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ANALYSIS OF NON-CLASSICAL FRACTURE PROBLEMS OF PRE-STRESSED BODIES WITH INTERACTING CRACKS

In this study two types of non-classical fracture mechanisms are considered, namely, the fracture of cracked bodies with initial (residual) stresses acting along the crack planes and fracture of solids under compression along parallel cracks. To investigate non-axisymmetric and axisymmetric problems for infinite solids containing two parallel co-axial cracks or a periodical set of co-axial parallel cracks we use a combined analytical-numerical method in the framework of three-dimensional linearized mechanics of solids. The analysis involves the representation of stresses and displacements of the linearized theory via harmonic potential functions. With the use of the integral Fourier – Hankel transformations the problems are reduced to resolving Fredholm integral equations of the second kind. This approach allows to investigate problems in a unified general form for compressible and non-compressible homogeneous isotropic or transversally isotropic elastic bodies with an arbitrary structure of the elastic potential, and the material specification of the model is carried out only at the stage of numerical calculation of resolving equations obtained in the general form. The effects of initial stresses on stress intensity factors are analyzed for highly elastic materials and layered composites (modeled as transversally isotropic elastic bodies). The «resonance-like» effects are found out when compressive initial stresses are reached the values that correspond to the local loss of material stability in the vicinity of the cracks, which, according to the combined method mentioned, allows one to determine critical (limiting) load parameters under compression of the body along the cracks. The conclusions concerning the dependences of stress intensity factors and critical (limiting) parameters of compression on geometrical parameters of the problems are formulated as well as on physical and mechanical characteristics of materials.

1. Introduction. Among the problems of fracture mechanics that need further research are, in particular, the analysis of the effect of initial (residual) stresses which, in practice, result from the inhomogeneity of linear or volume deformations in the neighboring regions of the material on the stress-strain state of cracked bodies and the study of fracture of bodies under compression [18, 24, 35]. Here, of particular interest are the problems in which initial stresses (or compressive forces) act along the surfaces of the cracks present in a body. In the terminology proposed in [13, 15, 27], these groups of problems are classified with the non-classical problems of fracture mechanics. The point is that within the framework of the linear brittle fracture mechanics based on Griffith – Irwin concept and approaches and their generalizations [19, 22, 23, 25], it is impossible to take into account the effect of the load components oriented in parallel to the crack plane on the fracture parameters. The solution of respective linear elasticity theory problems implies that the abovementioned load components do not affect stress intensity factors and crack-opening displacements, so they cannot be taken into account when relying on the classical fracture criteria, namely, the Griffith – Irwin criterion or critical crack-opening displacement criterion. But this is not consistent with the results of experiments in which the impact of load components along the cracks on the fracture parameters was found [32].

In [8, 9], a version of brittle fracture mechanics was proposed for materials with initial (residual) stresses acting along cracks that was based on three-dimensional linearized mechanics of solids [26]. In those works, the fundamentals of brittle fracture mechanics for initially stressed materials where initial stresses were much higher as compared to additional stress fields were elaborated, including formulations and methods of solving two- and three-dimensional problems, as well as formulations of a brittle fracture criterion analogous to that of Griffith – Irwin [14]. Solutions were obtained for some classes of static and dynamic problems, mainly those concerning isolated cracks in infinite bodies with initial stresses (brief reviews of these works were presented in [3, 13, 20, 30]). New mechanical effects (of both quantitative and qualitative nature) related to the effect of initial (residual) stresses on stress-strain states in materials were discovered.

In the works [12, 15] at studying the phenomenon of fracture of solids compressed along planes of parallel cracks it was proposed to consider the local loss of material stability in the vicinity of cracks as a failure mechanism. According to this approach, critical compression parameters can be determined by solving respective problems on eigenvalues within the framework of the three-dimensional linearized theory of deformable solids stability [11, 26]. Reviews of the works relying on this approach were presented in [10, 21,28, 36].

The very first works in the mechanics of brittle fracture of materials with initial stresses [8, 9] that investigated planar and spatial problems for an infinite pre-stressed material with an isolated non-interacting crack found a new mechanical effect due to the influence of initial stresses on the distribution of stress and strain fields in the vicinity of crack and, respectively, on the values of fracture loads. It is shown, in particular, that when initial (residual) stresses tend to values corresponding to surface instability of a half-plane (for planar problems) or half-space (for spatial ones), phenomena of «resonance» nature occur at the crack tip. Those consist in a part of stresses and displacements determined from the linearized relations quickly tending to «infinity». Respectively, in the case of a «free» crack in bodies with initial stresses the values of fracture loads within the framework of the linearized theory tend to zero when initial (residual) compressive stresses approach the values corresponding to the surface instability of the half-plane or half-space.

On the other hand, the studies of problems on compression of infinite materials along an isolated crack, when the fracture mechanism is due to the local loss of the stability of equilibrium near the crack, discovered a mechanical effect [9, 12] that consists in coinciding the critical loads during compression along the crack with the load values that realize the surface instability of the half-plane (for planar problems) or half-space (for spatial ones). Here, the loss of material stability in the local domain near the crack is of the surface instability type.

The two abovementioned mechanical effects indicate that both in the mechanics of brittle fracture of materials with initial stresses acting along cracks and in the mechanics of material fracture due to compression along cracks the phenomenon of surface instability of the half-plane or half-space is of fundamental nature. The situation could be explained by the following physical considerations. When initial stresses acting along the crack reach the values corresponding to the surface instability of the half-plane or half-space a state of neutral equilibrium develops near the crack tip. In this situation a minor change in the external load is sufficient to upset the neutral equilibrium and start the fracture process characterized by the local loss of material stability in the vicinity of the crack.

Taking into account this physical interpretation, one can assume that other geometries of crack location in pre-stressed materials will also lead to similar «resonance» phenomena – i.e., when initial compressive stress approaches the values corresponding to the local loss of material stability in respective problems on body compression along parallel cracks, stresses and displacements in the vicinity of the cracks change in the dramatic, «resonance-like» manner. One should also emphasize that the common point in analyzing the two abovementioned groups of problems is the use of allied mathematical apparatus in the framework of the three-dimensional linearized mechanics of solids. In this case, problems of fracture mechanics for materials with initial stresses acting along cracks are inhomogeneous linearized ones and those of mechanics of material fracture due to compression along parallel cracks are homogeneous linearized problems.

In view of these considerations, to ensure a substantial reduction in complex computations, a better account and correct interpretation of all mechanical effects, we think it reasonable to carry out a combined investigation of fracture mechanics problems on solids with initial stresses and problems on fracture of cracked materials with compression along cracks in the framework of linearized mechanics of solids. Such a joint approach allows us to propose a new, simpler and more effective in practical use, method to determine critical parameters of loading in problems on solids compression along the cracks which they contain when there is no need for individual investigation of eigenvalues problems within three-dimensional linearized stability theory. The parameters mentioned are calculated in solving respective inhomogeneous relations of the fracture mechanics of materials with initial stresses as values of initial compressing forces on achieving which a dramatic «resonance-like» change in stresses and displacements occurs in the vicinity of cracks [4–7, 29]. It is also evident that in investigating problems for fracture of materials with initial stresses there is a natural constraint on the values of initial compressive stresses when they cannot exceed the values corresponding to the local loss of material stability in the crack vicinity. It should be noted that the above method is similar to an approach in the theory of oscillations of mechanical systems. Namely, to determine the natural frequencies of a system, one can examine forced oscillations with continuous change of frequency of the external load. In this case, the frequency of the natural oscillations of the system is determined (or calculated) as the frequency of the external load, on achieving which a sharp «resonance» change of the amplitude values occurs and they tend to infinity.

The method proposed allows the investigation of the problems in a unified form for isotropic and transversally isotropic compressible and incompressible elastic bodies with the elastic potential of arbitrary structure both for the theory of large (finite) initial deformations and the theory of small initial deformations. Specific models of materials (e.g., the use of the elastic potentials of the specific structure) are only applied at the stage of the numerical analysis of the characteristic equations, solving the integral equations, etc.

The present work, relying on the aforementioned method, provides mathematical statements of problems on pre-stressed solids that contain interacting circular cracks. It solves problems on an infinite solid containing two parallel coaxial cracks and on a space with the periodical set of coaxial cracks. The cases of two parallel coaxial cracks and the periodic set of parallel cracks that allow us to estimate the effect of the mutual interaction of the cracks on the fracture parameters are the limiting cases for problems concerning the fracture of materials with an arbitrary finite number of coaxial parallel cracks.

Some patterns of loading on crack faces are considered. The effects of initial stresses on stress intensity factors are analyzed for highly elastic materials and layered composites (modeled in continuum approximation as transversally isotropic elastic bodies). The critical parameters of the fracture of solids containing interacting cracks under compression along the cracks are calculated. The influence of problems' geometrical parameters as well as physical and mechanical properties of materials on the critical parameters is analyzed.

2. Problem formulation. We consider an infinite elastic body with initial stresses $S_{11}^0 = S_{22}^0$ that are applied in the Oy_1y_2-plane and act along cracks located in parallel planes $y_3 = \text{const}$. This results in uniform initial stress-strain state

$$S_{33}^0 = 0, \quad S_{11}^0 = S_{22}^0 = \text{const} \neq 0, \quad \lambda_j = \text{const}, \quad \lambda_1 = \lambda_2 \neq \lambda_3,$$

$$u_j^0 = \lambda_j^{-1}(\lambda_j - 1) y_j, \qquad j = 1, 2, 3. \tag{1}$$

In Eqs. (1) and in what follows we use the following notations: y_j are Lagrangian coordinates, which, in the initial state (caused by the initial stresses) coincide with Cartesian coordinates; S_{ij}^0 are the components of the symmetric stress tensor measured per unit area in the undeformed state; λ_j are elongation (or contraction) ratios along the coordinate axes determined by the initial stretching (or compressing) stresses S_{ij}^0; Q'_{ij} are the components of the non-symmetric stress tensor measured per unit area of the body in the initial state and u_j are the components of the vector of displacements that correspond to them.

In the case where additional (with respect to the initial stress and strain state) forces are applied to the body, perturbations of the stress-strain state caused by their action are assumed much smaller than the corresponding values of the initial stress-strain state, which enables us to apply relations of the linearized theory of elasticity to the solution of the stated problems [11, 26]. In [9, 13] the general solutions of linearized equations of equilibrium for the uniform initial stress-strain state in Eqs. (1) are obtained in terms of potential functions. These solutions depend on the roots n_1 and n_2 of the governing characteristic equations. In the case of the non-equal roots $n_1 \neq n_2$ the solution is given by

$$u_r = \frac{\partial(\varphi_1 + \varphi_2)}{\partial r} - \frac{1}{r}\frac{\partial \varphi_3}{\partial \theta}, \qquad u_\theta = \frac{1}{r}\frac{\partial(\varphi_1 + \varphi_2)}{\partial \theta} + \frac{\partial \varphi_3}{\partial r},$$

$$u_3 = \frac{m_1}{\sqrt{n_1}}\frac{\partial \varphi_1}{\partial z_1} + \frac{m_2}{\sqrt{n_2}}\frac{\partial \varphi_2}{\partial z_2},$$

$$Q'_{3r} = C_{44}\left\{d_1 n_1^{-1/2}\frac{\partial^2}{\partial r \partial z_1}\varphi_1 + d_2 n_2^{-1/2}\frac{\partial^2}{\partial r \partial z_2}\varphi_2 - n_3^{-1/2}\frac{1}{r}\frac{\partial^2}{\partial \theta \partial z_3}\varphi_3\right\},$$

$$Q'_{3\theta} = C_{44}\left\{d_1 n_1^{-1/2}\frac{1}{r}\frac{\partial^2}{\partial \theta \partial z_1}\varphi_1 + d_2 n_2^{-1/2}\frac{1}{r}\frac{\partial^2}{\partial \theta \partial z_2}\varphi_2 + n_3^{-1/2}\frac{\partial^2}{\partial r \partial z_3}\varphi_3\right\},$$

$$Q'_{33} = C_{44}\left[d_1 \ell_1 \frac{\partial^2}{\partial z_1^2}\varphi_1 + d_2 \ell_2 \frac{\partial^2}{\partial z_2^2}\varphi_2\right], \tag{2}$$

where (r, θ, z_3) are cylindrical coordinates obtained from Cartesian coordinates $y_j, j = 1, 2, 3$; $z_j \equiv n_j^{-1/2} y_3$ and $\varphi_j(r, z_i)$, $j = 1, 2, 3$, are harmonic potential functions. The parameters C_{44}, n_j, $j = 1, 2, 3$, d_i and ℓ_i, $i = 1, 2$, included in (2) depend on the initial stresses as well as on the material properties [9, 13].

In what follows, we present a detailed computation in the case where the roots of the characteristic equation are non-equal. In the case where the roots are equal, calculations are performed in a similar way.

For the complete formulation of the problems, equations (1) and (2) should be complemented with boundary conditions. Below we present a detailed computation for the problem on the fracture of a body with two parallel co-axial cracks. In the problem on the space containing a periodic set of parallel co-axial cracks, calculations are performed in a similar way.

Let us consider two circular cracks with equal radii a, which are located in parallel planes $y_3 = 0$ and $y_3 = -2h$ with centers on the axis Oy_3. The origin of the cylindrical coordinates coincides with the centers of the cracks. For such cracks location there is a symmetry of geometrical and stress-strain schemes of the problem with the plane $y_3 = -h$. Therefore, the problem on the space containing two parallel cracks may be formulated in terms of a problem on the half-space $y_3 \geq -h$ with one near-surface crack that is located in the plane $y_3 = 0$. Below, we consider separately the boundary conditions corresponding to opening- and shear-mode cracks.

2.1. Mode I cracks. On the faces of the cracks, we set supplementary (with respect to the initial stress-strain state) fields of normal tensile stresses $\sigma(r,\theta)$ (symmetrical with respect to the plane $y_3 = 0$). Considering the upper half-space $y_3 \geq -h$ we have boundary conditions on the crack faces and on the plane $y_3 = -h$:

$$Q'_{33} = -\sigma(r,\theta), \quad Q'_{3r} = 0, \quad Q'_{3\theta} = 0,$$
$$0 \leq r < a, \quad y_3 = \pm 0, \quad 0 \leq \theta < 2\pi, \tag{3}$$

$$u_3 = 0, \quad Q'_{3r} = 0, \quad Q'_{3\theta} = 0,$$
$$0 \leq r < \infty, \quad y_3 = -h, \quad 0 \leq \theta < 2\pi. \tag{4}$$

We conventionally split the half-space $y_3 \geq -h$ into two sub-regions, namely, sub-domain «1», which is a half-space $y_3 \geq 0$, and sub-domain «2», which is a layer $-h \leq y_3 \leq 0$, and denote the quantities that correspond to these domains by superscripts. On the boundary of these sub-domains, outside the crack, the conditions of continuity of the components of the stress tensor and the vectors of displacements are satisfied. This requires the following conditions

$$u_3^{(2)} = 0, \quad Q'^{(2)}_{3r} = 0, \quad Q'^{(2)}_{3\theta} = 0,$$
$$y_3 = -h, \quad 0 \leq r < \infty, \quad 0 \leq \theta < 2\pi, \tag{5}$$

$$u_3^{(1)} = u_3^{(2)}, \quad u_r^{(1)} = u_r^{(2)}, \quad u_\theta^{(1)} = u_\theta^{(2)},$$
$$y_3 = 0, \quad a < r < \infty, \quad 0 \leq \theta < 2\pi, \tag{6}$$

$$Q'^{(1)}_{33} = Q'^{(2)}_{33}, \quad Q'^{(1)}_{3r} = Q'^{(2)}_{3r}, \quad Q'^{(1)}_{3\theta} = Q'^{(2)}_{3\theta},$$
$$y_3 = 0, \quad 0 \leq r < \infty, \quad 0 \leq \theta < 2\pi, \tag{7}$$

$$Q'^{(2)}_{33} = -\sigma(r,\theta), \quad Q'^{(2)}_{3r} = 0, \quad Q'^{(2)}_{3\theta} = 0,$$
$$y_3 = 0, \quad 0 \leq r < a, \quad 0 \leq \theta < 2\pi. \tag{8}$$

Moreover, the stresses and displacements in the half-space $y_3 \geq 0$ must vanish for large values of y_3. For the problem on compression of a body containing two parallel crack by forces acting along the cracks planes the boundary conditions will have the form (5)–(8) with $\sigma(r,\theta) \equiv 0$. Using the representations (2), we can reformulate the boundary conditions (5)–(8) in terms of the potential functions $\varphi_j(r, z_i)$, $j = 1,2,3$. For the axisymmetric case in relations (3)–(8) we should set $u_\theta = Q'_{3\theta} = 0$.

2.2. Mode II cracks. Similarly, in the case when radial stresses $\tau(r)$ (anti-symmetrical with respect to the plane $y_3 = 0$) are specified on the crack faces, the boundary conditions are (for problem with axisymmetry):

$$u_r^{(2)} = 0, \quad Q_{33}'^{(2)} = 0, \quad y_3 = -h, \quad 0 \le r < \infty,$$

$$u_3^{(1)} = u_3^{(2)}, \quad u_r^{(1)} = u_r^{(2)}, \quad y_3 = 0, \quad a < r < \infty,$$

$$Q_{33}'^{(1)} = Q_{33}'^{(2)}, \quad Q_{3r}'^{(1)} = Q_{3r}'^{(2)}, \quad y_3 = 0, \quad 0 \le r < \infty,$$

$$Q_{33}'^{(2)} = 0, \quad Q_{3r}'^{(2)} = -\tau(r), \quad y_3 = 0, \quad 0 \le r \le a. \tag{9}$$

2.3. Mode III cracks. We consider the axisymmetric case, when tangential torsion loads $\tau_\theta(r)$ (anti-symmetrical with respect to the plane $y_3 = 0$) are applied to the crack faces. In this case, only components of displacement vector u_θ and stress tensor $Q_{3\theta}'$ are nonzero and the representations (2) take the form

$$u_r = 0, \quad u_\theta = \frac{\partial \varphi_3}{\partial r}, \quad u_3 = 0,$$

$$Q_{33}' = Q_{3r}' = 0, \quad Q_{3\theta}' = C_{44} n_3^{-1/2} \frac{\partial^2 \varphi_3}{\partial r \partial z_3}. \tag{10}$$

Taking into account the abovementioned splitting of the half-space $y_3 \ge -h$ into two sub-domains, we obtain the boundary conditions for Mode III cracks:

$$u_\theta^{(2)}(r, y_3) = 0, \qquad y_3 = -h, \quad 0 \le r < \infty,$$

$$Q_{3\theta}'^{(1)}(r, y_3) = Q_{3\theta}'^{(2)}(r, y_3), \qquad y_3 = 0, \quad 0 \le r < \infty,$$

$$u_\theta^{(1)}(r, y_3) = u_\theta^{(2)}(r, y_3), \qquad y_3 = 0, \quad a < r < \infty,$$

$$Q_{3\theta}'^{(2)}(r, y_3) = -\tau_\theta(r), \qquad y_3 = 0, \quad r \le a.$$

3. Fredholm integral equations. The problems formulated can be reduced to systems of dual integral equations and then to Fredholm integral equations of the second kind. Below, we present a detailed computation for the problem of fracture of a body with two parallel Mode I cracks. In the cases of other problem formulations, calculations are performed in a similar way.

3.1. Mode I cracks. Consider a function of the intensity of external loads on the crack faces $\sigma(r, \theta)$ in the form of Fourier series in the angular coordinate θ, assuming for the simplicity of calculations that it is an even function (when it is an odd function of θ, here and in the further analysis, cosines should be replaced with sines and *vice versa*. But in the general case it is necessary to use the superposition of solutions.):

$$\sigma(r, \theta) = \sum_{n=0}^{\infty} \sigma^{(n)}(r) \cos n\theta, \tag{11}$$

where the coefficients $\sigma^{(n)}(r)$ have the form

$$\sigma^{(0)}(r) = \frac{1}{\pi} \int_0^\pi \sigma(r, \theta)\, d\theta, \qquad \sigma^{(n)}(r) = \frac{2}{\pi} \int_0^\pi \sigma(r, \theta) \cos n\theta\, d\theta, \; n = 1, 2, 3 \ldots.$$

We also represent the harmonic potential functions $\varphi_j(r, z_i)$, $j = 1, 2, 3$, in each sub-domain, namely, «1» and «2», in the form of Fourier series with respect to the angular coordinate θ with coefficients in the form of Hankel integral expansions with respect to the radial coordinate r of the order that corresponds to the order of the harmonic with respect to the coordinate θ

$$\varphi_1^{(1)}(r,\theta,z_1) = \sum_{n=0}^{\infty} \cos n\theta \int_0^{\infty} A_n(\lambda) e^{-\lambda z_1} J_n(\lambda r) \frac{d\lambda}{\lambda},$$

$$\varphi_2^{(1)}(r,\theta,z_2) = \sum_{n=0}^{\infty} \cos n\theta \int_0^{\infty} B_n(\lambda) e^{-\lambda z_2} J_n(\lambda r) \frac{d\lambda}{\lambda},$$

$$\varphi_3^{(1)}(r,\theta,z_3) = \sum_{n=0}^{\infty} \sin n\theta \int_0^{\infty} C_n(\lambda) e^{-\lambda z_3} J_n(\lambda r) \frac{d\lambda}{\lambda},$$

$$\varphi_1^{(2)}(r,\theta,z_1) = \sum_{n=0}^{\infty} \cos n\theta \int_0^{\infty} \left[A_n^{(1)}(\lambda) \operatorname{ch} \lambda(z_1 + h_1) + \right.$$

$$\left. + A_n^{(2)}(\lambda) \operatorname{sh} \lambda(z_1 + h_1)\right] J_n(\lambda r) \frac{\partial\lambda}{\lambda \operatorname{sh} \lambda h_1},$$

$$\varphi_2^{(2)}(r,\theta,z_2) = \sum_{n=0}^{\infty} \cos n\theta \int_0^{\infty} \left[B_n^{(1)}(\lambda) \operatorname{ch} \lambda(z_2 + h_2) + \right.$$

$$\left. + B_n^{(2)}(\lambda) \operatorname{sh} \lambda(z_2 + h_2)\right] J_n(\lambda r) \frac{\partial\lambda}{\lambda \operatorname{sh} \lambda h_2},$$

$$\varphi_3^{(2)}(r,\theta,z_3) = \sum_{n=0}^{\infty} \sin n\theta \int_0^{\infty} \left[C_n^{(1)}(\lambda) \operatorname{ch} \lambda(z_3 + h_3) + \right.$$

$$\left. + C_n^{(2)}(\lambda) \operatorname{sh} \lambda(z_3 + h_3)\right] J_n(\lambda r) \frac{\partial\lambda}{\lambda \operatorname{sh} \lambda h_3}, \qquad (12)$$

where $z_i \equiv n_i^{-1/2} y_i$, $h_i = h n_i^{-1/2}$, $i = 1,2,3$, and $A_n^{(j)}(\lambda)$, $B_n^{(j)}(\lambda)$, $C_n^{(j)}(\lambda)$, $j = 1,2$, are new arbitrary unknown functions. Note that the representation of the potential functions in the form of (12) provides regularity conditions for the stresses and displacements in the case where $y_3 \to \infty$.

Then we substitute the representation of the harmonic potential functions (12) and external load on the faces of the crack (11) into the boundary conditions (5)–(8). In this case, from conditions (5) and (7), which are set on the whole plane $y_3 = \text{const}$, we obtain six relations for nine unknown functions

$$A_n = \frac{1}{k}(k_2 + k_1 \operatorname{cth} \mu_1) A_n^{(1)} + \frac{d_2 \ell_2}{d_1 \ell_1} \frac{k_1}{k} (1 + \operatorname{cth} \mu_2) B_n^{(1)},$$

$$C_n = -C_n^{(1)}, \quad A_n^{(2)} = 0, \quad B_n^{(2)} = 0, \quad C_n^{(2)} = 0, \qquad (13)$$

where $\mu_i = \lambda h_i, i = 1,2$, $k_1 = \ell_1 n_2^{-1/2}$, $k_2 = \ell_2 n_1^{-1/2}$, $k = k_1 - k_2$.

Using the relations

$$\frac{2n}{\lambda r} J_n(\lambda r) = J_{n-1}(\lambda r) + J_{n+1}(\lambda r), \quad 2\frac{\partial J_n(\lambda r)}{\partial(\lambda r)} = J_{n-1}(\lambda r) - J_{n+1}(\lambda r),$$

from (6) and (8) separately for each nth harmonic with respect to the angular coordinate θ, we get a system of six dual integral equations

$$\int_0^\infty \left[d_1\ell_1 \mathrm{cth}\mu_1 A_n^{(1)}(\lambda) + d_2\ell_2 \mathrm{cth}\mu_2 B_n^{(1)}(\lambda)\right]\lambda J_n(\lambda r)\,d\lambda = -\frac{1}{C_{44}}\sigma^{(n)}(r), \quad r < a,$$

$$\int_0^\infty \left[n_1^{-1/2} d_1 A_n^{(1)}(\lambda) + n_2^{-1/2} d_2 B_n^{(1)}(\lambda) - n_3^{-1/2} C_n^{(1)}(\lambda)\right]\lambda J_{n-1}(\lambda r)\,d\lambda = 0, \quad r < a,$$

$$\int_0^\infty \left[n_1^{-1/2} d_1 A_n^{(1)}(\lambda) + n_2^{-1/2} d_2 B_n^{(1)}(\lambda) + n_3^{-1/2} C_n^{(1)}(\lambda)\right]\lambda J_{n-1}(\lambda r)\,d\lambda = 0, \quad r < a,$$

$$\int_0^\infty X_1 J_{n+1}(\lambda r)\,d\lambda = 0, \quad \int_0^\infty X_2 J_{n-1}(\lambda r)\,d\lambda = 0, \quad \int_0^\infty X_3 J_n(\lambda r)\,d\lambda = 0, \quad r > a, \tag{14}$$

where

$$X_1 = \left(1 - \frac{d_1\ell_1}{d_2\ell_2}\right)\frac{k_2}{k}(1 + \mathrm{cth}\,\mu_1)A_n^{(1)}(\lambda) - \left(1 - \frac{d_2\ell_2}{d_1\ell_1}\right)\frac{k_1}{k}(1 + {}$$

$$+ \mathrm{cth}\,\mu_2)B_n^{(1)}(\lambda) - (1 + \mathrm{cth}\,\mu_3)C_n^{(1)}(\lambda),$$

$$X_2 = \left(1 - \frac{d_1\ell_1}{d_2\ell_2}\right)\frac{k_2}{k}(1 + \mathrm{cth}\,\mu_1)A_n^{(1)}(\lambda) - \left(1 - \frac{d_2\ell_2}{d_1\ell_1}\right)\frac{k_1}{k}(1 + {}$$

$$+ \mathrm{cth}\,\mu_2)B_n^{(1)}(\lambda) + (1 + \mathrm{cth}\,\mu_3)C_n^{(1)}(\lambda),$$

$$X_3 = d_1\ell_1(1 + \mathrm{cth}\,\mu_1)A_n^{(1)}(\lambda) + d_2\ell_2(1 + \mathrm{cth}\,\mu_2)B_n^{(1)}(\lambda), \quad n = 1, 2, 3, \ldots. \tag{15}$$

In what follows, we assume that $n \geq 1$ in (14) and (15). The axisymmetric case $n = 0$ is a specific one because the number of equations and unknown function for it decreases and therefore it should be considered separately.

Let us solve the system of dual integral equations (14) with the substitution method [16] for the case when the dual integral equations contain Bessel functions of different orders. Correspondingly, we choose a solution of the system of dual integral equations (14) in the form

$$X_1 = \left(\frac{\pi}{2}\right)^{1/2}\lambda^{3/2}\int_0^a t^{1/2}\varphi(t)J_{n+1/2}(\lambda t)\,dt =$$

$$= -\left(\frac{\pi}{2}\right)^{1/2}\lambda^{1/2}\int_0^a \tilde{\varphi}(t)\left[a^{-n+1/2}J_{n-1/2}(\lambda a) - t^{-n+1/2}J_{n-1/2}(\lambda t)\right]dt,$$

$$X_2 = \left(\frac{\pi}{2}\right)^{1/2}\lambda^{1/2}\int_0^a t^{1/2}\psi(t)J_{n-1/2}(\lambda t)\,dt =$$

$$= \left(\frac{\pi}{2}\right)^{1/2}\lambda^{1/2}\int_0^a t^{1/2}\psi(t)J_{n-1/2}(\lambda t)\,dt,$$

$$X_3 = \left(\frac{\pi}{2}\right)^{1/2}\lambda^{1/2}\int_0^a t^{1/2}\omega(t)J_{n+1/2}(\lambda t)\,dt =$$

$$= -\left(\frac{\pi}{2}\right)^{1/2}\lambda^{-1/2}\int_0^a \tilde{\omega}(t)\left[a^{-n+1/2}J_{n-1/2}(\lambda a) - t^{-n+1/2}J_{n-1/2}(\lambda t)\right]dt, \tag{16}$$

where $\varphi(t)$, $\psi(t)$ and $\omega(t)$ are unknown functions that are continuous as well as their first derivatives on the segment $[0, a]$, and introduce the notation

$$\tilde{\varphi}(t) = \frac{d}{dt}[t^n \varphi(t)], \qquad \tilde{\omega}(t) = \frac{d}{dt}[t^n \omega(t)]. \tag{17}$$

Without going into details and applying a technique similar to [2], we can reduce system (14) to a system of Fredholm integral equations of the second kind in the dimensionless form

$$\frac{1}{2}\left(s\frac{k}{k_1} + q\right) f_1(\xi) + \frac{1}{2}\left(s\frac{k}{k_1} - q\right) f_2(\xi) + \frac{2}{\pi}\int_0^1 f_1(\eta)\mathcal{K}_{11}(\xi,\eta)\,d\eta +$$

$$+\frac{2}{\pi}\int_0^1 f_2(\eta)\mathcal{K}_{12}(\xi,\eta)\,d\eta + \frac{2}{\pi}\int_0^1 f_3(\eta)\mathcal{K}_{13}(\xi,\eta)\,d\eta = 0\,,$$

$$\frac{1}{2}\left(s\frac{k}{k_1} - q\right) f_1(\xi) + \frac{1}{2}\left(s\frac{k}{k_1} + q\right) f_2(\xi) + \frac{2}{\pi}\int_0^1 f_1(\eta)\mathcal{K}_{21}(\xi,\eta)\,d\eta +$$

$$+\frac{2}{\pi}\int_0^1 f_2(\eta)\mathcal{K}_{22}(\xi,\eta)\,d\eta + \frac{2}{\pi}\int_0^1 f_3(\eta)\mathcal{K}_{23}(\xi,\eta)\,d\eta = 0\,,$$

$$\frac{1}{2}s\frac{k}{k_2} f_3(\xi) + \frac{2}{\pi}\int_0^1 f_1(\eta)\mathcal{K}_{31}(\xi,\eta)\,d\eta + \frac{2}{\pi}\int_0^1 f_2(\eta)\mathcal{K}_{32}(\xi,\eta)\,d\eta +$$

$$+\frac{2}{\pi}\int_0^1 f_3(\eta)\mathcal{K}_{33}(\xi,\eta)\,d\eta = \frac{4}{\pi}\xi\int_0^{\pi/2} u'(\xi\sin\theta)\,d\theta\,, \tag{18}$$

where

$$u(\xi) = \frac{k_1}{C_{44}k_2}\xi^n\sigma^{(n)}(a\xi), \quad u'(\xi) = \frac{d}{d\xi}[u(\xi)], \quad s = n_2^{-1/2}d_2\left(1 - \frac{d_2\ell_2}{d_1\ell_1}\right)^{-1},$$

$$q = n_3^{-1/2}, \quad \xi = xa^{-1}, \quad \eta = ta^{-1}, \quad \beta = ha^{-1}, \quad f_1(\xi) = a^{-n-1}\frac{d}{dx}[x^n\varphi(x)],$$

$$f_2(\xi) = a^{-n-1}x^n\psi(x), \quad f_3(\xi) = a^{-n}\frac{d}{dx}[x^n\omega(x)]. \tag{19}$$

The kernels of the integral equations (18) are

$$\mathcal{K}_{11}(\xi,\eta) = n\xi^{n-1}\left\{-2s\frac{k_2}{k_1}\beta_1[\eta^{-n-1}S_n(z_{11}) - S_n(z_{21})] +\right.$$

$$+ 2s\beta_2[\eta^{-n-1}S_n(z_{12}) - S_n(z_{22})] + 2q\beta_3[\eta^{-n-1}S_n(z_{13}) -$$

$$\left.- S_n(z_{23})]\right\} + \sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\xi^{2n}\left\{-s\frac{k_2}{k_1}[R_n(2\beta_1,\eta) -\right.$$

$$- R_n(2\beta_1,1)] + s[R_n(2\beta_2,\eta) - R_n(2\beta_2,1)] +$$

$$\left.+ q[R_n(2\beta_3,\eta) - R_n(2\beta_3,1)]\right\}, \qquad \text{etc.,}$$

where

$$\beta_j = n_j^{-1/2}\beta, \qquad j = 1,2,3\,,$$

$$z_{1j} = \frac{(2\beta_j)^2 + \eta^2 + \xi^2}{2\eta\xi}, \qquad z_{2j} = \frac{(2\beta_j)^2 + 1 + \xi^2}{2\xi},$$

$$S_n(z) = \frac{1}{4}(z^2 - 1)^{-1}[Q_n(z) - zQ_n(z)], \qquad R_n(b,t) = \frac{1}{2}b(b^2 + t^2)^{-n-1},$$

and $Q_n(z)$ are Legendre functions of the second kind.

For the axisymmetric case ($n = 0$) we obtain the following solvable system of Fredholm integral equations of the second kind

$$f(\xi) + \frac{2}{\pi k}\int_0^1 f(\eta)\mathcal{K}_{11}(\xi,\eta)\,d\eta + \frac{2}{\pi k}\int_0^1 g(\eta)\mathcal{K}_{12}(\xi,\eta)\,d\eta = \frac{4}{\pi}\int_0^{\pi/2} u(\xi\sin\theta)\,d\theta\,,$$

$$g(\xi) + \frac{2}{\pi k}\int_0^1 f(\eta)\mathcal{K}_{21}(\xi,\eta)\,d\eta + \frac{2}{\pi k}\int_0^1 g(\eta)\mathcal{K}_{22}(\xi,\eta)\,d\eta = 0\,, \tag{20}$$

where $u(\xi) = -\dfrac{\xi\sigma(a\xi)}{C_{44}d_2\ell_2}$ with the kernels

$$\mathcal{K}_{11}(\xi,\eta) = k_1 I_1(2\beta_1,\eta) - k_2 I_1(2\beta_2,\eta)\,,$$

$$\mathcal{K}_{12}(\xi,\eta) = k_1\{[I_0(2\beta_1,1) - I_0(2\beta L_2,1)] - \eta^{-1}[I_0(2\beta L_1,\eta) - I_0(2\beta_2,\eta)]\}\,,$$

$$\mathcal{K}_{21}(\xi,\eta) = -k_2\xi[I_2(2\beta_1,\eta) - I_2(2\beta_2,\eta)]\,,$$

$$\mathcal{K}_{22}(\xi,\eta) = -\xi\{[k_2 I_1(2\beta_1,1) - k_1 I_1(2\beta_2,1)] - \eta^{-1}[k_2 I_1(2\beta_1,\eta) - k_1 I_1(2\beta_1,\eta)]\}\,, \tag{21}$$

where

$$I_0(\beta,\eta) = \frac{1}{4}\ln\frac{z+1}{z-1}\,,\qquad I_1(\beta,\eta) = \frac{\beta}{2\xi\eta(z^2-1)}\,,$$

$$I_2(\beta,\eta) = I_1(\beta,\eta)\left[4zI_1(\beta,\eta) - \frac{1}{\beta}\right],\qquad z = \frac{\beta^2+\xi^2+\eta^2}{2\xi\eta}\,.$$

3.2. Mode II cracks. For the axisymmetric case we get the following solvable system of Fredholm integral equations of the second kind:

$$f(\xi) - \frac{2}{\pi k}\int_0^1 f(\eta)\mathcal{K}_{11}(\xi,\eta)\,d\eta - \frac{2}{\pi k}\int_0^1 g(\eta)\mathcal{K}_{12}(\xi,\eta)\,d\eta = 0\,,$$

$$g(\xi) - \frac{2}{\pi k}\int_0^1 f(\eta)\mathcal{K}_{21}(\xi,\eta)\,d\eta - \frac{2}{\pi k}\int_0^1 g(\eta)\mathcal{K}_{22}(\xi,\eta)\,d\eta = -\frac{4}{\pi}\xi\int_0^{\pi/2} v'(\xi\sin\theta)\,d\theta\,, \tag{22}$$

where $v(\xi) = -\dfrac{\xi\tau(a\xi)}{C_{44}n_2^{-1/2}d_2}$ and the kernels $\mathcal{K}_{ij}$ take the form (21).

3.3. Mode III cracks. We obtain the solvable Fredholm integral equation of the second kind

$$f(\xi) + \frac{1}{\pi}\int_0^1 f(\eta)\mathcal{K}(\xi,\eta)\,d\eta = \frac{4}{\pi}\xi\int_0^{\pi/2} w'(\xi\sin\theta)\,d\theta,\qquad w(\xi) = \frac{\xi\tau_\theta(a\xi)}{C_{44}n_3^{-1/2}}\,, \tag{23}$$

with the kernel

$$\mathcal{K}(\xi,\eta) = 8\beta_3\xi^2\left[\frac{1}{(4\beta_3^2+\xi^2+\eta^2)^2 - 4\xi^2\eta^2} - \frac{1}{(4\beta_3^2+\xi^2+1)^2 - 4\xi^2}\right].$$

4. Stress intensity factors. Similarly to the classical case [22, 33], we determine the stress intensity factors as coefficients with singularities in the stress components near the tips of cracks

$$K_I = \lim_{r\to+a}\sqrt{2\pi(r-a)}\,Q'_{33}(r,0),\qquad K_{II} = \lim_{r\to+a}\sqrt{2\pi(r-a)}\,Q'_{3r}(r,0)\,,$$

$$K_{III} = \lim_{r\to+a}\sqrt{2\pi(r-a)}\,Q'_{3\theta}(r,0)\,. \tag{24}$$

4.1. Two parallel co-axial cracks.

4.1.1. Mode I cracks. From the solution of the system of Fredholm integral equations of the second kind (18), taking into account expressions (19), (17), (16), (15), (13), (12), and the representations of the solutions of the linearized equilibrium equations (2), we can calculate distributions of stresses and displacements in the material via the potential functions. Let us consider the values of the components of the stress tensor in the domain $y_3 \geq 0$, $r > a$ (i.e., in the plane of the crack, in domain «2»). From (2) we have

$$Q'^{(2)}_{33}(r,\theta,0) = \frac{1}{4} C_{44} \frac{k}{k_1} s \sum_{n=0}^{\infty} \cos n\theta \frac{1}{r^n \sqrt{r^2 - a^2}} \int_0^a \tilde{\omega}(t)\, dt + O(1),$$

$$Q'^{(2)}_{3r}(r,\theta,0) =$$

$$= \frac{1}{4} C_{44} \frac{k}{k_1} s \sum_{n=0}^{\infty} \cos n\theta \left[\frac{\tilde{\varphi}(a)}{a} + a^{n-1} \psi(a) \right] \frac{1}{r^{n-1} \sqrt{r^2 - a^2}} + O(1),$$

$$Q'^{(2)}_{3\theta}(r,\theta,0) =$$

$$= \frac{1}{4} C_{44} q \sum_{n=0}^{\infty} \sin n\theta \left[\frac{\tilde{\varphi}(a)}{a} - a^{n-1} \psi(a) \right] \frac{1}{r^{n-1} \sqrt{r^2 - a^2}} + O(1), \qquad (25)$$

where the regular components that do not have singularities by $r \to a$ are denoted by the symbol $O(1)$. Substituting (25) into the expressions for the stress intensity factors (24) and taking into account (19), we obtain

$$K_I = \frac{1}{4} C_{44} s \frac{k}{k_1} \sqrt{\pi a} \sum_{n=0}^{\infty} \cos n\theta \int_0^1 f_3(\eta)\, d\eta,$$

$$K_{II} = \frac{1}{4} C_{44} s \frac{k}{k_1} \sqrt{\pi a} \sum_{n=0}^{\infty} \cos n\theta \left[f_1(1) + f_2(1) \right],$$

$$K_{III} = \frac{1}{4} C_{44} q \sqrt{\pi a} \sum_{n=0}^{\infty} \sin n\theta \left[f_1(1) - f_2(1) \right], \qquad (26)$$

where the functions $f_1(\xi)$, $f_2(\xi)$ and $f_3(\xi)$ are determined from the system of Fredholm integral equations of the second kind (18).

In a similar way, in the case of axisymmetry one can obtain

$$K_I = -\frac{1}{2} C_{44} d_2 \ell_2 \sqrt{\pi a}\, f(1), \quad K_{II} = \frac{1}{2} C_{44} d_2 n_2^{-1/2} \sqrt{\pi a} \int_0^1 g(\xi)\, d\xi, \quad K_{III} = 0, \qquad (27)$$

where the functions $f(\xi)$ and $g(\xi)$ are determined from system (20).

4.1.2. Mode II cracks. In this case the expressions for SIFs coincide with (27) where functions $f(\xi)$ and $g(\xi)$ should be determined from the system of Fredholm integral equations (22).

4.2.3. Mode III cracks. The expressions for SIFs take the form

$$K_I = 0, \qquad K_{II} = 0, \qquad K_{III} = \frac{1}{2} C_{44} n_3^{-1/2} \sqrt{\pi a} \int_0^1 f(\eta)\, d\eta, \qquad (28)$$

where $f(\xi)$ should be determined from the Fredholm integral equation (23).

It follows from (27), (28) that the interaction of two parallel cracks leads to the non-trivial stress intensity factor K_{II} for Mode I cracks (when the radial shear is equal to zero) (for a Mode I crack in an infinite solid $K_{II} = 0$ [13)]). On the other hand, for two parallel cracks under radial shear and zero normal stress, K_I is nontrivial (for a Mode II crack in an infinite solid $K_I = 0$ [13]). Meanwhile, each of the SIFs K_I, K_{II} and K_{III} is affected by the initial stresses $S_{11}^0 = S_{22}^0$ (or elongation ratio $\lambda_1 = \lambda_2$) and also depends on the distance h (or β) between cracks, since the solutions of the Fredholm integral equations depend on these parameters.

4.2. Periodic set of parallel co-axial cracks. We consider an infinite body containing a periodic set of parallel coaxial penny-shaped cracks with equal radii a, which are located in parallel planes $y_3 = \text{const}$: $\{r < a,\ 0 \le \theta < 2\pi,\ y_3 = 2hn,\ n = 0, \pm 1, \pm 2, \ldots\}$ with centers on the axis Oy_3.

4.2.1. Mode I cracks. On the faces of the cracks, we presribe fields of normal tensile stresses $\sigma(r,\theta)$. Using the procedure similar to the one presented in [2, 6], we obtain the following expression for the stress intensity factors

$$K_I = -\sqrt{\pi}\, C_{44} d_1 \ell_1 \frac{k}{k_1} \sum_{n=1}^{\infty} \cos(n\theta) a^{-n-1/2} \int_0^a \tilde{\omega}_n(t)\, dt\,,$$

$$K_{II} = 0, \qquad K_{III} = 0\,,$$

where functions $\tilde{\omega}_n(t)$ should be determined from Fredholm integral equations

$$\tilde{\omega}_n(x) - \frac{2}{\pi}\int_0^a \tilde{\omega}_n(t)\mathcal{K}_n(x,t)\, dt = \frac{2}{\pi} x \int_0^{\pi/2} \Sigma_n'(x\sin\theta)\, d\theta,\ 0 \le x \le a,\ \ n = 0, 1, 2, \ldots\,,$$

$$\Sigma_n(x) = -\frac{1}{C_{44} d_1 \ell_1} \frac{k_1}{k} x^n \sigma^{(n)}(x)$$

with the kernel

$$\mathcal{K}_n(x,t) = \frac{x^{n+1/2}}{k} \int_0^{\infty} \left[t^{-n+1/2} J_{n-1/2}(\lambda t) J_{n-1/2}(\lambda x) - \right.$$

$$\left. - a^{-n+1/2} J_{n-1/2}(\lambda a) J_{n-1/2}(\lambda x)\right]\left[k_1 \frac{e^{-\mu_1}}{\operatorname{sh}\mu_1} - k_2 \frac{e^{-\mu_2}}{\operatorname{sh}\mu_2}\right]\lambda\, d\lambda\,.$$

When the load on crack faces takes the form

$$\sigma(r,\theta) = \sigma_1(r)\cos\theta\,, \tag{29}$$

by introducing the normalized on radii of cracks variables and functions

$$\xi \equiv a^{-1}x, \qquad \eta \equiv a^{-1}t, \qquad f_1(\xi) \equiv a^{-1}\tilde{\omega}_1(a\xi) = a^{-1}\tilde{\omega}_1(x)\,,$$

one can obtain the Fredholm integral equation in the dimensionless form:

$$f_1(\xi) - \frac{2}{\pi}\int_0^1 f_1(\eta)\mathcal{K}_1(\xi,\eta)\, d\eta = \frac{2}{\pi}\xi \int_0^{\pi/2} \Sigma_1'(\xi\sin\theta)\, d\theta, \qquad 0 \le \xi \le 1\,,$$

with the kernel

$$\mathcal{K}_1(\xi,\eta) = \xi\{\eta^{-1}[R(\xi-\eta) - R(\xi+\eta)] - [R(\xi-1) - R(\xi+1)]\}\,,$$

where

$$R(z) = \frac{1}{k}\left[\frac{k_1}{\beta_1}\operatorname{Re}\psi\left(1 + \frac{iz}{2\beta_1}\right) - \frac{k_2}{\beta_2}\operatorname{Re}\psi\left(1 + \frac{iz}{2\beta_2}\right)\right] \tag{30}$$

and $\operatorname{Re}\psi\left(1+\frac{iz}{2\beta_j}\right)$, $j = 1, 2$, is the real part of ψ-function $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$.

In this case, K_I takes the form

$$K_I = -\sqrt{\pi a}\, C_{44} d_1 \ell_1 \frac{k}{k_1} \int_0^1 f_1(\eta)\, d\eta \cos\theta .$$

For the problem with axisymmetry the representations for SIFs are:

$$K_I = -C_{44} d_1 \ell_1 \frac{k}{k_1} \sqrt{\pi a}\, f(1), \qquad K_{II} = 0, \qquad K_{III} = 0,$$

and function f should be determined from the Fredholm integral equation

$$f(\xi) - \frac{1}{\pi}\int_0^1 f(\eta)\mathcal{K}(\xi,\eta)\, d\eta = \frac{2}{\pi}\xi \int_0^{\pi/2} u(\xi\sin\theta)\sin\theta\, d\theta, \quad u(\xi) \equiv -\frac{k_1\sigma(a\xi)}{kC_{44}d_1\ell_1},$$

with the kernel $\mathcal{K}(\xi,\eta) = R(\xi-\eta) - R(\xi+\eta)$, where $R(z)$ is determined from (30).

4.2.2. Mode II cracks. In the case when radial stresses $\tau(r)$ are specified on crack faces, the expressions for SIFs are

$$K_I = 0, \qquad K_{II} = C_{44} d_1 n_1^{-1/2} \frac{k}{k_2} \sqrt{\pi a} \int_0^1 f(\eta)\, d\eta, \qquad K_{III} = 0,$$

where function $f(\xi)$ should be determined from the Fredholm integral equation

$$f(\xi) - \frac{1}{\pi}\int_0^1 f(\eta)\mathcal{K}(\xi,\eta)\, d\eta = \frac{2}{\pi}\xi \int_0^{\pi/2} v'(\xi\sin\theta)\, d\theta, \quad v(\xi) = -\frac{\xi k_2 \tau(a\xi)}{kC_{44}d_1 n_1^{-1/2}},$$

with the kernel

$$\mathcal{K}(\xi,\eta) = \xi\eta^{-1}[R_1(\xi-\eta) - R_1(\xi+\eta)] - \xi[R_1(\xi-1) - R_1(\xi+1)],$$

where

$$R_1(z) \equiv \frac{1}{k}\left[\frac{k_1}{\beta_2}\operatorname{Re}\psi\left(1+\frac{iz}{2\beta_2}\right) - \frac{k_2}{\beta_1}\operatorname{Re}\psi\left(1+\frac{iz}{2\beta_1}\right)\right].$$

4.2.3. Mode III cracks. We consider the case when tangential torsion loads $\tau_\theta(r)$ are applied on the cracks faces. The stress intensity factors take the form

$$K_I = 0, \qquad K_{II} = 0, \qquad K_{III} = -C_{44} n_3^{-1/2} \sqrt{\pi a} \int_0^1 f(\eta)\, d\eta,$$

and function $f(\xi)$ should be obtained from the Fredholm integral equation

$$f(\xi) - \frac{1}{\pi}\int_0^1 f(\eta)\mathcal{K}(\xi,\eta)\, d\eta = -\frac{2}{\pi}\xi \int_0^{\pi/2} w'(\xi\sin\theta)\, d\theta, \quad w(\xi) = \frac{\xi\tau_{3\theta}(a\xi)}{C_{44}n_3^{-1/2}},$$

with the kernel

$$\mathcal{K}(\xi,\eta) = \xi\beta_3^{-1}\{\eta^{-1}[R_3(\xi-\eta) - R_3(\xi+\eta)] - [R_3(\xi-1) - R_3(\xi+1)]\},$$

where

$$R_3(z) = \operatorname{Re}\psi\left(1+\frac{iz}{2\beta_3}\right).$$

5. Influence of initial stresses on stress intensity factors. Below we present the results of numerical analysis of initial stress effect on the stress intensity factors for highly elastic materials with Treloar potential [34], Bartenev – Khazanovich potential [1], the potential of harmonic type [31] (these potentials are used to model elastomers (rubber-like materials) with finite strain) and for a laminated composite consisting of two alternating isotropic layers of elastic materials. In the study of the composite, we will consider developed cracks whose minimum dimensions exceed the values of the geometric parameters characterizing the structure of the composite material, i.e., macrocracks. Also, we will only consider the processes of fracture which do not demonstrate the properties of the composite as a piecewise-homogeneous medium (the type of fracture at the interface between the media, etc.). Under these assumptions one can apply the known continuum model of the composite [17, 24] with the overall mechanical characteristics of a transversely isotropic body, the plane of isotropy is parallel to the planes of cracks location.

The parameters included in (2) for highly elastic materials are given in [6]. For the laminated composite they take the form

$$n_{1,2} = \frac{1}{2}(\mu_{13} + S_{11}^0)^{-1}(A_{11} + S_{11}^0)^{-1}\{(A_{11}A_{33} + S_{11}^0 A_{33} + S_{11}^0\mu_{13} - 2A_{13}\mu_{13} - A_{13}^2) \pm [(A_{11}A_{33} + S_{11}^0 A_{33} + S_{11}^0\mu_{13} - 2A_{13}\mu_{13} - A_{13}^2)^2 - 4(A_{11} + S_{11}^0)(\mu_{13} + S_{11}^0)\mu_{13}A_{33}]^{1/2}\},$$

$$n_3 = \mu_{13}(\mu_{12} + S_{11}^0)^{-1}, \quad m_j = [(A_{11} + S_{11}^0)n_j - \mu_{13}](A_{13} + \mu_{13})^{-1},$$

$$\ell_j = [n_j(A_{11}A_{33} + S_{11}^0 A_{33} - A_{13}^2 - A_{13}\mu_{13}) - A_{33}\mu_{13}] \times [n_j(A_{11} + S_{11}^0) + A_{13}]^{-1}(n_j)^{-1}\mu_{13}^{-1},$$

$$d_j = 1 + m_j, \quad j = 1,2, \quad C_{44} = \mu_{13},$$

where the parameters A_{ij}, μ_{ij} are expressed via technical constants in the form [17]

$$A_{11} = E(1 - \nu'\nu'')A^{-1}, \quad A_{33} = E'(1 - \nu^2)A^{-1}, \quad A_{13} = E\nu'(1 + \nu)A^{-1},$$

$$A = 1 - \nu^2 - 2\nu'\nu'' - 2\nu\nu'\nu'',$$

$$\mu_{12} = G \equiv G_{12} = \frac{1}{2}E(1 + \nu)^{-1}, \quad \mu_{13} = G' \equiv G_{13},$$

$$\nu \equiv \nu_{12}, \quad \nu' \equiv \nu_{31}, \quad \nu'' \equiv \nu_{13},$$

$$E \equiv E_1, \quad E' \equiv E_3.$$

For a periodic set of parallel coaxial Mode I cracks subjected to uniform normal tensile stress $\sigma(r,\theta) = \sigma_1(r)\cos\theta$ (where $\sigma_1(r) = \sigma = \mathrm{const}$), the dependence of the stress intensity factor ratio K_I/K_I^∞ (here K_I^∞ is the SIF for isolated Mode I crack in an infinite body, see [13]) on the initial elongation (or contraction) ratio λ_1 and on the dimensionless distance β between the cracks are shown, in Fig. 1 and Fig. 2, respectively, for the material with *Bartenev – Khazanovich potential.* It is seen that the SIF ratio depends greatly on the values of initial elongation (or contraction) ratio λ_1, especially in the area of compressive initial stress.

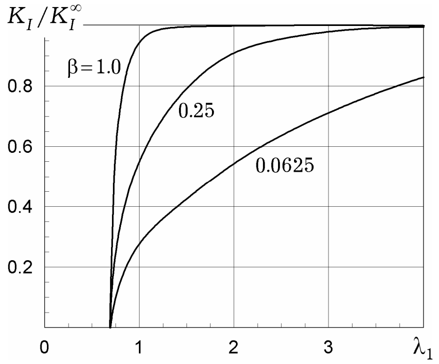

Fig. 1. Dependence of stress intensity factor ratio K_I/K_I^∞ on elongation (or contraction) ratio λ_1 for Bartenev – Khazanovich potential (periodic set of parallel coaxial Mode I cracks)

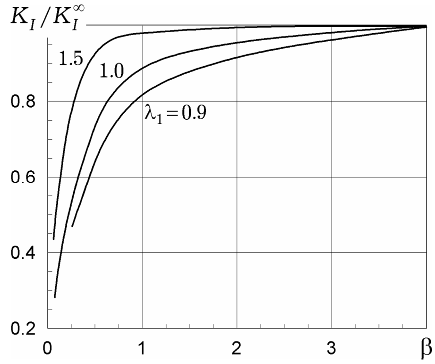

Fig. 2. Dependence of stress intensity factor ratio K_I/K_I^∞ on the dimensionless distance β between the cracks for Bartenev – Khazanovich potential (periodic set of parallel coaxial Mode I cracks)

Also, from Fig. 2 we can see that the mutual influence of cracks in the body with an initial stress leads to a decrease (especially significant in the case of small distances between cracks) in the stress intensity factor as compared to K_I^∞. E.g., for $\lambda_1 = 0.9$ the value of K_I at $\beta = 0.25$ is smaller than K_I^∞ by a factor of 2.2. With increasing distances between the cracks their mutual influence is weakening, and the corresponding values of SIF tend to the value K_I^∞.

Figs. 3 and 4 illustrate the dependence of the stress intensity factor ratio K_I/K_I^∞ on the parameter λ_1 for a material with the *Treloar potential* (with different values of β) and a material with the *potential of harmonic type* (with different values of Poisson's ratio ν and $\beta = 0.25$), respectively. As we can see in the latter picture, the compressibility of the material with the potential of harmonic type, which is characterized by Poisson's ratio, has a noticeable effect on the SIF values. Thus, for $\lambda_1 = 0.7$ the value of K_I for the Poisson's ratio $\nu = 0.1$ exceeds the value of K_I for $\nu = 0.5$ by $20\,\%$.

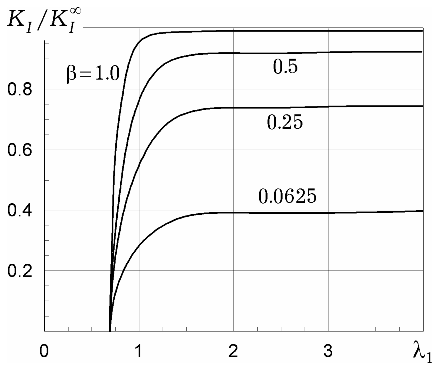

Fig. 3. Dependence of stress intensity factor ratio K_I/K_I^∞ on elongation (or contraction) ratio λ_1 for Treloar potential (periodic set of parallel Mode I cracks)

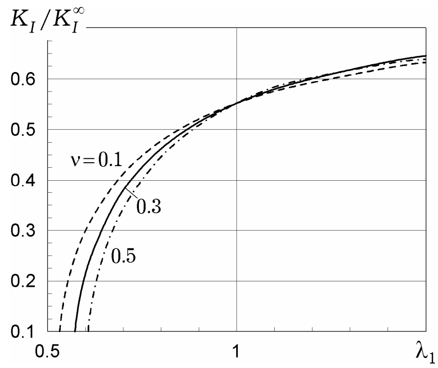

Fig. 4. Dependence of stress intensity factor ratio K_I/K_I^∞ on elongation (or contraction) ratio λ_1 for potential of harmonic type (periodic set of parallel coaxial Mode I cracks)

In Figs. 5, 6 and 7, we present the results of numerical calculations for the axisymmetric problem on a body with two parallel coaxial Mode I cracks, when the faces of the cracks are loaded by uniform normal tensile stress $\sigma(r) = \sigma = \text{const}$.

For the material with the *Bartenev – Khazanovich potential*, variations of the stress intensity factor ratio K_I/K_I^∞ and $-K_{II}/K_I^\infty$ with the initial elongation (or contraction) ratio λ_1 are shown in Fig. 5 and Fig. 6, respectively, for the values of $\beta = 0.25$, $\beta = 0.5$ and $\beta = 1.0$. In the figures we can see that the stress intensity factors also substantially depend on the initial stresses. The dependences shown in Fig. 6 have vertical asymptotes that correspond to the «resonance-like» effect, which occurs when the initial compressive stresses reach the values at which the local loss of material stability takes place in the vicinity of the crack (for the form that is symmetric with respect to the crack plane).

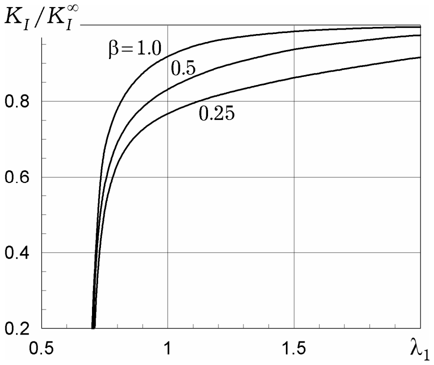

Fig. 5. Dependence of stress intensity factor ratio K_I/K_I^∞ on elongation (or contraction) ratio λ_1 for Bartenev – Khazanovich potential (two parallel coaxial Mode I cracks)

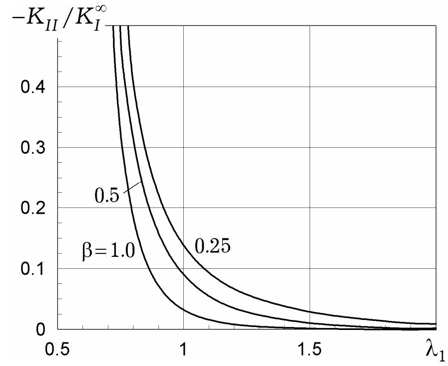

Fig. 6. Dependence of stress intensity factor ratio $-K_{II}/K_I^\infty$ on elongation (or contraction) ratio λ_1 for Bartenev – Khazanovich potential (two parallel coaxial Mode I cracks)

In Fig. 7, for the *laminated composite* made of aluminum / boron / silicate glass with epoxy-maleinic resin [17], we show the dependence of the SIFs ratio K_I/K_I^∞ on the glass concentration ratio c_1 for different values of the initial stress parameter λ_1 (for $\beta = 0.25$), indicating the influence of initial stress and physical-mechanical characteristics of the composite on SIF values.

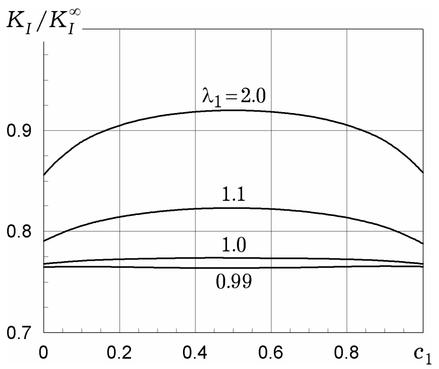

Fig. 7. Dependence of stress intensity factor ratio K_I/K_I^∞ on the glass concentration ratio c_1 for laminated composite (composition of aluminum / boron / silicate glass with epoxy-maleinic resin) (two parallel coaxial Mode I cracks).

Figs. 8, 9 and 10 illustrate the results for the case of axisymmetric problems on solids with Mode II cracks, when radial stresses $\tau(r) = \tau = \text{const}$ are specified on the crack faces.

For the material with *Bartenev – Khazanovich potential*, the dependences of the stress intensity factor ratio K_{II}/K_{II}^{∞} (here K_{II}^{∞} is the SIF for an isolated Mode II crack in an infinite body) on the initial elongation (or contraction) ratio λ_1 are shown in Fig. 8 for the values of $\beta = 0.25$, $\beta = 0.5$ and $\beta = 1.0$ (solid lines are given for the case of a periodic set of parallel coaxial cracks, dashed lines – for the case of two parallel coaxial cracks). Variations of the stress intensity factor ratio K_I/K_{II}^{∞} with the initial elongation (or contraction) ratio λ_1 for this potential are displayed in Fig. 9 (for the case of two parallel Mode II cracks). As we can see in the Figs. 8 and 9, the stress intensity factor substantially depends on the initial stresses and geometric parameters of the problem (the distance between the cracks and the radii of the cracks). Besides, Fig. 9 demonstrates that mutual influence of two parallel Mode II cracks results in non-trivial stress intensity factor K_I.

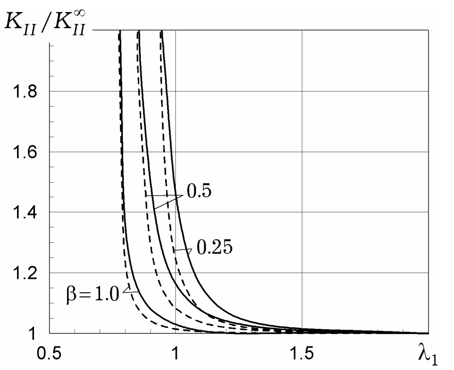

Fig. 8. Dependence of stress intensity factor ratio K_{II}/K_{II}^{∞} on elongation (or contraction) ratio λ_1 for Bartenev – Khazanovich potential (Mode II cracks)

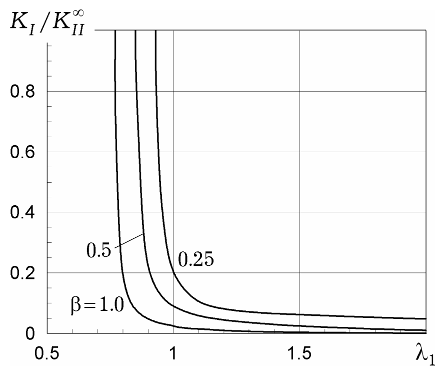

Fig. 9. Dependence of stress intensity factor ratio K_I/K_{II}^{∞} on elongation (or contraction) ratio λ_1 for Bartenev – Khazanovich potential (Mode II cracks).

The dependences shown in Figs. 8 and 9 have vertical asymptotes which correspond to abrupt increases in the SIF values when the initial compressive stresses reach the values that correspond to the local loss of stability of the material in the vicinities of the cracks.

In Fig. 10, for the periodic set of parallel cracks the dependences of the stress intensity factor ratio K_{II}/K_{II}^{∞} on the dimensionless distance β between the cracks are given for $\lambda_1 = 2.0$ (it corresponds to tensile initial stresses), $\lambda_1 = 0.9$ (this corresponds to compressive initial stresses) and $\lambda_1 = 1.0$ (the case with no initial stresses) in the case of highly elastic material with *Treloar potential*. The figure shows that the interaction between cracks leads to an increase in the value of stress intensity factor as compared to the case of an isolated crack in an infinite body. With increasing distance between cracks the values K_{II} decrease and tend to the values K_{II}^{∞}. For the values of $\beta > 3$ the mutual influence of cracks can be neglected in practical calculations, since in this case the difference of the SIF values in the vicinity of the cracks for the case of the periodic system of cracks is different from the SIF values for an isolated crack in an infinite body by less than 3 %.

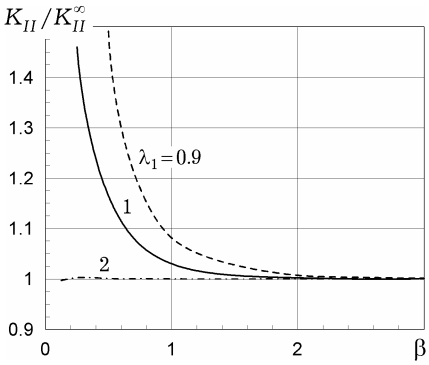

Fig. 10. Dependences of stress intensity factor ratio K_{II}/K_{II}^{∞} on the dimensionless distance β between the cracks for Treloar potential (periodic set of parallel coaxial Mode II cracks).

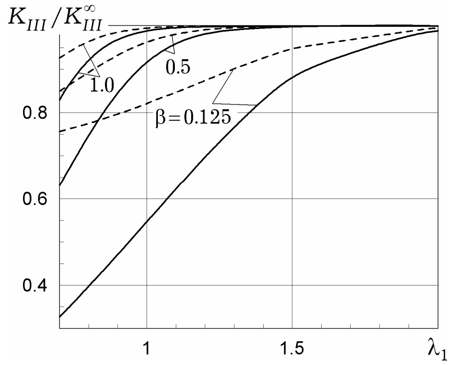

Fig. 11. Dependences of stress intensity factor ratio K_{III}/K_{III}^{∞} on elongation (or contraction) ratio λ_1 for Treloar potential (Mode III cracks).

For Mode III cracks subjected to uniform tangential stress $\tau_\theta(r) = \tau = \text{const}$, the dependences of the SIF ratio K_{III}/K_{III}^{∞} (here K_{III}^{∞} is the SIF for an isolated Mode III crack in an infinite body) on the initial elongation (or contraction) ratio λ_1 are shown in Fig. 11 (solid lines correspond to the case of a periodic set of parallel coaxial cracks, dashed lines – to the case of two parallel coaxial cracks) in the case of material with *Treloar potential*. It can be seen that the initial stresses significantly affect the values of the stress intensity factor. However, in contrast with the above cases, on pre-stressed solids with Mode I or Mode II cracks the effect of «resonance» SIF changes has not been detected in the problem on the solid with cracks under tangential torsion.

Fig. 12 for the same material illustrates the variations of the SIF ratio K_{III}/K_{III}^{∞} with the dimensionless distance between cracks β in a body with Mode III cracks (solid lines correspond to the case of a periodic set of parallel coaxial cracks, dashed lines – to the case of two parallel coaxial cracks).

In Fig. 13, for the *laminated composite*, we show the dependences of the stress intensity factor ratio K_{III}/K_{III}^{∞} on the ratio of the elasticity modules of the layers $E^{(1)}/E^{(2)}$ for $\beta = 0.25$ in the case of the periodic set of parallel cracks. The dependences are presented for different values of initial elongation (contraction) ratio λ_1. As we can see, the values K_{III}/K_{III}^{∞} increase monotonously with increasing $E^{(1)}/E^{(2)}$.

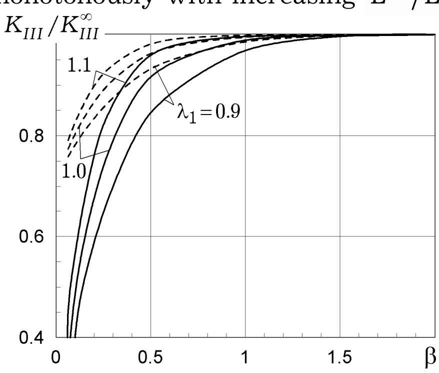

Fig. 12. Dependences of stress intensity factor ratio K_{III}/K_{III}^{∞} on the dimensionless distance β between cracks for Treloar potential (Mode III cracks).

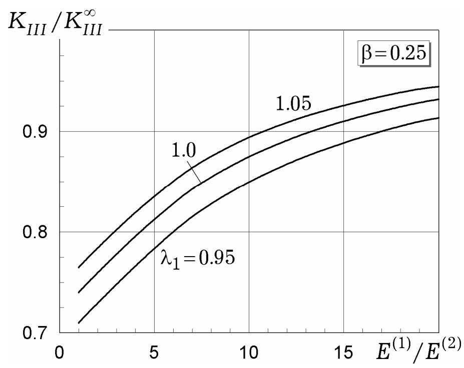

Fig. 13. Dependences of stress intensity factor ratio K_{III}/K_{III}^{∞} on $E^{(1)}/E^{(2)}$ for laminated composite (periodic set of parallel coaxial Mode III cracks).

6. Critical parameters of loading in compression. According to the method described in the Introduction, the critical parameters of compression that correspond to the local loss of stability of a material under compression along the cracks planes are determined from the solution of the above-mentioned inhomogeneous problem on the stress-strain state of a cracked solid with initial stresses as values of the initial compressive stresses whose achieving is followed by sharp «resonance-like» changes in the stress intensity factors.

Fig. 14 shows the results of calculations relying on the above method of the values of the relative critical contraction $\varepsilon_1 = 1 - \lambda_1$ of the material with *Bartenev – Khazanovich potential*, which correspond to the local instability of the body with circular cracks under a compressive load oriented parallel to the crack planes. The figure illustrates the dependence of the values ε_1 on the parameter $\beta = h/a$ characterizing the relative sizes of cracks, for the periodic set of parallel co-axial cracks (solid line), two parallel co-axial cracks (dashed line) and a single isolated crack (dotted line) (in this case, for the *Bartenev – Khazanovich potential*, the critical compressive parameter is $\varepsilon_1^* = 0.307$ [15]). Note that the results are shown for the bending (anti-symmetric) form of the local loss of stability (obtained from numerical solutions of the problems on solids containing parallel Mode II cracks), since for the symmetric form of stability loss the critical values ε_1 (obtained from numerical solutions of the problems for solids containing parallel Mode I cracks) significantly exceed the critical values for the bending form. The figure shows that the mutual influence between the cracks leads to a significant reduction of the values of relative critical contraction ε_1 (and thus lowers the values of the critical compressive load) as compared to the case of a single isolated crack in an infinite body. Furthermore, the entire range of parameter values in the case of the periodic system of cracks are higher than for the case of two parallel cracks but lower than for the case of a single isolated crack, which is consistent with physical considerations.

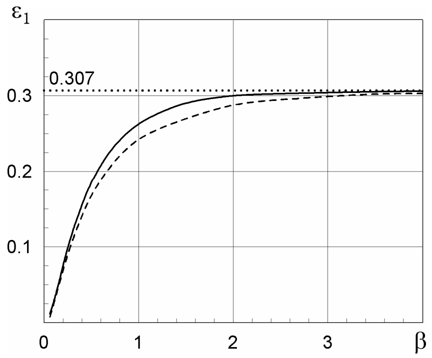

Fig. 14. Dependence of the relative critical reduction ε_1 on distance ratio β for Bartenev – Khazanovich potential.

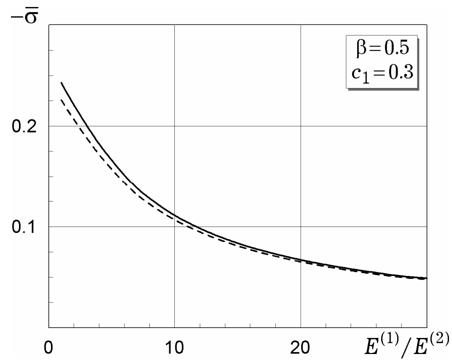

Fig. 15. Dependence of the dimensionless critical compressive stresses $\bar{\sigma} = S_{11}^0/E$ on the ratio $E^{(1)}/E^{(2)}$ for laminated composite.

Fig. 15 for the *laminated composite with isotropic layers* shows the dependence of the dimensionless critical compressive stresses $\bar{\sigma} = S_{11}^0/E$ (stresses related to the reduced modulus of elasticity of the composite under consideration) on the ratio $E^{(1)}/E^{(2)}$ of the elastic moduli of the layers (solid line is for the problem on the periodic set of cracks, dashed line – for two parallel cracks).

Table 1 for the material with the *potential of harmonic type* gives values ε_1 for different values of the dimensionless distance β between the cracks and the Poisson ratio ν, for the case of periodic set of parallel co-axial cracks. For sufficiently large values of β we obtain the values ε_1 that coincide with the critical values $\varepsilon_1^* = (2+\nu)^{-1}$ obtained in the problem on an isolated crack in an infinite body [15].

Table 1. Values of relative critical reduction ε_1 for an elastic material with the potential of harmonic type

ν \ β	0.0625	0.125	0.25	0.50	0.75	1.00	2.00	5.00	10.00
0.1	0.0159	0.0529	0.1377	0.2631	0.338	0.3842	0.4565	0.4756	0.4762
0.2	0.0145	0.0481	0.1247	0.2399	0.3107	0.3562	0.4312	0.4538	0.4545
0.3	0.0133	0.0439	0.113	0.2182	0.2849	0.3291	0.4067	0.4337	0.4347
0.4	0.0123	0.0401	0.102	0.1974	0.2597	0.3023	0.3822	0.4151	0.4166
0.5	0.0114	0.0365	0.0916	0.1769	0.2343	0.2749	0.3567	0.3975	0.3999

7. Conclusions. In this paper, the study of problems of fracture mechanics of bodies with cracks under the loadings directed along the cracks planes is carried out using the method proposed in [8, 12] within the three-dimensional linearized mechanics of solids. It, as against the methods of the classical linear fracture mechanics, has allowed to reveal the influence of the stresses oriented parallel to the crack plane on the fracture parameters. Within the above general method we proposed an approach to combined investigation of fracture mechanics problems on solids with initial stresses and problems on fracture of cracked materials with compression along cracks. Such a unified approach allows not only to investigate the effect of initial stress on the stress intensity factors, but also to effectively determine the critical parameters under solid compression along the cracks. It should be noted that the proposed approach allows the study in a uniform manner for different material models (the material model being only specified at the stage of numerical calculation of resolvant equations obtained in the general form).

The numerical results obtained for highly elastic materials with some types of elastic potential and a layered composite (modeled in continuum approximation as transversally isotropic elastic bodies) allow the following conclusions to be made:

- for all the problems considered the stress intensity factors substantially depend on the initial stresses;
- the values of stress intensity factors change abruptly (the «resonance-like» effect) when the initial reduction ratio λ_1 tends to the value at which the local loss of stability occurs in cracks vicinities. According to the combined approach proposed here this effect allows the determination of critical loading parameters in problems on solids compression along the cracks they contain;
- the mutual influence of two parallel cracks in pre-stressed infinite body leads to the nontrivial SIF K_{II} for cracks under normal stress and the nontrivial K_I for cracks under radial shear;
- the geometrical parameters of the problems (the distances between cracks and cracks radii) have a great effect both on the values of stress intensity factors and on the relative critical contractions.

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ДОСЛІДЖЕННЯ НЕКЛАСИЧНИХ ПРОБЛЕМ РУЙНУВАННЯ ПОПЕРЕДНЬО НАПРУЖЕНИХ ТІЛ ІЗ ВЗАЄМОДІЮЧИМИ ТРІЩИНАМИ

Розглянуто два типи некласичних механізмів руйнування, а саме: руйнування тіл з тріщинами в умовах дії спрямованих вздовж площин розташування тріщин початкових (залишкових) напружень і руйнування тіл при стисканні вздовж паралельних тріщин. При дослідженні неосесиметричних та осесиметричних задач для нескінченних тіл, що містять дві паралельні співвісні тріщини чи періодичну систему співвісних паралельних тріщин, застосовується комбінований аналітично-чисельний метод в рамках тривимірної лінеаризованої механіки деформівних тіл. З використанням подань напружень та переміщень в рамках лінеаризованої теорії через гармонічні потенціальні функції та шляхом застосування інтегральних перетворень Фур'є – Ганкеля задачі зводяться до розв'язуючих інтегральних рівнянь Фредгольма другого роду. Підхід дозволяє проводити дослідження задач в єдиній загальній формі для стисливих і нестисливих однорідних ізотропних або трансверсально-ізотропних тіл, а конкретизація моделі матеріалу здійснюється лише на стадії чисельного розв'язку отримуваних в загальній формі розв'язуючих рівнянь. Проаналізовано вплив початкових напружень на коефіцієнти інтенсивності напружень для високоеластичних матеріалів та шаруватих композитів, які моделюються трансверсально-ізотропними тілами. При досягненні стискаючими початковими напруженнями значень, що відповідають локальній втраті стійкості матеріалу в околі тріщин, проявляються «резонансоподібні» ефекти, які, відповідно до зазначеного комбінованого методу, дозволяють визначати критичні (граничні) параметри навантаження при стисканні тіла вздовж тріщин. Сформульовано висновки про залежність коефіцієнтів інтенсивності напружень та критичних (граничних) параметрів стиску від геометричних параметрів задач та фізико-механічних характеристик матеріалів.

ИССЛЕДОВАНИЕ НЕКЛАССИЧЕСКИХ ПРОБЛЕМ РАЗРУШЕНИЯ ПРЕДВАРИТЕЛЬНО НАПРЯЖЕННЫХ ТЕЛ СО ВЗАИМОДЕЙСТВУЮЩИМИ ТРЕЩИНАМИ

Рассмотрены два типа неклассических механизмов разрушения, а именно: разрушение тел с трещинами в условиях действия направленных вдоль плоскостей расположения трещин начальных (остаточных) напряжений и разрушение тел при сжатии вдоль параллельных трещин. При исследовании неосесимметричных и осесимметричных задач для бесконечных тел, содержащих две параллельные соосные трещины или периодическую систему соосных параллельных трещин, применяется комбинированный аналитико-числовой метод в рамках трехмерной линеаризированной механики деформируемых тел. С использованием представлений напряжений и перемещений в рамках линеаризированной теории через гармонические потенциальные функции и путем применения интегральных преобразований Фурье – Ханкеля, задачи сводятся к разрешающим интегральным уравнениям Фредгольма второго рода. Подход позволяет проводить исследование задач в единой общей форме для сжимаемых и несжимаемых однородных изотропных или трансверсально-изотропных упругих тел, а конкретизация модели материала осуществляется лишь на стадии численного решения получаемых в общей форме разрешающих уравнений. Проанализировано влияние начальных напряжений на коэффициенты интенсивности напряжений для высокоэластических материалов и слоистых композитов, моделируемых трансверсально-изотропными упругими телами. При достижении сжимающими начальными напряжениями значений, соответствующих локальной потере устойчивости материала в окрестности трещин, обнаруживаются «резонасоподобные» эффекты, которые, согласно указанному комбинированному методу, позволяют определять критические (предельные) параметры нагружения при сжатии тела вдоль трещин. Сформулированы выводы о зависимости коэффициентов интенсивности напряжений и критических (предельных) параметров сжатия от геометрических параметров задач и физико-механических характеристик материалов.

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