UDK 517.519: 517.96

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NEW TYPE OF INSTABILITY IN FRACTIONAL REACTION-DIFFUSION SYSTEMS

The linear stage of the two-component fractional reaction-diffusion system stability is studied. It is shown that for certain value of fractional derivative index a new type of instability takes place and the system becomes unstable towards perturbations of finite wave number for given value of fractional derivative. As a result, inhomogeneous oscillations with this wave number become unstable and lead to non-linear oscillations which result in spatial oscillatory structure formation. Computer simulation of the system for cubic non-linearity is performed.

Introduction. Fractional reaction-diffusion systems (FRDS) have been used in the study of the new type of self-organization phenomena [1–7]. The analysis of the structures in FRDS evolves both from the standpoint of the qualitative analysis and from the computer simulation. Namely these two problems are the goal of our present investigation.

Let us consider the reaction-diffusion system for activator n_1, inhibitor n_2 in the following equations:

$$\tau_1 \frac{\partial^\alpha n_1(x,t)}{\partial t^\alpha} = l^2 \frac{\partial^2}{\partial x^2} n_1(x,t) + W(n_1, n_2), \tag{1}$$

$$\tau_2 \frac{\partial^\alpha n_2(x,t)}{\partial t^\alpha} = L^2 \frac{\partial^2}{\partial x^2} n_2(x,t) + Q(n_1, n_2, \mathcal{A}), \tag{2}$$

subject to

(*i*) Neumann

$$\left.\frac{dn_i}{dx}\right|_{x=0} = \left.\frac{dn_i}{dx}\right|_{x=l_x} = 0, \quad i = 1,2, \tag{3}$$

or

(*ii*) periodic

$$n_i(t,0) = n_i(t,l_x), \ \frac{dn_i}{dx}\Big|_{x=0} = \frac{dn_i}{dx}\Big|_{x=l_x}, \qquad i = 1,2, \tag{4}$$

boundary conditions and with the certain initial condition $n_i\big|_{t=0} = n_i^0(x)$. Here $x : 0 \le x \le l_x ; (x,t) \in \mathbb{R} \times \mathbb{R}_+$; τ_1, τ_2, l, L are the characteristic times and lengths of the system, $\mathcal{A}$ is a bifurcation parameter.

Fractional derivatives $\dfrac{\partial^\alpha n_i(x,t)}{\partial t^\alpha}$ on the left hand side of equations (1), (2), instead of standard time derivatives, are the Caputo fractional derivatives in time of the order $0 < \alpha < 2$ and are represented as [9, 10]

$$\frac{\partial^\alpha}{\partial t^\alpha} n_i(t) := \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{n_i^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, \qquad m-1 < \alpha < m, \qquad m \in 1,2 .$$

It should be noted that equations (1), (2) at $\alpha = 1$ correspond to standard reaction-diffusion systems (RDS) [8, 11]. At $\alpha < 1$, they describe anomalous subdiffusion and at $\alpha > 1$ – anomalous super-diffusion [12–14].

Linear stability analysis. Stability of the steady-state constant solutions of the system (1), (2) corresponds to homogeneous equilibrium state

$$W(n_1, n_2) = 0, \qquad Q(n_1, n_2, \mathcal{A}) = 0 \tag{5}$$

and can be analyzed by linearization of the system nearby this solution. In this case the system (1), (2) can be transformed into a linear system at this equilibrium point. As a result, we have

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(u)u(x,t), \tag{6}$$

where $u(x,t) = \begin{Vmatrix} \Delta n_1(x,t) \\ \Delta n_2(x,t) \end{Vmatrix}$, $F(u) = \begin{Vmatrix} (l^2\nabla^2 + a_{11})/\tau_1 & a_{12}/\tau_1 \\ a_{21}/\tau_2 & (L^2\nabla^2 + a_{22})/\tau_2 \end{Vmatrix}$ is a Frechet derivative with respect to $u(x,t)$, $a_{11} = W'_{n_1}$, $a_{12} = W'_{n_2}$, $a_{21} = Q'_{n_2}$, $a_{22} = Q'_{n_2}$ (all derivatives are taken at homogeneous equilibrium state). By substituting the solution in the form $u(x,t) = \begin{Vmatrix} \Delta n_1(t) \\ \Delta n_2(t) \end{Vmatrix} \cos kx = u(t)\cos kx$, $k = \frac{\pi}{l_x} j$, $j = 1, 2, \ldots$, into FRDS (6) we can get the system of linear ordinary differential equations (6) with the matrix

$$F = \begin{Vmatrix} (a_{11} - k^2 l^2)/\tau_1 & a_{12}/\tau_1 \\ a_{21}/\tau_2 & (a_{22} - k^2 L^2)/\tau_2 \end{Vmatrix}.$$

By simple linear transformation, equations (6) can be converted into the simplest possible matrix representation, i. e. Jordan canonical form

$$\frac{d^\alpha \eta(t)}{dt^\alpha} = G\eta(t), \tag{7}$$

where G is a diagonal matrix for F: $G = P^{-1}FP = \begin{Vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{Vmatrix}$; eigenvalues $\lambda_{1,2}$ are determined by the characteristic equation of the matrix F: $\lambda_{1,2} = \frac{1}{2}\Big(\operatorname{tr} F \pm \pm \sqrt{\operatorname{tr}^2 F - 4\det F}\Big)$; $\eta(t) = \frac{1}{P} u(t)$; P is the change of the basis matrix corresponding to diagonalization of the matrix F. For stability analysis, it is sufficient to investigate the spectrum of eigenvalues λ_i for the evolution matrix G. In this case, the solution of the vector equation (7) is given by Mittag − Leffler functions [9, 10]

$$\Delta n_i(t) = \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(k\alpha + 1)} \Delta n_i(0) = \mathrm{G}_\alpha(\lambda_i t^\alpha)\Delta n_i(0), \qquad i = 1, 2. \tag{8}$$

Using the result obtained in the papers [7], we can conclude that if for any of the roots

$$|\operatorname{Arg}(\lambda_i)| < \frac{\pi}{2}\alpha \tag{9}$$

the solution has an increasing function component, and then the system is asymptotically unstable.

Analyzing the roots of the characteristic equations, we can see that at $4\det F - \operatorname{tr}^2 F > 0$ eigenvalues $\lambda_{1,2}$ are complex inside the parabola $4\det F = \operatorname{tr}^2 F$. For integer $\alpha : \alpha = 1$ the fixed points are the spiral sources ($\operatorname{tr} F > 0$) or spiral sinks ($\operatorname{tr} F < 0$). In general case $\alpha : 0 < \alpha < 2$ we can introduce marginal value $\alpha : \alpha = \alpha_0 = \frac{2}{\pi}|\operatorname{Arg}(\lambda_i)|$ which follows from the conditions (9) and determines the stability conditions [2]

$$\alpha_0 = \begin{cases} \dfrac{2}{\pi}\arctan\sqrt{4\dfrac{\det F}{\mathrm{tr}^2 F}-1}, & \mathrm{tr}\,F \geq 0, \\ 2-\dfrac{2}{\pi}\arctan\sqrt{4\dfrac{\det F}{\mathrm{tr}^2 F}-1}, & \mathrm{tr}\,F \leq 0. \end{cases} \tag{10}$$

Let us consider the parameters which keep the system inside the parabola $\det F = \frac{1}{4}\mathrm{tr}^2 F$. It is a well-known fact, that at $\alpha = 1$, the domain on the right hand side of the parabola ($\mathrm{tr}\,F > 0$) is unstable with the existing limit cycle, while the domain on the left hand side ($\mathrm{tr}\,F < 0$) is stable. By crossing the axis $\mathrm{tr}\,F = 0$, the Hopf bifurcation conditions become true.

In the general case of $\alpha : 0 < \alpha < 2$, for every point inside the parabola, there exists a marginal value of α_0 where the system changes its stability. The value of α is a certain bifurcation parameter which switches the stable and unstable state of the system. At lower $\alpha : \alpha < \alpha_0 = \frac{2}{\pi}|\mathrm{Arg}(\lambda_i)|$, the system has oscillatory modes but they are stable. Increasing the value of $\alpha < \alpha_0 = \frac{2}{\pi}|\mathrm{Arg}(\lambda_i)|$ leads to instability.

Different limits of instability. Let us consider stability conditions for different possible limits. It is widely known that for integer time derivatives, the system (1), (2) becomes unstable according to either Turing or Hopf bifurcations.

The conditions for the Turing instability are:

$$\mathrm{tr}\,F < 0, \quad \det F(k=0) > 0, \qquad \det F(k_0) < 0. \tag{11}$$

By rewriting the last condition we have

$$\left[\frac{(a_{11}-k^2 l^2)}{\tau_1} - \frac{(a_{22}-k^2 L^2)}{\tau_2}\right]^2 > -4\frac{a_{12}a_{21}}{\tau_1\tau_2}. \tag{12}$$

In this case, the eigenvalues are real and at $a_{11} > 0$, $a_{22} < 0$, $a_{12}a_{21} < 0$, $l \ll L$, the conditions of Turing instability for $k_0 \neq 0$ lead to spatial pattern formation.

Let us consider the conditions for Hopf bifurcation which are held at $k = 0$ if

$$\mathrm{tr}\,F > 0, \qquad \det F(k=0) > 0. \tag{13}$$

By rewriting the last expression explicitly we have

$$-4\frac{a_{12}a_{21}}{\tau_1\tau_2} > \left[\frac{a_{11}}{\tau_1} - \frac{a_{22}}{\tau_2}\right]^2. \tag{14}$$

This condition holds at $a_{11} > 0$, $a_{22} < 0$, $a_{12}a_{21} < 0$, $\tau_1 < \tau_2$ and leads to homogeneous oscillations.

In the case of fractional derivative index, Hopf bifurcation is not connected with the condition $a_{11} > 0$ and can hold at certain value of α when fractional derivative index is sufficiently large [1, 2]. In this case, the easiest way to satisfy this conditions, is when the right hand side of (14) is close to zero and in this case

$$\alpha_0 \approx 2 - \frac{2}{\pi}\arctan\left(-\frac{a_{12}a_{21}\tau_1}{a_{11}^2\tau_2}\right)^{1/2} \approx 2 - \frac{2}{\pi}\arctan\left(-\frac{a_{12}a_{21}\tau_2}{a_{22}^2\tau_1}\right)^{1/2}.$$

Let us consider a new possible situation when

$$\operatorname{tr} F < 0, \quad 4\det F(0) < \operatorname{tr}^2 F(0)\,,$$

$$4\det F(k_0) > \operatorname{tr}^2 F(k_0)\,. \tag{15}$$

Analysis of expressions (15) shows that at $k = 0$ we have two real and smaller then zero eigenvalues and the system is certainly stable. If the latter inequality takes place for certain value of $k_0 \neq 0$, we can obtain for matrix F two complex eigenvalues. As a result, in the case of fractional derivatives, a new type of instability, connected with the interplay between the determinant and trace of the linear system, emerges. By obtaining such type of eigenvalues, it is possible to find out the value of fractional derivative index when the system becomes unstable.

In fact, the last two conditions can be rewritten as

$$\left|\frac{a_{11}}{\tau_1} - \frac{a_{22}}{\tau_2}\right| > \left(-4\,\frac{a_{12}a_{21}}{\tau_1\tau_2}\right)^{1/2}, \tag{16}$$

$$\left(-4\,\frac{a_{12}a_{21}}{\tau_1\tau_2}\right)^{1/2} > \left|\frac{a_{11} - k^2 l^2}{\tau_1} - \frac{a_{22} - k^2 L^2}{\tau_2}\right|. \tag{17}$$

Here we require that $a_{12}a_{21} < 0$, $a_{11} < 0$, $a_{22} < 0$. In order to satisfy the last condition, the easiest way would be to estimate the best value of $k = k_0$:

$$k_0^2 = \left|\left(\frac{a_{11}}{\tau_1} - \frac{a_{22}}{\tau_2}\right)\cdot\left(\frac{L^2}{\tau_2} - \frac{l^2}{\tau_1}\right)^{-1}\right|. \tag{18}$$

Considering (18), let us estimate the expression

$$4\frac{\det F}{\operatorname{tr}^2 F} - 1 = \frac{-4\,\dfrac{a_{12}a_{21}}{\tau_1\tau_2}}{\left[\left|\left(\dfrac{a_{11}}{\tau_1} - \dfrac{a_{22}}{\tau_2}\right)\left(\dfrac{l^2}{\tau_1} + \dfrac{L^2}{\tau_2}\right)\cdot\left(\dfrac{L^2}{\tau_2} - \dfrac{l^2}{\tau_1}\right)^{-1}\right| + \left(-\dfrac{a_{11}}{\tau_1} - \dfrac{a_{22}}{\tau_2}\right)\right]^2}\,. \tag{19}$$

The last expression determines the value of α_0 as a function of all parameters of the system. The greater is the value of expression, the smaller is the value α_0. In order to have the maximum possible value of (19), we can see that it goes to zero if either τ_1 or τ_2 goes to zero and, as a result, $\alpha_0 \to 2$. In the intermediate situation, when $\tau_1 \approx \tau_2$, the expression reaches its maximum.

Let us simplify the expression for the case $\tau_1 = \tau_2$. From (19) we have

$$4\,\frac{\det F}{\operatorname{tr}^2 F} - 1 = \frac{-4a_{12}a_{21}}{\left[\left|(a_{11} - a_{22})(l^2 + L^2)\cdot(L^2 - l^2)^{-1}\right| + (-a_{11} - a_{22})\right]^2}\,.$$

Analyzing the last expression, we can see that at $L \approx l$ the denominator is large and the right hand side tends to zero. In the cases of different lengths $L \gg l$, or $L \ll l$, the expression looks sufficiently simple and determines instability conditions for inhomogeneous wave number (18)

$$\alpha_0 = 2 - \frac{2}{\pi}\arctan\left\{\frac{-4a_{12}a_{21}}{\left[\left|(a_{11} - a_{22})\right| + (-a_{11} - a_{22})\right]^2}\right\}^{1/2}.$$

In the case of marginal system parameters (16), we can estimate the last expression as:

$$\alpha_0 \cong 2 - \frac{2}{\pi}\arctan\frac{1}{2} \approx 1.7\,.$$

This value seems to be very close to the minimum of α_0.

Computer simulation of the stability curves and inhomogeneous oscillatory structures. We consider a very well-known example of the RDS with cubical nonlinearity [8, 11] which is probably the simplest one used in RDS modeling. Let us, for example, consider the isoclines for the model with cubical nonlinearity for activator variable $W = n_1 - n_1^3 - n_2$ and linear for inhibitor one $Q = -n_2 + \beta n_1 + \mathcal{A}$. Their null isoclines ($W = Q = 0$) are represented on Fig. 1*a*. In this case, a homogeneous solution can be determined from the solution of the system of equations $W = Q = 0$, and is given by the cubic algebraic equation

$$(\beta - 1)\bar{n}_1 + \frac{1}{3}\bar{n}_1^3 + \mathcal{A} = 0\,. \tag{20}$$

Simple calculation of the derivatives $a_{11} = \dfrac{1 - \bar{n}_1^2}{\tau_1} > 0$, $a_{12} = -\dfrac{1}{\tau_1} < 0$, $a_{21} = \dfrac{\beta}{\tau_2} > 0$, $a_{22} = -\dfrac{1}{\tau_2} < 0$ makes it possible to write the expression at $\tau_1 = \tau_2 = 1$:

$$\alpha_0 = 2 - \frac{2}{\pi}\arctan\frac{2\sqrt{\beta}}{\left|(\bar{n}_1^2 - 2)(l^2 + L^2)/(L^2 - l^2)\right| + \bar{n}_1^2}\,.$$

The real and imaginary parts of the eigenvalues for this case at $l \ll L$ are represented on Fig. 1*b*.

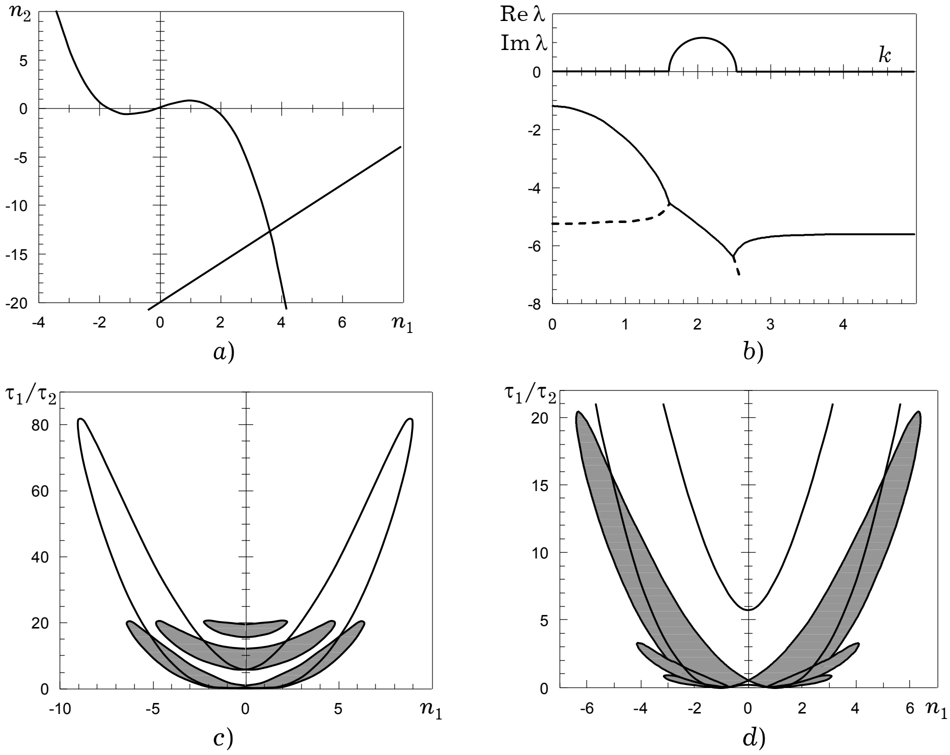

Fig. 1

We can see that real part of the roots is always less than zero and the imaginary one in some interval of wave number k becomes nonzero. In this case, if fractional derivative index becomes greater than α_0 determined by condition (9), instability holds true. In this case, the instability conditions are possible to realize for some interval $k_{\min} < k < k_{\max}$. This means that only the perturbations with namely this wave number are unstable, and they are unstable for oscillatory fluctuation. The stability domains for different l and k, $k = 0, 1, 2, 3$, are represented on Fig. 1*c*, *d*. With the solid bold line, we indicate the marginal curve for stability domain with $k = 0$ which corresponds to homogenous oscillations. Inside this curve, the system is unstable for homogenous oscillations [2], while in the domains which bulge out from the solid curve, we have a new type of instability for the given values of k. This situation is qualitatively different from the classical RDS, whether either Turing ($k \neq 0$) or Hopf bifurcation ($k = 0$) takes place, and this depends on which conditions are realized easier. In the system under consideration, we can choose the parameter when we don't have Turing and Hopf bifurcations (for $k = 0$) at all. Nevertheless, we obtain conditions for a new bifurcation which can be realized for non-homogeneous wave numbers only.

The numerical study of the initial value problem of the system (1), (2) was performed for the conditions at which the inhomogeneous perturbations become unstable. The system with corresponding initial and boundary conditions was integrated numerically using implicit schemes with respect to time and centered difference approximation for spatial derivatives. The fractional derivatives were approximated using the scheme on the basis of Grunwald – Letnikov definition for $1 < \alpha < 2$ [9, 10]. The results of the computer simulation of the oscillatory inhomogeneous dissipative structures for different values of α are presented on Fig. 2. Dynamics of variable n_1 (Fig. 2*a*) and n_2 (Fig. 2*b*) is represented on the time interval $t \in (0, 30)$ for $\alpha = 1.94$; $l_x = 6.28$; $\mathcal{A} = -50$; $\tau_1 = 12$, $\tau_2 = 1.0$; $l^2 = 0.05$, $L^2 = 1$; $\beta = 2$. We used initial conditions in the form of small perturbation of homogeneous state

$$\bar{n}_{10} = \bar{n}_1 + 0.05\cos(k_0 x), \qquad \bar{n}_{20} = \bar{n}_2 + 0.05\cos(k_0 x).$$

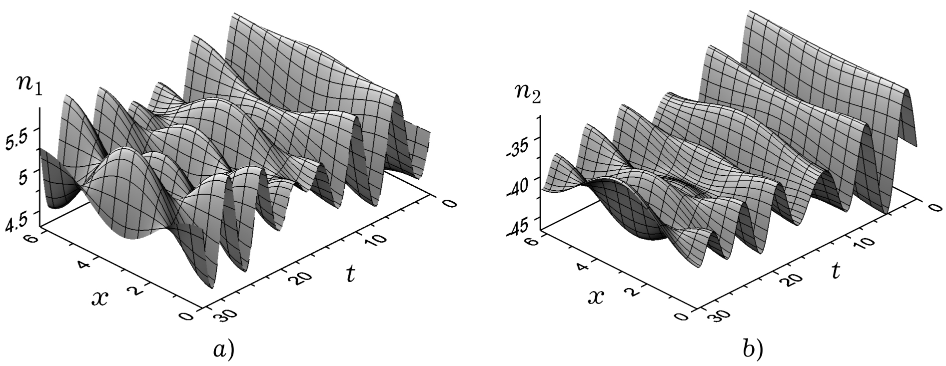

Fig. 2

In contrast to standard RDS, the inhomogeneous distributions are unstable according to certain wave number and lead to space time oscillation. With the increase in the parameter α, the amplitude of the oscillatory structures increases. The emergence of inhomogeneous oscillations, which destroy the stationary state, leads to a new form of pattern formation. The resulting structures are rather similar to sending waves, than to standard structures already investigated in autowave media.

Conclusion. In this article we consider a new mechanism of instability in reaction-diffusion systems with fractional derivatives. It was shown that at a sufficient value of fractional derivative index α, the system becomes unstable according to inhomogeneous perturbation ($k \neq 0$) with eigenvalues with imaginary part. As a result of this instability, pattern formation can be represented as oscillatory structures, similar to inhomogeneous standing waves in linear systems.

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НОВИЙ ТИП НЕСТІЙКОСТІ В СИСТЕМАХ РЕАКЦІЇ-ДИФУЗІЇ З ПОХІДНИМИ ДРОБОВОГО ПОРЯДКУ

Досліджено лінійну стадію стійкості двокомпонентної системи реакції-дифузії з дробовими похідними. Показано, що при певному значенні порядку дробової похідної має місце новий тип нестійкості й система стає нестійкою стосовно деякого хвильового числа для цього значення дробової похідної. В результаті нестійкості збурень з цим хвильовим числом у системі формуються просторово-неоднорідні коливні структури. Проведено комп'ютерне моделювання системи для кубічної нелінійності.

НОВЫЙ ТИП НЕУСТОЙЧИВОСТИ В СИСТЕМАХ РЕАКЦИИ-ДИФФУЗИИ С ПРОИЗВОДНЫМИ ДРОБНОГО ПОРЯДКА

Исследована линейная стадия устойчивости двухкомпонентной системы реакции-диффузии с дробными производными. Показано, что при определенном значении порядка дробной производной имеет место новый вид неустойчивости и система становится неустойчивой по отношению к определенному волновому числу для заданного значения дробной производной. В результате неоднородные возмущения с этим волновым числом становятся неустойчивыми и приводят к формированию пространственно-неоднородных колебательных структур в системе. Проведено компьютерное моделирование системы для кубической нелинейности.

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Received
27.10.06