\Big\},$$

where $[\cdot,\cdot]_{C^\infty(S^T)} : C^\infty(S^T)^* \times C^\infty(S^T) \to \mathbb{C}$ is the dual pairing.

Given $\ell \in \mathbb{N}_0$, denote

$$H^{\ell,\gamma} = H^\ell(0,1; H^\gamma(S^T)) = \Big\{u(\cdot,t) : (0,1) \to$$

$$\to H^\gamma(S^T) \mid \| u \|^2_{H^{\ell,\gamma}} = \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \sum_{m=0}^{\ell} \int_0^1 \left|\frac{d^m}{dx^m} u_k(x)\right|^2 dx < \infty\Big\},$$

where

$$u_k(x) = T^{-1}[u(x,\cdot), \varphi_{-k}]_{C^\infty(S^T)}, \qquad k \in \mathbb{Z}, \tag{4}$$

denote the t-Fourier coefficients of $u \in H^{\ell,\gamma}$.

Finally, for each $\gamma \in \mathbb{R}$ we define the spaces W^γ and V^γ by

$$W^\gamma = H^{0,\gamma} \times H^{0,\gamma}$$

and

$$V^\gamma = W^{\gamma+1} \cap [H^{1,\gamma} \times H^{1,\gamma}].$$

These spaces will be endowed with norms

$$\| (u,v) \|^2_{W^\gamma} = \| u \|^2_{H^{0,\gamma}} + \| v \|^2_{H^{0,\gamma}}$$

and

$$\| (u,v) \|_{V^\gamma}^2 = \| u \|_{H^{0,\gamma+1}}^2 + \| v \|_{H^{0,\gamma+1}}^2 + \| u \|_{H^{1,\gamma}}^2 + \| v \|_{H^{1,\gamma}}^2 .$$

We now collect useful properties of the function spaces introduced above.

Lemma 1 [2]. *W^γ is a Hilbert space.*

Define a Euclidian space

$$E^\gamma = \left\{ (u_k(x))_{k\in\mathbb{Z}} \mid u_k(x) \in L^2(0,1) \quad \text{for each } k, \; \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \| u_k \|_{L^2(0,1)}^2 < \infty \right\}$$

with inner product

$$\left\langle (u_k)_k, (w_k)_k \right\rangle = \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \int_0^1 u_k(x) \overline{w_k(x)}\, dx .$$

Lemma 2 [2]. *E^γ is a Hilbert space.*

Lemma 3 [2]. *The map $u \to (u_k(x))_{k\in\mathbb{Z}}$ is a Hilbert space isomorphism from $H^{0,\gamma}$ onto E^γ.*

Corollary 1. *For any $u, v \in H^{\ell,\gamma}$ there exist unique sequences $(u_k)_{k\in\mathbb{Z}}$, $(v_k)_{k\in\mathbb{Z}}$ in $H^\ell(0,1)$ given by (4) such that the series*

$$\sum_{k\in\mathbb{Z}} u_k \varphi_k , \qquad \sum_{k\in\mathbb{Z}} v_k \varphi_k \tag{5}$$

converge, respectively, to u and v in $H^{\ell,\gamma}$. Vice versa, for any sequences $(u_k)_{k\in\mathbb{Z}}$, $(v_k)_{k\in\mathbb{Z}}$ in $H^\ell(0,1)$ such that

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \| u_k \|_{H^\ell(0,1)}^2 < \infty \quad \text{and} \quad \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \| v_k \|_{H^\ell(0,1)}^2 < \infty$$

there exist unique $u, v \in H^{\ell,\gamma}$ with u_k and v_k being their t-Fourier coefficients.

In what follows, we will identify distributions $u \in H^{\ell,\gamma}$ and sequences $(u_k(x))_{k\in\mathbb{Z}}$ in $H^\ell(0,1)$ corresponding to these distributions by Corollary 1.

Lemma 4. *Let $(u,v) \in V^\gamma$. Then for any $x \in [0,1]$ the traces $u(x,\cdot)$ and $v(x,\cdot)$ are distributions in $H^\gamma(S^T)$ and satisfy the estimate*

$$\| (u(x,\cdot), v(x,\cdot)) \|_{[H^\gamma(S^T)]^2}^2 \le C \| (u,v) \|_{V^\gamma}^2 ,$$

where C does not depend on x, u, and v.

P r o o f. The corollary follows from the continuous embedding

$$V^\gamma \hookrightarrow H^1(0,1;H^\gamma(S^T)) \hookrightarrow C(0,1;H^\gamma(S^T)) . \qquad \Diamond$$

3. A priori estimates. We here give conditions ensuring the uniqueness and the continuous dependence of generalized solutions to (1)–(3).

Definition 1. A function $(u,v) \in V^\gamma$ is called a *strong generalized solution* to the problem (1)–(3) if it satisfies (1) in $H^{0,\gamma}$ and (3) in $H^\gamma(S^T)$.

Assume that

$$\operatorname{Re}\lambda > 0, \qquad \operatorname{Im}\lambda > 0, \qquad \operatorname{Re}\mu > 0, \qquad \operatorname{Im}\mu > 0 \tag{6}$$

and

$$\operatorname{Re}\lambda = \operatorname{Re}\mu. \tag{7}$$

We can state and prove our result as well for some other signs of $\operatorname{Re}\lambda$, $\operatorname{Im}\lambda$, $\operatorname{Re}\mu$, $\operatorname{Im}\mu$, say, if these numbers are all negative. However, some restriction on these signs distribution is necessary, say, our argument does not work if the signs of $\operatorname{Re}\lambda \operatorname{Im}\lambda$ and $\operatorname{Re}\mu \operatorname{Im}\mu$ are different. What about the condition (7), it can be removed at all and imposed only to simplify technicalities.

Let δ_0 be an arbitrary fixed real in the range $0 < \delta_0 < 1$ and set

$$p = \delta_0 - \log|r_1|, \qquad q = \log|r_0| - \delta_0. \tag{8}$$

Fix $N \in \mathbb{N}$ so large that the following estimates are fulfilled

$$\|\operatorname{Im} a\|_{L^\infty} + \|\operatorname{Im} d\|_{L^\infty} + \|b\|_{L^\infty} \max_{x\in[0,1]} \{1, e^{-px-q(1-x)}\} + \\ + \|c\|_{L^\infty} \max_{x\in[0,1]} \{1, e^{px+q(1-x)}\} < \frac{N\omega}{8}, \tag{9}$$

$$\|a\|_{L^\infty} (1+|\lambda||p-q|) + \|d\|_{L^\infty} + \|b\|_{L^\infty} \max_{x\in[0,1]} \{1, e^{-px-q(1-x)}\} + \\ + \|c\|_{L^\infty} \max_{x\in[0,1]} \{1, e^{px+q(1-x)}\} < N \operatorname{Re}\lambda, \tag{10}$$

$$\frac{(\operatorname{Re}\lambda + \operatorname{Im}\lambda + \operatorname{Im}\mu)^2 (\operatorname{Im}\lambda + \operatorname{Im}\mu + \omega)}{\operatorname{Re}\lambda \min\{\operatorname{Im}\lambda, \operatorname{Im}\mu\}} \Big(\|a\|_{L^\infty} (1+|\lambda||p-q|) + \\ + \|d\|_{L^\infty} + \|b\|_{L^\infty} \max_{x\in[0,1]} \{1, e^{-px-q(1-x)}\} + \\ + \|c\|_{L^\infty} \max_{x\in[0,1]} \{1, e^{px+q(1-x)}\} \Big) < \frac{N\omega^2}{64}. \tag{11}$$

To formulate the main result of this section, we will make the following assumption about the coefficients of the differential equations and the reflection coefficients r_0 and r_1: For all k in the range $|k| \le N$

$$(|\lambda^{-1}|\|b\|_{L^\infty} + |\mu^{-1}|\|c\|_{L^\infty}) [1 + (1+|r_0|)(1 + \\ + |r_1|)|\Delta_k^{-1}|] \exp\{2(|\lambda^{-1}| + |\mu^{-1}|)(1 + \|a\|_{L^\infty} + \|d\|_{L^\infty})\} < 1, \tag{12}$$

where

$$\alpha(x) = \int_0^x a(y)\,dy, \qquad \delta(x) = \int_0^x d(y)\,dy,$$

$$\Delta_k = e^{\mu^{-1}(ik\omega + \delta(1))} - r_0 r_1 e^{-\lambda^{-1}(ik\omega + \alpha(1))}.$$

Theorem 1. *Let $\gamma \in \mathbb{R}$ be a fixed real, $a, b, c, d \in L^\infty(0,1)$, and $(f, g) \in W^\gamma$. Assume that $|r_0| < 1$, $|r_1| < 1$, and the estimate* (12) *is fulfilled. If for all k in the range $|k| \le N$*

$$|r_0 r_1| \ne \exp\left\{ -k\omega(\lambda^{-1} + \mu^{-1}) + \int_0^1 [\operatorname{Re}(a\lambda^{-1}) + \operatorname{Re}(d\mu^{-1})]\,dx \right\}, \tag{13}$$

then every strong generalized solution to the problem (1)–(3) *satisfies the a priory estimate*

$$\|(u, v)\|_{V^\gamma} \le C \|(f, g)\|_{W^\gamma} \tag{14}$$

for some $C > 0$ not depending on (f, g).

P r o o f. Due to the assumptions imposed on the functions f and g, they allow the following series representations:

$$\sum_{k\in\mathbb{Z}} f_k\varphi_k, \qquad \sum_{k\in\mathbb{Z}} g_k\varphi_k, \tag{15}$$

where $f_k(x) = T^{-1}[f(x,\cdot),\varphi_{-k}]_{C^\infty(S^T)}$ and $g_k(x) = T^{-1}[g(x,\cdot),\varphi_{-k}]_{C^\infty(S^T)}$. Clearly, $f_k, g_k \in L^2(0,1)$,

$$\sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \| f_k(x)\|^2_{L^2(0,1)} < \infty, \qquad \sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \| g_k(x)\|^2_{L^2(0,1)} < \infty, \tag{16}$$

and the series (15) converge to f and g in $H^{0,\gamma}$. Assume that (u,v) is a strong generalized solution to the problem (1)–(3). Represent u and v as series (5). Hence u_k, v_k for each $k\in\mathbb{Z}$ are in $H^1(0,1)$ and satisfy the boundary value problem

$$\lambda u_k' = f_k(x) - (a(x) + ik\omega)u_k - b(x)v_k,$$

$$\mu v_k' = -g_k(x) + (d(x) + ik\omega)v_k + c(x)u_k, \tag{17}$$

$$u_k(0) = r_0 v_k(0), \qquad v_k(1) = r_1 u_k(1). \tag{18}$$

Our aim is to show that

$$\sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma+1}\left[\| u_k(x)\|^2_{L^2(0,1)} + \| v_k(x)\|^2_{L^2(0,1)}\right] < \infty, \tag{19}$$

$$\sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma}\left[\| u_k'(x)\|^2_{L^2(0,1)} + \| v_k'(x)\|^2_{L^2(0,1)}\right] < \infty. \tag{20}$$

The estimate (20) follows from (16), (17), and (19). It remains to prove (19).

We will distinguish three cases.

Case 1. *The estimate* (19) *is fulfilled with* $k \ge N$ *in place of* $k\in\mathbb{Z}$. Fix $k\ge N$. Multiplying the equations of the system (17) by $\bar{u}_k$ and $\bar{v}_k$, respectively, and then subtracting the resulting equalities and their complex conjugations, we arrive at the system

$$\operatorname{Re}\lambda\int_0^1(\bar{u}_k u_k' - u_k\bar{u}_k')\,dx + i\operatorname{Im}\lambda\int_0^1(\bar{u}_k u_k' + u_k\bar{u}_k')\,dx +$$
$$+ 2i\int_0^1(\operatorname{Im} a + k\omega)|u_k|^2\,dx = \int_0^1(f_k\bar{u}_k - \bar{f}_k u_k)\,dx -$$
$$- \int_0^1(b\bar{u}_k v_k - \bar{b}u_k\bar{v}_k)\,dx,$$

$$\operatorname{Re}\lambda\int_0^1(\bar{v}_k v_k' - v_k\bar{v}_k')\,dx + i\operatorname{Im}\mu\int_0^1(\bar{v}_k v_k' + v_k\bar{v}_k')\,dx -$$
$$- 2i\int_0^1(\operatorname{Im} d + k\omega)|v_k|^2\,dx = \int_0^1(\bar{g}_k v_k - g_k\bar{v}_k)\,dx +$$
$$+ \int_0^1(c\bar{v}_k u_k - \bar{c}v_k\bar{u}_k)\,dx. \tag{21}$$

Subtraction of the second equality of (21) from the first one and multiplication of the resulting equality by $\dfrac{-ik^{1+\gamma}}{\omega}$ yields

$$2k^{2+\gamma}\int_0^1 (|u_k|^2 + |v_k|^2)\,dx = \frac{k^{1+\gamma}}{\omega}\Bigg[-2\int_0^1 (\operatorname{Im} a\,|u_k|^2 + \operatorname{Im} d\,|v_k|^2)\,dx +$$

$$+ i\operatorname{Re}\lambda\int_0^1 (\bar{u}_k u_k' - u_k\bar{u}_k' + v_k\bar{v}_k' - \bar{v}_k v_k')\,dx -$$

$$- \operatorname{Im}\lambda\int_0^1 (\bar{u}_k u_k' + u_k\bar{u}_k')\,dx + \operatorname{Im}\mu\int_0^1 (v_k\bar{v}_k' + \bar{v}_k v_k')\,dx -$$

$$- i\int_0^1 (f_k\bar{u}_k - \bar{f}_k u_k - g_k\bar{v}_k + \bar{g}_k v_k)\,dx +$$

$$+ i\int_0^1 (b\bar{u}_k v_k - \bar{b}u_k\bar{v}_k + c\bar{v}_k u_k - \bar{c}v_k\bar{u}_k)\,dx\Bigg]. \tag{22}$$

We will make use of the following simple inequalities for $m, \varepsilon_0, K > 0$:

$$k^{1+\gamma}\int_0^1 \left|\operatorname{Im} a\,|u_k|^2 + \operatorname{Im} d\,|v_k|^2\right| dx \le$$

$$\le \frac{k^{2+\gamma}}{K}\left[\|\operatorname{Im} a\|_{L^\infty}\int_0^1 |u_k|^2\,dx + \|\operatorname{Im} d\|_{L^\infty}\int_0^1 |v_k|^2\,dx\right],$$

$$k^{1+\gamma}\int_0^1 |u_k'\bar{u}_k|\,dx \le \frac{k^\gamma}{m}\int_0^1 |u_k'|^2\,dx + k^{2+\gamma}m\int_0^1 |u_k|^2\,dx,$$

$$k^{1+\gamma}\int_0^1 |f_k\bar{u}_k|\,dx \le \frac{k^\gamma}{\varepsilon_0}\int_0^1 |f_k|^2\,dx + \varepsilon_0 k^{2+\gamma}\int_0^1 |u_k|^2\,dx,$$

$$k^{1+\gamma}\int_0^1 |b\bar{u}_k v_k|\,dx \le \|b\|_{L^\infty}\frac{k^{2+\gamma}}{2K}\left(\int_0^1 |u_k|^2\,dx + \int_0^1 |v_k|^2\,dx\right), \tag{23}$$

the latter inequality being true for all $k \ge K > 0$. We estimate all other integrals in the right-hand side of (22) similarly. Set

$$m = \frac{\omega}{8(\operatorname{Re}\lambda + \operatorname{Im}\lambda + \operatorname{Im}\mu)}.$$

We will consider $K \ge N$, where N was fixed to satisfy (9)–(11). Then, on the account of (9), (22), and (23), we are able to choose ε_0 so small that the following estimate is true:

$$\frac{3}{2}k^{2+\gamma}\int_0^1 (|u_k|^2 + |v_k|^2)\,dx \le$$

$$\le k^\gamma\,\frac{16(\operatorname{Re}\lambda + \operatorname{Im}\lambda + \operatorname{Im}\mu)^2}{\omega^2}\int_0^1 (|u_k'|^2 + |v_k'|^2)\,dx +$$

$$+ Ck^\gamma\int_0^1 (|f_k|^2 + |g_k|^2)\,dx, \tag{24}$$

where $C > 0$ is a constant which depends on ε_0 but neither on k nor on K.

To estimate the first summand in the right hand side of the latter inequality, we multiply equations of the system (17) by $\bar{u}_k$ and $\bar{v}_k$, respectively, and then sum up the resulting equalities with their complex conjugations. Combining this with the following equalities, obtained by integration by parts, and with the boundary conditions (18):

$$\int_0^1 (\bar{u}_k u_k' + u_k \bar{u}_k')\,dx = |u_k(1)|^2 - |r_0|^2\,|v_k(0)|^2,$$

$$\int_0^1 (\bar{v}_k v_k' + v_k \bar{v}_k')\,dx = |r_1|^2\,|u_k(1)|^2 - |v_k(0)|^2,$$

we get

$$\begin{aligned}
&\operatorname{Re}\lambda\big[(1-|r_1|^2)\,|u_k(1)|^2 + (1-|r_0|^2)\,|v_k(0)|^2\big] + \\
&\quad + 2\int_0^1 (\operatorname{Re} a\,|u_k|^2 + \operatorname{Re} d\,|v_k|^2)\,dx + i\operatorname{Im}\lambda\int_0^1 (\bar{u}_k u_k' - u_k \bar{u}_k')\,dx + \\
&\quad + i\operatorname{Im}\mu\int_0^1 (v_k \bar{v}_k' - \bar{v}_k v_k')\,dx = \int_0^1 (f_k \bar{u}_k + \bar{f}_k u_k)\,dx + \\
&\quad + \int_0^1 (g_k \bar{v}_k + \bar{g}_k v_k)\,dx - \int_0^1 (b\bar{u}_k v_k + \bar{b} u_k \bar{v}_k)\,dx - \\
&\quad - \int_0^1 (c\bar{v}_k u_k + \bar{c} v_k \bar{u}_k)\,dx. \qquad (25)
\end{aligned}$$

For the third and the fourth summands in the left hand side we will use the representation obtained from the equations of the system (17) multiplied by $\bar{u}_k'$ and $\bar{v}_k'$, respectively. More precisely, we make of use the following equalities:

$$\begin{aligned}
2\operatorname{Re}\lambda\int_0^1 |u_k'|^2\,dx &= \int_0^1 (f_k \bar{u}_k' + \bar{f}_k u_k')\,dx - \int_0^1 (a u_k \bar{u}_k' + \bar{a}\bar{u}_k u_k')\,dx - \\
&\quad - \int_0^1 (b v_k \bar{u}_k' + \bar{b}\bar{v}_k u_k')\,dx + ik\omega\int_0^1 (\bar{u}_k u_k' - u_k \bar{u}_k')\,dx,
\end{aligned}$$

$$\begin{aligned}
2\operatorname{Re}\lambda\int_0^1 |v_k'|^2\,dx &= -\int_0^1 (g_k \bar{v}_k' + \bar{g}_k v_k')\,dx + \int_0^1 (d v_k \bar{v}_k' + \bar{d}\bar{v}_k v_k')\,dx + \\
&\quad + \int_0^1 (c u_k \bar{v}_k' + \bar{c}\,\bar{u}_k v_k')\,dx + ik\omega\int_0^1 (v_k \bar{v}_k' - \bar{v}_k v_k')\,dx.
\end{aligned}$$

Combining this with (25), we arrive at

$$\begin{aligned}
&k^{1+\gamma}\operatorname{Re}\lambda\big[(1-|r_1|^2)\,|u_k(1)|^2 + (1-|r_0|^2)\,|v_k(0)|^2\big] + \\
&\quad + k^{\gamma}\,\frac{2\operatorname{Re}\lambda\operatorname{Im}\lambda}{\omega}\int_0^1 |u_k'|^2\,dx + k^{\gamma}\,\frac{2\operatorname{Re}\lambda\operatorname{Im}\mu}{\omega}\int_0^1 |v_k'|^2\,dx = \\
&\quad = -2k^{1+\gamma}\int_0^1 (\operatorname{Re} a\,|u_k|^2 + \operatorname{Re} d\,|v_k|^2)\,dx +
\end{aligned}$$

$$+ k^{1+\gamma}\int_0^1 (f_k\bar u_k + \bar f_k u_k + g_k\bar v_k + \bar g_k v_k)\,dx -$$

$$- k^{1+\gamma}\int_0^1 (b\bar u_k v_k + \bar b u_k\bar v_k + c\bar v_k u_k + \bar c v_k\bar u_k)\,dx -$$

$$- k^{\gamma}\frac{\operatorname{Im}\lambda}{\omega}\int_0^1 (-a\bar u_k u_k' - \bar a u_k\bar u_k' + b v_k\bar u_k' + \bar b\,\bar v_k u_k')\,dx +$$

$$+ k^{\gamma}\frac{\operatorname{Im}\mu}{\omega}\int_0^1 (d v_k\bar v_k' + \bar d\bar v_k v_k' + c u_k\bar v_k' + \bar c\bar u_k v_k')\,dx -$$

$$- k^{\gamma}\frac{\operatorname{Im}\lambda}{\omega}\int_0^1 (f_k\bar u_k' + \bar f_k u_k')\,dx + k^{\gamma}\frac{\operatorname{Im}\mu}{\omega}\int_0^1 (g_k\bar v_k' + \bar g_k v_k')\,dx. \quad (26)$$

Since $|r_0| < 1$ and $|r_1| < 1$, the sum of the boundary terms is positive. We will drop them in the subsequent estimates. In addition to the above simple inequalities (23) (but now with a new ε) we will use the following estimates. Given $\varepsilon_0 > 0$ and $K > 0$, for all $k \ge K$ we have

$$\int_0^1 |f_k\bar u_k'|\,dx \le \frac{1}{\varepsilon_0}\int_0^1 |f_k|^2\,dx + \varepsilon_0\int_0^1 |\bar u_k'|^2\,dx,$$

$$k^{\gamma}\int_0^1 |a\bar u_k u_k'|\,dx \le \|a\|_{L^\infty}\left[\frac{k^{\gamma}}{2K}\int_0^1 |u_k'|^2\,dx + \frac{k^{\gamma+2}}{2K}\int_0^1 |u_k|^2\,dx\right].$$

If $K \ge N$, then, on the account of (10), the equality (25) yields

$$k^{\gamma}\int_0^1 \left[|u_k'|^2 + |v_k'|^2\right]dx \le \frac{2k^{\gamma+2}(\operatorname{Im}\lambda + \operatorname{Im}\mu + \omega)}{K\operatorname{Re}\lambda\min\{\operatorname{Im}\lambda, \operatorname{Im}\mu\}}\Big[\|b\|_{L^\infty} +$$

$$+ \|c\|_{L^\infty} + \|a\|_{L^\infty} + \|d\|_{L^\infty} + K\varepsilon_0\Big]\int_0^1 (|u_k|^2 + |v_k|^2)\,dx +$$

$$+ C k^{\gamma}\int_0^1 (|f_k|^2 + |g_k|^2)\,dx, \quad (27)$$

which is true for some $C > 0$ depending on ε_0 but neither on k nor on K. As above, we consider $K \ge N$. We are now able to choose ε_0 so small that the claim is a consequence of (11), (24), and (27). Therewith we are done.

Case 2. *The estimate* (19) *is fulfilled with* $k \le -N$ *in place of* $k \in \mathbb{Z}$. We start from the observation that (u_k, v_k) is a solution to the problem (17), (18) if (w_k, v_k), where $e^{px+q(1-x)}w_k = u_k$ and $p, q \in \mathbb{R}$ are fixed reals, is a solution to the problem (for negative k)

$$\lambda w_k' = e^{-px-q(1-x)}f_k(x) - \big(a(x) - i|k|\omega + \lambda(p-q)\big)w_k - e^{-px-q(1-x)}b(x)v_k,$$

$$\mu v_k' = -g_k(x) + \big(d(x) - i|k|\omega\big)v_k + e^{px+q(1-x)}c(x)w_k, \quad (28)$$

$$e^q w_k(0) = r_0 v_k(0),$$

$$v_k(1) = r_1 e^p w_k(1). \quad (29)$$

The proof of the claim follows the same scheme as the proof of Claim 1. We indicate only the differences. Let us write down analogues of the equalities (22) and (26) with respect to functions w_k and v_k satisfying (28), (29):

$$2|k|^{2+\gamma}\int_0^1(|w_k|^2+|v_k|^2)\,dx=\frac{|k|^{1+\gamma}}{\omega}\Bigg|-2\int_0^1(\operatorname{Im}a|w_k|^2+\operatorname{Im}d|v_k|^2)\,dx+$$

$$+i\operatorname{Re}\lambda\int_0^1(\bar w_k w_k'-w_k\bar w_k'+v_k\bar v_k'-\bar v_k v_k')\,dx-$$

$$-\operatorname{Im}\lambda\int_0^1(\bar w_k w_k'+w_k\bar w_k')\,dx+\operatorname{Im}\mu\int_0^1(v_k\bar v_k'+\bar v_k v_k')\,dx-$$

$$-i\int_0^1(f_k\bar w_k-\bar f_k w_k)\,e^{-px-q(1-x)}\,dx-i\int_0^1(g_k\bar v_k+\bar g_k v_k)\,dx+$$

$$+i\int_0^1(b\bar w_k v_k-\bar b w_k\bar v_k)\,e^{-px-q(1-x)}\,dx+$$

$$+\int_0^1(c\bar v_k w_k-\bar c v_k\bar w_k)\,e^{px+q(1-x)}\,dx\Bigg|$$

and

$$|k|^{1+\gamma}\operatorname{Re}\lambda\big[(1-|r_1|^2e^{2p})|w_k(1)|^2+(1-|r_0|^2e^{-2q})|v_k(0)|^2\big]-$$

$$-|k|^{\gamma}\frac{2\operatorname{Re}\lambda\operatorname{Im}\lambda}{\omega}\int_0^1|w_k'|^2\,dx-|k|^{\gamma}\frac{2\operatorname{Re}\lambda\operatorname{Im}\mu}{\omega}\int_0^1|v_k'|^2\,dx=$$

$$=-2|k|^{1+\gamma}\int_0^1\big[(\operatorname{Re}a+\operatorname{Re}\lambda(p-q))|w_k|^2+\operatorname{Re}d|v_k|^2\big]\,dx+$$

$$+|k|^{1+\gamma}\int_0^1\big[(f_k\bar w_k+\bar f_k w_k)\,e^{-px-q(1-x)}+g_k\bar v_k+\bar g_k v_k\big]\,dx-$$

$$-|k|^{1+\gamma}\int_0^1\big[(b\bar w_k v_k+\bar b w_k\bar v_k)\,e^{-px-q(1-x)}+$$

$$+(c\bar v_k w_k+\bar c v_k\bar w_k)\,e^{px+q(1-x)}\big]\,dx-$$

$$-|k|^{\gamma}\frac{\operatorname{Im}\lambda}{\omega}\int_0^1\big(-(a+\lambda(p-q))\,\bar w_k w_k'-(\bar a+\bar\lambda(p-q))\,w_k\bar w_k'+$$

$$+(bv_k\bar w_k'+\bar b\,\bar v_k w_k')\,e^{-px-q(1-x)}\big)\,dx+$$

$$+|k|^{\gamma}\frac{\operatorname{Im}\mu}{\omega}\int_0^1\big(dv_k\bar v_k'+\bar d\bar v_k v_k'+(cw_k\bar v_k'+\bar c\bar w_k v_k')e^{px+q(1-x)}\big)\,dx-$$

$$-|k|^{\gamma}\frac{\operatorname{Im}\lambda}{\omega}\int_0^1(f_k\bar w_k'+\bar f_k w_k')\,e^{-px-q(1-x)}\,dx+$$

$$+|k|^{\gamma}\frac{\operatorname{Im}\mu}{\omega}\int_0^1(g_k\bar v_k'-\bar g_k v_k')\,dx\,.$$

Take p and q to be defined by the formula (8) for an arbitrary fixed $0 < \delta_0 < 1$. This clearly forces

$$(1 - |r_1|^2 e^{2p})|w_k(1)|^2 + (1 - |r_0|^2 e^{-2q})|v_k(0)|^2 < 0 .$$

Further we use similar argument as in the proof of Claim 1. The claim follows.

Case 3. *The estimate* (19) *is fulfilled with* $|k| \le N$ *in place of* $k \in \mathbb{Z}$. It suffices to show that, given k in the range $|k| \le N$, $u_k \in H^1(0,1)$.

By a straightforward calculation, the problem (17), (18) has in $H^1(0,1)$ the following equivalent integral representation:

$$u_k(x) = e^{-\lambda^{-1}(ik\omega x + \alpha(x))}\left(\frac{1}{\lambda}\int_0^x e^{\lambda^{-1}(ik\omega y + \alpha(y))}(f_k(y) - b(y)v_k)\,dy + \right.$$

$$\left. + \frac{r_0}{\Delta_k} w_k(f_k - bv_k, -g_k + cu_k)\right),$$

$$v_k(x) = e^{\mu^{-1}(ik\omega x + \delta(x))}\left(\frac{1}{\mu}\int_0^x e^{-\mu^{-1}(ik\omega y + \delta(y))}(-g_k(y) + c(y)u_k)\,dy + \right.$$

$$\left. + \frac{1}{\Delta_k} w_k(f_k - bv_k, -g_k + cu_k)\right), \qquad (30)$$

where

$$w_k(f_k, g_k) = r_1 e^{-\lambda^{-1}(ik\omega + \alpha(1))}\lambda^{-1}\int_0^1 e^{\lambda^{-1}(ik\omega y + \alpha(y))} f_k(y)\,dy -$$

$$- e^{\mu^{-1}(ik\omega + \delta(1))}\mu^{-1}\int_0^1 e^{-\mu^{-1}(ik\omega y + \delta(y))} g_k(y)\,dy .$$

Here we used assumption (13), which implies $\Delta_k \ne 0$ for all $|k| \le N$. Fix any $k \in \mathbb{N}$. Since $H^1(0,1)$ is a Banach space and, by (12), the operator of the problem (30) is contractible, application of the Banach fixed point theorem to the system (30) gives us the unique solvability of the latter in $H^1(0,1)$. Since $k \in \mathbb{N}$ is arbitrary, the claim follows.

The estimate (19) now follows from Claims 1–3. This finishes the proof of the theorem. ◊

The following corollaries are straightforward.

Corollary 2. *Under the conditions of Theorem* 1 *a strong generalized solution to the problem* (1)–(3) (*if such exists*) *is unique.*

Corollary 3. *Under the conditions of Theorem* 1 *any strong generalized solution to the problem* (1)–(3) *continuously depends on the right hand sides of* (1).

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АПРІОРНІ ОЦІНКИ ДЛЯ ПЕРІОДИЧНИХ ЛІНІЙНИХ ЕЛІПТИЧНИХ СИСТЕМ ПЕРШОГО ПОРЯДКУ

Розглянуто змішану періодичну задачу для майже лінійних еліптичних систем першого порядку та доведено для неї теорему про єдиність і неперервну залежність розв'язків у повній шкалі соболівських просторів періодичних функцій. Виведено апріорні оцінки в просторах розв'язків і правих частин, що дають оптимальне співвідношення регулярності для розглядуваної задачі.

АПРИОРНЫЕ ОЦЕНКИ ДЛЯ ПЕРИОДИЧЕСКИХ ЛИНЕЙНЫХ ЭЛЛИПТИЧЕСКИХ СИСТЕМ ПЕРВОГО ПОРЯДКА

Рассмотрена смешанная периодическая задача для почти линейных эллиптических систем первого порядка и доказана для нее теорема о единственности и непрерывной зависимости решений в полной шкале Соболевских пространств периодических функций. Выведены априорные оценки в пространствах решений и правых частей, обеспечивающие оптимальное соотношение регулярности для рассматриваемой задачи.

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