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To the 60-th anniversary of our teacher and our scientific colleague I. I. Guran

O. V. Gutik[1], O. V. Ravsky[2]

PSEUDOCOMPACTNESS, PRODUCTS AND TOPOLOGICAL BRANDT λ^0-EXTENSIONS OF SEMITOPOLOGICAL MONOIDS

In the paper we study the preservation of pseudocompactness (respectively, countable compactness, sequential compactness, ω-boundedness, totally countable compactness, countable pracompactness, sequential pseudocompactness) by Tychonoff products of pseudocompact (and countably compact) topological Brandt λ_i^0-extensions of semitopological monoids with zero. In particular we show that if $\{(B^0_{\lambda_i}(S_i),\tau^0_{B(S_i)}) : i\in\mathcal{I}\}$ is a family of Hausdorff pseudocompact topological Brandt λ_i^0-extensions of pseudocompact semitopological monoids with zero such that the Tychonoff product $\prod\{S_i : i\in\mathcal{I}\}$ is a pseudocompact space then the direct product $\prod\{(B^0_{\lambda_i}(S_i),\tau^0_{B(S_i)}) : i\in\mathcal{I}\}$ endowed with the Tychonoff topology is a Hausdorff pseudocompact semitopological semigroup.

Introduction and preliminaries. Further we shall follow the terminology of [7, 9, 12, 25, 27]. Via $\mathbb{N}$ we shall denote the set of all positive integers.

A semigroup is a non-empty set with a binary associative operation. A semigroup S is called *inverse* if for any $x\in S$ there exists a unique $y\in S$ such that $x\cdot y\cdot x = x$ and $y\cdot x\cdot y = y$. Such the element y in S is called *inverse* of x and is denoted by x^{-1}. The map assigning to each element x of an inverse semigroup S its inverse x^{-1} is called the *inversion*.

For a semigroup S by $E(S)$ we denote the subset of idempotents of S, and by S^1 (respectively, S^0) we denote the semigroup S with the adjoined unit (respectively, zero) (see [9, Section 1.1]). Also if a semigroup S has zero 0_S, then for any $A\subseteq S$ we denote $A^* = A\setminus\{0_S\}$.

For a semilattice E the semilattice operation on E determines the partial order $\leq$ on E:

$$e\leq f \quad\text{if and only if}\quad ef = fe = e.$$

This order is called *natural*. An element e of a partially ordered set X is called *minimal* if $f\leq e$ implies $f = e$ for $f\in X$. An idempotent e of a semigroup S without zero (with zero) is called *primitive* if e is a minimal element in $E(S)$ (in $(E(S))^*$).

Let S be a semigroup with zero and $\lambda\geq 1$ be a cardinal. On the set $B_\lambda(S) = (\lambda\times S\times\lambda)\sqcup\{0\}$ we define a semigroup operation as follows

$$(\alpha,a,\beta)\cdot(\gamma,b,\delta) = \begin{cases}(\alpha,ab,\delta), & \beta=\gamma,\\ 0, & \beta\neq\gamma,\end{cases}$$

and $(\alpha,a,\beta)\cdot 0 = 0\cdot(\alpha,a,\beta) = 0\cdot 0 = 0$, for all $\alpha,\beta,\gamma,\delta\in\lambda$ and $a,b\in S$. If S is a monoid, then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup S* [1]. Obviously, $\mathcal{J} = \{0\}\cup\{(\alpha,\mathcal{O},\beta) : \mathcal{O}$ is the zero of $S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S) = B_\lambda(S)/\mathcal{J}$ and we shall call $B^0_\lambda(S)$ the *Brandt λ^0-extension of the semigroup S with zero* [16]. Further, if $A\subseteq S$ then we shall denote $A_{\alpha,\beta} = \{(\alpha,s,\beta) : s\in A\}$ if A does not contain zero, and $A_{\alpha,\beta} =$

$= \{(\alpha, s, \beta) : s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in \lambda$. If $\mathcal{I}$ is a trivial semigroup (i.e., $\mathcal{I}$ contains only one element), then by $\mathcal{I}^0$ we denote the semigroup $\mathcal{I}$ with the adjoined zero. Obviously, for any $\lambda \geq 2$ the Brandt λ^0-extension of the semigroup $\mathcal{I}^0$ is isomorphic to the semigroup of $\lambda \times \lambda$-matrix units and any Brandt λ^0-extension of a semigroup with zero contains the semigroup of $\lambda \times \lambda$-matrix units. Further by B_λ we shall denote the semigroup of $\lambda \times \lambda$-matrix units and by $B^0_\lambda(1)$ the subsemigroup of $\lambda \times \lambda$-matrix units of the Brandt λ^0-extension of a monoid S with zero.

A semigroup S with zero is called 0*-simple* if $\{0\}$ and S are its only ideals and $S^2 \neq \{0\}$, and *completely* 0*-simple* if it is 0-simple and has a primitive idempotent [9]. A completely 0-simple inverse semigroup is called a *Brandt semigroup* [25]. By Theorem II.3.5 of [25], a semigroup S is a Brandt semigroup if and only if S is isomorphic to a Brandt λ-extension $B_\lambda(G)$ of a group G.

A non-trivial inverse semigroup is called a *primitive inverse semigroup* if all its non-zero idempotents are primitive [25]. A semigroup S is a primitive inverse semigroup if and only if S is an orthogonal sum of Brandt semigroups [25, Theorem II.4.3].

In this paper all topological spaces are Hausdorff. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\mathrm{cl}_Y(A)$ and $\mathrm{int}_Y(A)$ we denote the topological closure and interior of A in Y, respectively.

A subset A of a topological space X is called *regular open* if $\mathrm{int}_X(\mathrm{cl}_X(A)) = A$.

We recall that a topological space X is said to be

- *semiregular* if X has a base consisting of regular open subsets;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_i\}_{i\in\mathbb{N}}$ of X has a convergent subsequence in X;
- ω*-bounded* if every countably infinite set in X has the compact closure [15];
- *totally countably compact* if every countably infinite set in X contains an infinite subset with the compact closure [14];
- *countably compact* if each open countable cover of X has a finite subcover;
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A [4];
- *sequentially pseudocompact* if for each sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of the space X there exist a point $x \in X$ and an infinite set $S \subset \mathbb{N}$ such that for each neighborhood U of the point x the set $\{n \in S : U_n \cap U = \varnothing\}$ is finite [21];
- H*-closed* if X is Hausdorff and X is a closed subspace of every Hausdorff space in which it is contained [3];
- *pseudocompact* if each locally finite open cover of X is finite.

According to Theorem 3.10.22 of [12], a Tychonoff topological space X is pseudocompact if and only if each continuous real-valued function on X is bounded. Also, a Hausdorff topological space X is pseudocompact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is pseudocompact (see [4]). We observe that pseudocompact spaces in topological literature also are called *lightly compact* or *feebly compact* (see [5, 13. 28]).

We recall that the Stone – Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f} : \beta X \to Y$ [12].

A (*semi*)*topological semigroup* is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [11, Proposition II.1]). A Hausdorff topology τ on a (inverse) semigroup S is called (*inverse*) *semigroup* if (S,τ) is a topological (inverse) semigroup. A *paratopological* (*semitopological*) *group* is a Hausdorff topological space with a jointly (separately) continuous group operation. A paratopological group with continuous inversion is a *topological group*.

Let $\mathfrak{STSG}_0$ be a class of semitopological semigroups.

Definition 1 [**1**]. Let $\lambda \ge 1$ be a cardinal and $(S,\tau) \in \mathfrak{STSG}_0$ be a semitopological monoid with zero. Let τ_B be a topology on $B_\lambda(S)$ such that

(***a***) $(B_\lambda(S), \tau_B) \in \mathfrak{STSG}_0$;

(***b***) for some $\alpha \in \lambda$ the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to (S,τ).

Then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt* λ*-extension* of (S,τ) in $\mathfrak{STSG}_0$.

Definition 2 [**16**]. Let $\lambda \ge 1$ be a cardinal and $(S,\tau) \in \mathfrak{STSG}_0$. Let τ_B be a topology on $B^0_\lambda(S)$ such that

(***a***) $(B^0_\lambda(S), \tau_B) \in \mathfrak{STSG}_0$;

(***b***) the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to (S,τ) for some $\alpha \in \lambda$.

Then $(B^0_\lambda(S), \tau_B)$ is called a *topological Brandt* λ^0*-extension* of (S,τ) in $\mathfrak{STSG}_0$.

Later, if $\mathfrak{STSG}_0$ coincides with the class of all semitopological semigroups we shall say that $(B^0_\lambda(S), \tau_B)$ (respectively, $(B_\lambda(S), \tau_B)$) is called a *topological Brandt* λ^0*-extension* (respectively, a *topological Brandt* λ*-extension*) of (S,τ).

Algebraic properties of Brandt λ^0-extensions of monoids with zero, non-trivial homomorphisms between them, and a category whose objects are ingredients of the construction of such extensions were described in [22]. Also, in [19] and [22] a category whose objects are ingredients in the constructions of finite (respectively, compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros were described.

Gutik and Repovš proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-

extension $B_\lambda(H)$ of a countably compact topological group H in the class of all topological inverse semigroups for some finite cardinal $\lambda \geq 1$ [23]. Also, every 0-simple pseudocompact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(H)$ of a pseudocompact topological group H in the class of all topological inverse semigroups for some finite cardinal $\lambda \geq 1$ [2]. Next Gutik and Repovš showed in [23] that the Stone – Čech compactification $\beta(T)$ of a 0-simple countably compact topological inverse semigroup T has a natural structure of a 0-simple compact topological inverse semigroup. It was proved in [2] that the same is true for 0-simple pseudocompact topological inverse semigroups.

In the paper [6] the structure of compact and countably compact primitive topological inverse semigroups was described and was showed that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

Comfort and Ross in [10] proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact topological groups is a pseudocompact topological group. Also, they proved there that the Stone – Čech compactification of a pseudocompact topological group has a natural structure of a compact topological group. Ravsky in [26] generalized Comfort – Ross Theorem and proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact paratopological groups is pseudocompact.

In the paper [17] it is described the structure of pseudocompact primitive topological inverse semigroups and it is shown that the Tychonoff product of an arbitrary non-empty family of pseudocompact primitive topological inverse semigroups is pseudocompact. Also, there is proved that the Stone – Čech compactification of a pseudocompact primitive topological inverse semigroup has a natural structure of a compact primitive topological inverse semigroup.

In the paper [20] we studied the structure of inverse primitive pseudocompact semitopological and topological semigroups. We found conditions when a maximal subgroup of an inverse primitive pseudocompact semitopological semigroup S is a closed subset of S and described the topological structure of such semiregular semigroup. Also there we described structure of pseudocompact topological Brandt λ^0-extensions of topological semigroups and semiregular (quasi-regular) primitive inverse topological semigroups. In [20] we showed that the inversion in a quasi-regular primitive inverse pseudocompact topological semigroup is continuous. Also there, an analogue of Comfort – Ross Theorem is proved for such semigroups: the Tychonoff product of an arbitrary non-empty family of primitive inverse semiregular pseudocompact semitopological semigroups with closed maximal subgroups is a pseudocompact space, and we described the structure of the Stone – Čech compactification of a Hausdorff primitive inverse countably compact semitopological semigroup S such that every maximal subgroup of S is a topological group.

In this paper we study the preserving of Tychonoff products of the pseudocompactness (respectively, countable compactness, sequential compactness, ω-boundedness, totally countable compactness, countable pracompactness, sequential pseudocompactness) by pseudocompact (and countably compact) topological Brandt λ_i^0-extensions of semitopological monoids with zero. In particular we show that if $\{(B^0_{\lambda_i}(S_i), \tau^0_{B(S_i)}) : i \in \mathcal{I}\}$ is a family of Hausdorff pseudocompact topological Brandt λ_i^0-extensions of pseudocompact semitopological monoids with zero such that the Tychonoff product $\prod\{S_i : i \in \mathcal{I}\}$ is a pseudocompact space, then the direct product $\prod\{(B^0_{\lambda_i}(S_i), \tau^0_{B(S_i)}) : i \in \mathcal{I}\}$ with the Tychonoff topology is a Hausdorff pseudocompact semitopological semigroup.

Tychonoff products of pseudocompact topological Brandt λ^0-extensions of semitopological semigroups.

Later we need the following

Theorem 1 [**18**, Theorem 12]. *For any Hausdorff countably compact semitopological monoid (S,τ) with zero and for any cardinal $\lambda \geq 1$ there exists a unique Hausdorff countably compact topological Brandt λ^0-extension $(B^0_\lambda(S),\tau^S_B)$ of (S,τ) in the class of semitopological semigroups, and the topology τ^S_B is generated by the base $\mathscr{B}_B = \bigcup\{\mathscr{B}_B(t) : t \in B^0_\lambda(S)\}$, where:*

(*i*) $\mathscr{B}_B(t) = \{(U(s)\setminus\{0_S\})_{\alpha,\beta} : U(s) \in \mathscr{B}_S(s)\}$*, where* $t = (\alpha, s, \beta)$ *is a non-zero element of* $B^0_\lambda(S)$, $\alpha,\beta \in \lambda$;

(*ii*) $\mathscr{B}_B(0) = \left\{ U_F(0) = \bigcup_{(\alpha,\beta)\in(\lambda\times\lambda)\setminus F} S_{\alpha,\beta} \cup \bigcup_{(\gamma,\delta)\in F} (U(0_S))_{\gamma,\delta} : F \right.$ *is a finite subset of* $\lambda\times\lambda$ *and* $\left. U(0_S) \in \mathscr{B}_S(0_S) \right\}$*, where* 0 *is the zero of* $B^0_\lambda(S)$*, and* $\mathscr{B}_S(s)$ *is a base of the topology* τ *at the point* $s \in S$.

Lemma 1. *For any Hausdorff sequentially compact semitopological monoid (S,τ) with zero and for any cardinal $\lambda \geq 1$ the Hausdorff countably compact topological Brandt λ^0-extension $(B^0_\lambda(S),\tau^S_B)$ of (S,τ) in the class of semitopological semigroups is a sequentially compact space.*

P r o o f. In the case when $\lambda < \omega$ the statement of the lemma follows from Theorems 3.10.32 and 3.10.34 from [12].

Next we suppose that $\lambda \geq \omega$. Let $\mathcal{A}(\lambda)$ be the one point Alexandroff compactification of the discrete space of cardinality λ. Then $\mathcal{A}(\lambda)$ is scattered because $\mathcal{A}(\lambda)$ has only one non-isolated point, and hence by Theorem 5.7 from [30] the space $\mathcal{A}(\lambda)$ is sequentially compact. Since cardinal λ is infinite without loss of generality we can assume that $\lambda = \lambda\cdot\lambda$ and hence we can identify the space $\mathcal{A}(\lambda)$ with $\mathcal{A}(\lambda\times\lambda)$. Then by Theorem 3.10.35 from [12] the space $\mathcal{A}(\lambda\times\lambda)\times S$ is sequentially compact. Later we assume that a is non-isolated point of the space $\mathcal{A}(\lambda\times\lambda)$. We define the map $g : \mathcal{A}(\lambda\times\lambda)\times S \to B^0_\lambda(S)$ by the formulae

$$g(a) = 0 \quad \text{and} \quad g((\alpha,\beta,s)) = \begin{cases} (\alpha,s,\beta), & s \in S\setminus\{0_s\}, \\ 0, & s = 0_S. \end{cases}$$

Theorem 1 implies that so defined map g is continuous and hence by Theorem 3.10.32 of [12] we get that the topological Brandt λ^0-extension $(B^0_\lambda(S),\tau^S_B)$ of (S,τ) in the class of semitopological semigroups is a sequentially compact space. ♦

Lemma 1 and Theorem 3.10.35 from [12] imply the following

Theorem 2. *Let $\{B^0_{\lambda_i}(S_i) : i \in \omega\}$ be a countable family of Hausdorff countably compact topological Brandt λ^0_i-extensions of sequentially compact Hausdorff semitopological monoids. Then the direct product $\prod\{B^0_{\lambda_i}(S_i) : i\in\omega\}$ with the Tychonoff topology is a Hausdorff sequentially compact semitopological semigroup.*

Theorem 3. *Let $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ be a non-empty family of Hausdorff countably compact topological Brandt λ^0_i-extensions of countably compact*

Hausdorff semitopological monoids such that the Tychonoff product $\prod\{S_i : i \in \mathcal{I}\}$ *is a countably compact space. Then the direct product* $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ *with the Tychonoff topology is a Hausdorff countably compact semitopological semigroup.*

P r o o f. For every infinite cardinal λ_i, $i \in \mathcal{I}$, we shall repeat the construction proposed in the proof of Lemma 1. Let $\mathcal{A}(\lambda_i)$ be the one-point Alexandroff compactification of the discrete space of cardinality λ_i. Since cardinal λ_i is infinite without loss of generality we can assume that $\lambda_i = \lambda_i \cdot \lambda_i$ and hence we can identify the space $\mathcal{A}(\lambda_i)$ with $\mathcal{A}(\lambda_i \times \lambda_i)$. Later we assume that a_i is a non-isolated point of the space $\mathcal{A}(\lambda_i \times \lambda_i)$. We define the map $g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i)$ by the formulae

$$g_i(a_i) = 0_i \quad \text{and} \quad g_i((\alpha_i, \beta_i, s_i)) = \begin{cases} (\alpha_i, s_i, \beta_i), & s_i \in S \setminus \{0_{S_i}\}, \\ 0_i, & s = 0_{S_i}, \end{cases} \tag{1}$$

where 0_i and 0_{S_i} are zeros of the semigroup $B^0_{\lambda_i}(S_i)$ and the monoid S_i, respectively. Theorem 1 implies that so defined map g_i is continuous.

In the case when cardinal λ_i, $i \in \mathcal{I}$, is finite we put $\mathcal{A}(\lambda_i \times \lambda_i)$ is the discrete space of cardinality $\lambda_i^2 + 1$ with the fixed point $a_i \in \mathcal{A}(\lambda_i \times \lambda_i)$. Next we define the map $g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i)$ by the formulae (1), where 0_i and 0_{S_i} are zeros of the semigroup $B^0_{\lambda_i}(S_i)$ and the monoid S_i, respectively. Obviously, such defined map g_i is continuous. Then the space $\prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is homeomorphic to $\prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i\in\mathcal{I}} S_i$ and hence by Theorem 3.2.4 and Corollary 3.10.14 from [12] the Tychonoff product $\prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is countably compact. Later we define the map $g : \prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to \prod_{i\in\mathcal{I}} B^0_{\lambda_i}(S_i)$ by putting $g = \prod_{i\in\mathcal{I}} g_i$. Since for any $i \in \mathcal{I}$ the map $g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i)$ is continuous, Theorem 1 and Proposition 2.3.6 of [12] imply that g is continuous too. Therefore by Theorem 3.10.5 from [12] we obtain that the direct product $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a Hausdorff countably compact semitopological semigroup. ♦

Lemma 2. *For any Hausdorff totally countably compact semitopological monoid* (S, τ) *with zero and for any cardinal* $\lambda \geq 1$ *the Hausdorff countably compact topological Brandt* λ^0*-extension* $(B^0_\lambda(S), \tau^S_B)$ *of* (S, τ) *in the class of semitopological semigroups is a totally countably compact space.*

P r o o f. In the case when $\lambda < \omega$ the statement of the lemma is trivial. So we suppose that $\lambda \geq \omega$.

Let A be an arbitrary countably infinite subset of $(B^0_\lambda(S), \tau^S_B)$. Put $\mathcal{J} = \{(\alpha, \beta) \in \lambda \times \lambda : A \cap S_{\alpha,\beta} \neq \varnothing\}$. If the set $\mathcal{J}$ is finite then total countable compactness of the space (S, τ) and Lemma 2 of [18] imply the statement of the lemma. So we suppose that the set $\mathcal{J}$ is infinite. For each pair of indices $(\alpha, \beta) \in \mathcal{J}$ we choose a point $a_{\alpha,\beta} \in A \cap S_{\alpha,\beta}$ and put $K = \{0\} \cup \{a_{\alpha,\beta} : (\alpha, \beta) \in \mathcal{J}\}$.

Then the definition of the topology τ_B^S on $B_\lambda^0(S)$ implies that K is a compact subset of the $(B_\lambda^0(S),\tau_B^S)$ and $K\bigcap A$ is infinite. This completes the proof of the lemma. ♦

Lemma 2 and Theorem 4.3 from [14] imply the following

Theorem 4. *Let* $\{B_{\lambda_i}^0(S_i): i\in\omega\}$ *be a countable family of Hausdorff countably compact topological Brandt* λ_i^0*-extensions of totally countably compact Hausdorff semitopological monoids. Then the direct product* $\prod\{B_{\lambda_i}^0(S_i): i\in\omega\}$ *with the Tychonoff topology is a Hausdorff totally countably compact semitopological semigroup.*

Theorem 5. *Let* $\{B_{\lambda_i}^0(S_i): i\in\mathcal{I}\}$ *be a non-empty family of Hausdorff countably compact topological Brandt* λ_i^0*-extensions of Hausdorff totally countably compact semitopological monoids such that the Tychonoff product* $\prod\{S_i : i\in\mathcal{I}\}$ *is a totally countably compact space. Then the direct product* $\prod\{B_{\lambda_i}^0(S_i): i\in\mathcal{I}\}$ *with the Tychonoff topology is a totally countably compact semitopological semigroup.*

P r o o f. Let for every $i\in\mathcal{I}$, $\mathcal{A}(\lambda_i\times\lambda_i)$ be a space and $g_i:\mathcal{A}(\lambda_i\times\lambda_i)\times S_i\to B_{\lambda_i}^0(S_i)$ be a map defined in the proof of Theorem 3. Also, Theorem 1 implies that the map g_i is continuous for every $i\in\mathcal{I}$. Since the space $\prod_{i\in\mathcal{I}}\mathcal{A}(\lambda_i\times\lambda_i)\times S_i$ is homeomorphic to $\prod_{i\in\mathcal{I}}\mathcal{A}(\lambda_i\times\lambda_i)\times\prod_{i\in\mathcal{I}}S_i$ and by Theorem 4.3 from [14] we see that the Tychonoff product $\prod_{i\in\mathcal{I}}\mathcal{A}(\lambda_i\times\lambda_i)\times S_i$ is a totally countably compact space. Then by Theorem 1 and Proposition 2.3.6 of [12] the map $g:\prod_{i\in\mathcal{I}}\mathcal{A}(\lambda_i\times\lambda_i)\times S_i\to\prod_{i\in\mathcal{I}}B_{\lambda_i}^0(S_i)$ defined by the formula $g=\prod_{i\in\mathcal{I}}g_i$ is continuous. Simple verification implies that a continuous image of a totally countably compact space is a totally countably compact space too. Hence the direct product $\prod\{B_{\lambda_i}^0(S_i): i\in\mathcal{I}\}$ with the Tychonoff topology is a totally countably compact semitopological semigroup. ♦

Similarly to the proof of Lemma 2 we can prove the following

Lemma 3. *For any Hausdorff* ω*-bounded semitopological monoid* (S,τ) *with zero and for any cardinal* $\lambda\geq 1$ *the Hausdorff countably compact topological Brandt* λ^0*-extension* $(B_\lambda^0(S),\tau_B^S)$ *of* (S,τ) *in the class of semitopological semigroups is an* ω*-bounded space.*

Since by Lemma 4 of [15] the Tychonoff product of an arbitrary non-empty family of ω-bounded spaces is an ω-bounded space, similarly to the proof of Theorem 5 we can prove the following

Theorem 6. *Let* $\{B_{\lambda_i}^0(S_i): i\in\mathcal{I}\}$ *be a non-empty family of Hausdorff countably compact topological Brandt* λ_i^0*-extensions of Hausdorff* ω*-bounded semitopological monoids. Then the direct product* $\prod\{B_{\lambda_i}^0(S_i): i\in\mathcal{I}\}$ *with the Tychonoff topology is an* ω*-bounded semitopological semigroup.*

Theorems 1 and 6 imply the following

Corollary 1. *Let* $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ *be a non-empty family of Hausdorff totally countably compact topological Brandt* λ_i^0*-extensions of Hausdorff* ω*-bounded semitopological monoids. Then the direct product* $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ *with the Tychonoff topology is an* ω*-bounded semitopological semigroup.*

Later we shall use the following

Theorem 7 [**18**, Theorem 15]. *For any semiregular pseudocompact semitopological monoid* (S,τ) *with zero and for any cardinal* $\lambda \ge 1$ *there exists a unique semiregular pseudocompact topological Brandt* λ^0*-extension* $(B^0_\lambda(S), \tau^S_B)$ *of* (S,τ) *in the class of semitopological semigroups, and the topology* τ^S_B *is generated by the base* $\mathscr{B}_B = \bigcup\{\mathscr{B}_B(t) : t \in B^0_\lambda(S)\}$, *where:*

(***i***) $\mathscr{B}_B(t) = \{(U(s) \setminus \{0_S\})_{\alpha,\beta} : U(s) \in \mathscr{B}_S(s)\}$, *where* $t = (\alpha, s, \beta)$ *is a non-zero element of* $B^0_\lambda(S)$, $\alpha, \beta \in \lambda$;

(***ii***) $\mathscr{B}_B(0) = \{U_F(0) = \bigcup_{(\alpha,\beta)\in(\lambda\times\lambda)\setminus F} S_{\alpha,\beta} \cup \bigcup_{(\gamma,\delta)\in F} (U(0_S))_{\gamma,\delta} : F$ *is a finite subset of* $\lambda \times \lambda$ and $U(0_S) \in \mathscr{B}_S(0_S)\}$, *where* 0 *is the zero of* $B^0_\lambda(S)$, *and* $\mathscr{B}_S(s)$ *is a base of the topology* τ *at the point* $s \in S$.

Theorem 8. *Let* $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ *be a non-empty family of semiregular pseudocompact topological Brandt* λ_i^0*-extensions of semiregular pseudocompact semitopological monoids such that the Tychonoff product* $\prod\{S_i : i \in \mathcal{I}\}$ *is a pseudocompact space. Then the direct product* $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ *with the Tychonoff topology is a semiregular pseudocompact semitopological semigroup.*

P r o o f. Since by Lemma 20 from [26] the Tychonoff product of regular open sets is regular open we obtain that the Tychonoff product of semiregular topological spaces is semiregular.

Let for every $i \in \mathcal{I}$, $\mathcal{A}(\lambda_i \times \lambda_i)$ be a space and $g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i)$ be the map defined in the proof of Theorem 3. Theorem 7 implies that the map g_i is continuous for every $i \in \mathcal{I}$. Since the space $\prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is homeomorphic to $\prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i\in\mathcal{I}} S_i$, Theorem 3.2.4 from [12] and Corollary 9 from [20] imply that the Tychonoff product $\prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is a pseudocompact space. Then by Theorem 7 and Proposition 2.3.6 of [12] the map $g : \prod_{i\in\mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to \prod_{i\in\mathcal{I}} B^0_{\lambda_i}(S_i)$ defined by the formula $g = \prod_{i\in\mathcal{I}} g_i$ is continuous, and hence the direct product $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a semiregular pseudocompact semitopological semigroup. ♦

Proposition 1. *Let* X, Y *be Hausdorff countably pracompact spaces. Then the product* $X \times Y$ *is countably pracompact provided* Y *is a* k*-space or sequentially compact.*

P r o o f. Let the space X be countably compact at its dense subset D_X and the space Y be countably compact at its dense subset D_Y. The set $D_X \times D_Y$ is a dense subset of the space $X \times Y$. We claim that the space $X \times Y$ is countably compact at the set $D_X \times D_Y$. Indeed, let $A = \{(x_s, y_s) : s \in S\}$ be an infinite subset of the set $D_X \times D_Y$ such that $(x_s, y_s) \ne (x_{s'}, y_{s'})$ provided $s \ne s'$. Assume that the set A has no accumulation point in the space X.

If Y is a k-space then Lemma 3.10.12 from [12] implies that there exists an infinite subset $S_0 \subset S$ such that either the set $\{x_s : s \in S_0\}$ or the set $\{y_s : s \in S_0\}$ has no accumulation point. Then this set is finite. Without loss of generality, we can assume that there are a point $x \in X$ and an infinite subset S_1 of the set S_0 such that $x_s = x$ for each index $s \in S_1$. Since the space Y is countably compact at the set D_Y, there exists an accumulation point $y \in Y$ of the set $\{y_s : s \in S_1\}$. Then the point (x, y) is an accumulation point of the set $\{(x_s, y_s) : s \in S_1\}$, a contradiction. ♦

If Y is a sequentially compact space then the proof of the claim is similar to the proof of Theorem 3.10.36 from [12].

Proposition 1 implies the following

Corollary 2. *The product $X \times Y$ of Hausdorff countably pracompact space X and compactum Y is countably pracompact.*

Theorem 9. *Let $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ be a non-empty family of semiregular countably pracompact topological Brandt λ_i^0-extensions of countably pracompact semiregular semitopological monoids such that the Tychonoff product $\prod\{S_i : i \in \mathcal{I}\}$ is a countably pracompact space. Then the direct product $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a semiregular countably pracompact semitopological semigroup.*

P r o o f. Let for every $i \in \mathcal{I}$, $\mathcal{A}(\lambda_i \times \lambda_i)$ be a space and g_i be a map defined in the proof of Theorem 3, $g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i)$. Theorem 7 implies that the map g_i is continuous for every $i \in \mathcal{I}$. Since the space $\prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is homeomorphic to $\prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i \in \mathcal{I}} S_i$, Theorem 3.2.4 from [12] and Corollary 2 imply that the Tychonoff product $\prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is a countably pracompact space. Then by Theorem 7 and Proposition 2.3.6 of [12] the map $g : \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to \prod_{i \in \mathcal{I}} B^0_{\lambda_i}(S_i)$ defined by the formula $g = \prod_{i \in \mathcal{I}} g_i$ is continuous, and since by Lemma 8 from [18] every continuous image of a countably pracompact space is countably pracompact, we see that the direct product $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a semiregular pseudocompact semitopological semigroup. ♦

Since for any semitopological monoid (S, τ) with zero and for any finite cardinal $\lambda \geq 1$ there exists a unique topological Brandt λ^0-extension $(B^0_\lambda(S), \tau_B)$ of (S, τ) in the class of semitopological semigroups, the proof of the following theorem is similar to the proofs of Theorems 8 and 9.

Theorem 10. *Let $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ be a non-empty family of Hausdorff pseudocompact (countably pracompact) topological Brandt λ_i^0-extensions of Hausdorff pseudocompact (countably pracompact) semitopological monoids such that the Tychonoff product $\prod\{S_i : i \in \mathcal{I}\}$ is a Hausdorff pseudocompact (countably pracompact) space and every cardinal λ_i, $i \in \mathcal{I}$, is non-zero and finite. Then the direct product $\prod\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a Hausdorff pseudocompact (countably pracompact) semitopological semigroup.*

By Theorem 3 of [20] we have that a topological Brandt λ^0-extension $(B^0_\lambda(S),\tau_B)$ of a topological monoid (S,τ_S) with zero in the class of Hausdorff topological semigroups is pseudocompact if and only if cardinal λ is finite and the space (S,τ_S) is pseudocompact. Hence Theorem 3 of [20] and Theorem 10 imply the following

Theorem 11. *Let* $\{B^0_{\lambda_i}(S_i) : i\in\mathcal{I}\}$ *be a non-empty family of Hausdorff pseudocompact (countably pracompact) topological Brandt* λ_i^0*-extensions of Hausdorff pseudocompact (countably pracompact) topological monoids in the class of Hausdorff topological semigroups such that the Tychonoff product* $\prod\{S_i : i\in\mathcal{I}\}$ *is a Hausdorff pseudocompact (countably pracompact) space. Then the direct product* $\prod\{B^0_{\lambda_i}(S_i) : i\in\mathcal{I}\}$ *with the Tychonoff topology is a Hausdorff pseudocompact (countably pracompact) topological semigroup.*

The following lemma describes the main property of a base of the topology at zero of a Hausdorff pseudocompact topological Brandt λ^0-extension of a Hausdorff pseudocompact semitopological monoid in the class of Hausdorff semitopological semigroups.

Lemma 4. *Let* $(B^0_\lambda(S),\tau^S_B)$ *be any Hausdorff pseudocompact topological Brandt* λ^0*-extension of a pseudocompact semitopological monoid* (S,τ) *with zero in the class of semitopological semigroups. Then for every open neighborhood* $U(0)$ *of zero* 0 *in* $(B^0_\lambda(S),\tau^S_B)$ *there exist at most finitely many pairs of indices* $(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\in\lambda\times\lambda$ *such that* $S^*_{\alpha_i,\beta_i}\nsubseteq \operatorname{cl}_{B^0_\lambda(S)}(U(0))$ *for any* $i=1,\ldots,n$.

P r o o f. Suppose the contrary: there exist an open neighborhood $V(0)$ of zero 0 in $(B^0_\lambda(S),\tau^S_B)$ and infinitely many pairs of indices (α_1,β_1), $\ldots,(\alpha_n,\beta_n),\ldots\in\lambda\times\lambda$ such that $S^*_{\alpha_i,\beta_i}\nsubseteq \operatorname{cl}_{B^0_\lambda(S)}(U(0))$ for every positive integer i. Then by Proposition 1.1.1 of [12] for every positive integer i there exists a non-empty open subset W_{α_i,β_i} in $(B^0_\lambda(S),\tau^S_B)$ such that $W_{\alpha_i,\beta_i}\subseteq S^*_{\alpha_i,\beta_i}$ and $V(0)\cap W_{\alpha_i,\beta_i}=\varnothing$. Hence by Lemma 3 of [18] we have that $\{W_{\alpha_i,\beta_i} : i=1,2,3,\ldots\}$ is an infinite locally finite family in $(B^0_\lambda(S),\tau^S_B)$ which contradicts the pseudocompactness of the space $(B^0_\lambda(S),\tau^S_B)$. The obtained contradiction implies the statement of our lemma. ♦

Given a topological space (X,τ) Stone [29] and Katětov [24] consider the topology τ_r on X generated by the base consisting of all regular open sets of the space (X,τ). This topology is called the *regularization* of the topology τ. It is easy to see that if (X,τ) is a Hausdorff topological space then (X,τ_r) is a semiregular topological space.

Example 1. Let (S,τ) be any semitopological monoid with zero. Then for any infinite cardinal λ we define a topology τ^S_B on the Brandt λ^0-extension $(B^0_\lambda(S),\tau^S_B)$ of (S,τ) in the following way. The topology τ^S_B is generated by the base $\mathscr{B}_B=\bigcup\{\mathscr{B}_B(t) : t\in B^0_\lambda(S)\}$, where

- $\mathscr{B}_B(t)=\{(U(s)\setminus\{0_S\})_{\alpha,\beta} : U(s)\in\mathscr{B}_S(s)\}$, where $t=(\alpha,s,\beta)$ is a non-zero element of $B^0_\lambda(S)$, $\alpha,\beta\in\lambda$;

- $\mathscr{B}_B(0)=\{U_F(0)=\bigcup_{(\alpha,\beta)\in(\lambda\times\lambda)\setminus F} S_{\alpha,\beta}\cup\bigcup_{(\gamma,\delta)\in F}(U(0_S))_{\gamma,\delta} : F$ is a finite subset of $\lambda\times\lambda$ and $U(0_S)\in\mathscr{B}_S(0_S)\}$, where 0 is the zero of $B^0_\lambda(S)$, and $\mathscr{B}_S(s)$ is a base of the topology τ at the point $s\in S$.

We observe that the space $(B^0_\lambda(S),\tau^S_B)$ is Hausdorff (respectively, regular, Tychonoff, normal) if and only if the space (S,τ) is Hausdorff (respectively, regular, Tychonoff, normal) (see Propositions 21 and 22 in [18]).

Proposition 2. *Let λ be any infinite cardinal. If (S,τ) is a Hausdorff semitopological monoid with zero then $(B^0_\lambda(S),\tau^S_B)$ is a Hausdorff semitopological semigroup. Moreover, the space (S,τ) is pseudocompact if and only if so is $(B^0_\lambda(S),\tau^S_B)$.*

P r o o f. The Hausdorffness of the space $(B^0_\lambda(S),\tau^S_B)$ follows from Proposition 21 from [18].

Let a and b are arbitrary elements of S and $W(ab)$, $U(a)$, $V(b)$ be arbitrary open neighborhoods of the elements ab, a and b, respectively, such that $U(a)\cdot b\subseteq W(ab)$ and $a\cdot V(b)\subseteq W(ab)$. Then we have that the following conditions hold for each $\alpha,\beta,\gamma,\delta\in\lambda$:

(***i***) $(U(a))_{\alpha,\beta}\cdot(\beta,b,\gamma)\subseteq(W(ab))_{\alpha,\gamma}$;

(***ii***) $(\alpha,a,\beta)\cdot(V(b))_{\beta,\gamma}\subseteq(W(ab))_{\alpha,\gamma}$;

(***iii***) if $\beta\neq\gamma$ then $(U(a))_{\alpha,\beta}\cdot(\gamma,b,\delta)=\{0\}\subseteq W_F(0)$ and $(\alpha,a,\beta)\cdot(V(b))_{\gamma,\delta}=$ $=\{0\}\subseteq W_F(0)$ for every finite subset F of $\lambda\times\lambda$ and every $W(0_S)\in$ $\in\mathscr{B}_S(0_S)$;

(***iv***) $W_F(0)\cdot 0=\{0\}\subseteq W_F(0)$ and $0\cdot W_F(0)=\{0\}\subseteq W_F(0)$ for every finite subset F of $\lambda\times\lambda$ and every $W(0_S)\in\mathscr{B}_S(0_S)$;

(***v***) $(U(a))_{\alpha,\beta}\cdot 0=\{0\}\subseteq W_F(0)$ and $0\cdot(V(b))_{\beta,\gamma}=\{0\}\subseteq W_F(0)$ for every finite subset F of $\lambda\times\lambda$ and every $W(0_S)\in\mathscr{B}_S(0_S)$;

(***vi***) $(\alpha,a,\beta)\cdot V_{F_1}(0)\subseteq W_F(0)$ for every finite subset $\{\alpha_1,\dots,\alpha_k\}\subset\lambda$ and every $W(0_S)\in\mathscr{B}_S(0_S)$, where $F=\{\alpha,\alpha_1,\dots,\alpha_k\}\times\{\alpha_1,\dots,\alpha_k\}$ and $F_1=\{(\beta,\alpha_1),\dots,(\beta,\alpha_k)\}$;

(***vii***) $V_{F_1}(0)\cdot(\alpha,a,\beta)\subseteq W_F(0)$ for every finite subset $\{\alpha_1,\dots,\alpha_k\}\subset\lambda$ and every $W(0_S)\in\mathscr{B}_S(0_S)$, where $F=\{\alpha_1,\dots,\alpha_k\}\times\{\beta,\alpha_1,\dots,\alpha_k\}$ and $F_1=\{(\alpha_1,\alpha),\dots,(\alpha_k,\alpha)\}$.

This completes the proof of separate continuity of the semigroup operation in $(B^0_\lambda(S),\tau^S_B)$.

The implication $(\Leftarrow)$ of the last statement follows from Lemma 9 of [18]. To show the converse implication assume that $\{U_i : i\in\mathcal{I}\}$ is any locally finite family of open subsets of $(B^0_\lambda(S),\tau^S_B)$. Without loss of generality we can assume that $0\notin U_i$ for any $i\in\mathcal{I}$. Then the definition of the base of the topology τ^S_B at zero implies that there exists a finite family of pairs of indices $\{(\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)\}\subset\lambda\times\lambda$ such that almost all elements of the family $\{U_i : i\in\mathcal{I}\}$ are contained in the set $S^*_{\alpha_1,\beta_1}\cup\dots\cup S^*_{\alpha_k,\beta_k}$. Since a union of a finite family of pseudocompact spaces is pseudocompact, $S_{\alpha_1,\beta_1}\cup\dots\cup S_{\alpha_k,\beta_k}$ with the topology induced from $(B^0_\lambda(S),\tau^S_B)$ is pseudocompact space. This implies that the family $\{U_i : i\in\mathcal{I}\}$ is finite. ♦

Example 2. Let λ be any infinite cardinal. Let (S, τ_S) be a Hausdorff pseudocompact semitopological monoid with zero 0_S and $(B_\lambda^0(S), \tau_{B_S}^0)$ be a pseudocompact topological Brandt λ^0-extension of (S, τ_S) in the class of Hausdorff semitopological semigroups.

For every open neighborhood $U(0)$ of zero in $(B_\lambda^0(S), \tau_{B_S}^0)$ we put

$$F_{U(0)} = \{(\alpha, \beta) \in \lambda \times \lambda : S_{\alpha,\beta} \nsubseteq \mathrm{cl}_{B_\lambda^0(S)}(U(0))\}.$$

Let $\pi_{B_S} : B_\lambda(S) \to B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$ be the natural homomorphisms, where $\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) : 0_S$ is zero of $S\}$ is an ideal of the semigroup $B_\lambda(S)$.

We generate a topology $\hat{\tau}_{B_S}$ on the Brandt λ-extension $B_\lambda(S)$ by a base $\hat{\mathscr{B}}_B = \bigcup\{\hat{\mathscr{B}}_B(t) : t \in B_\lambda(S)\}$, where

- $\hat{\mathscr{B}}_B(\alpha, s, \beta) = \{(U(s))_{\alpha,\beta} : U(s) \in \mathscr{B}_S(s)\}$, for all $s \in S$ and $\alpha, \beta \in \lambda$;
- $\hat{\mathscr{B}}_B(0) = \{U_\pi(0) = \pi^{-1}(U(0)) \setminus \bigcup_{(\alpha,\beta)\in F_{U(0)}} S_{\alpha,\beta} : U(0)$ is an element of a base of the topology $\tau_{B_S}^0$ at zero 0 of $B_\lambda^0(S)\}$, and $\mathscr{B}_S(s)$ is a base of the topology τ at the point $s \in S$.

Proposition 3. *Let λ be any infinite cardinal. Let $(B_\lambda^0(S), \tau_{B_S}^0)$ be a pseudocompact topological Brandt λ^0-extension of a Hausdorff pseudocompact semitopological monoid with zero (S, τ_S) in the class of Hausdorff semitopological semigroup. Then $(B_\lambda(S), \hat{\tau}_{B_S})$ is a Hausdorff semitopological semigroup. Moreover, the space (S, τ) is pseudocompact if and only if so is $(B_\lambda(S), \hat{\tau}_{B_S})$.*

P r o o f. We observe that simple verifications show that the natural homomorphism $\pi_{B_S} : (B_\lambda(S), \hat{\tau}_{B_S}) \to (B_\lambda^0(S), \tau_{B_S}^0)$ is a continuous map.

Let a and b be arbitrary elements of the semitopological semigroup (S, τ_S). Let $W(ab)$, $U(a)$ and $V(b)$ be arbitrary open neighborhoods of the elements ab, a and b, respectively, such that $U(a) \cdot b \subseteq W(ab)$ and $a \cdot V(b) \subseteq W(ab)$. Then the following conditions hold for each $\alpha, \beta, \gamma, \delta \in \lambda$:

(***i***) $(U(a))_{\alpha,\beta} \cdot (\beta, b, \gamma) \subseteq (W(ab))_{\alpha,\gamma}$;

(***ii***) $(\alpha, a, \beta) \cdot (V(b))_{\beta,\gamma} \subseteq (W(ab))_{\alpha,\gamma}$;

(***iii***) if $\beta \neq \gamma$ then $(U(a))_{\alpha,\beta} \cdot (\gamma, b, \delta) = \{0\} \subseteq U_\pi(0)$ and $(\alpha, a, \beta) \cdot (V(b))_{\gamma,\delta} = \{0\} \subseteq$ $\subseteq U_\pi(0)$ for every open neighborhood $U(0)$ of zero in $(B_\lambda^0(S), \tau_{B_S}^0)$;

(***iv***) $U_\pi(0) \cdot 0 = \{0\} \subseteq U_\pi(0)$ and $0 \cdot U_\pi(0) = \{0\} \subseteq U_\pi(0)$ for every open neighborhood $U(0)$ of zero in $(B_\lambda^0(S), \tau_{B_S}^0)$;

(***v***) $(U(a))_{\alpha,\beta} \cdot 0 = \{0\} \subseteq U_\pi(0)$ and $0 \cdot (V(b))_{\beta,\gamma} = \{0\} \subseteq U_\pi(0)$ for open neighborhood $U(0)$ of zero in $(B_\lambda^0(S), \tau_{B_S}^0)$;

(***vi***) $(\alpha, a, \beta) \cdot U_\pi(0) \subseteq W_\pi(0)$ in $(B_\lambda(S), \hat{\tau}_{B_S})$ for $U_\pi(0), W_\pi(0) \in \hat{\mathscr{B}}_B(0)$ where $U(0)$ and $W(0)$ are elements of a base of the topology $\tau_{B_S}^0$ at zero 0 of $B_\lambda^0(S)$ such that $(\alpha, a, \beta) \cdot U(0) \subseteq W(0)$;

(***vii***) $U_\pi(0)\cdot(\alpha,a,\beta)\subseteq W_\pi(0)$ in $(B_\lambda(S),\hat{\tau}_{B_S})$ for $U_\pi(0),W_\pi(0)\in\hat{\mathscr{B}}_B(0)$ where $U(0)$ and $W(0)$ are elements of a base of the topology $\tau^0_{B_S}$ at zero 0 of $B^0_\lambda(S)$ such that $U(0)\cdot(\alpha,a,\beta)\subseteq W(0)$.

The proof of the last statement is similar to the proof of the second statement of Proposition 2. ♦

Remark 1. Also, we may consider the semitopological semigroup $(B_\lambda(S),\hat{\tau}_{B_S})$ as a topological Brandt λ^0-extension of a Hausdorff pseudocompact semitopological monoid $T = S\sqcup 0_T$ with « new » isolated zero 0_T.

Theorem 12. *Let* $\{(B^0_{\lambda_i}(S_i),\tau^0_{B(S_i)}) : i\in\mathcal{I}\}$ *be a non-empty family of Hausdorff pseudocompact topological Brandt* λ_i^0*-extensions of Hausdorff pseudocompact semitopological monoids with zero such that the Tychonoff product* $\prod\{S_i : i\in\mathcal{I}\}$ *is a pseudocompact space. Then the direct product* $\prod\{(B^0_{\lambda_i}(S_i),\tau^0_{B(S_i)}) : i\in\mathcal{I}\}$ *with the Tychonoff topology is a Hausdorff pseudocompact semitopological semigroup.*

P r o o f. We consider two cases: ***1°***) λ_i is an infinite cardinal, and ***2°***) λ_i is a finite cardinal, $i\in\mathcal{I}$.

1°) Let $i\in\mathcal{I}$ be an index such that λ_i is an infinite cardinal. Then we put $\hat{\tau}_{B(S_i)}$ is the topology on the Brandt λ_i-extension $B_{\lambda_i}(S_i)$ defined in *Example 2*. Then by Proposition 3, $(B_{\lambda_i}(S_i),\hat{\tau}_{B(S_i)})$ is a Hausdorff pseudocompact semitopological semigroup. By Remark 1 we have that the semitopological semigroup $(B_{\lambda_i}(S_i),\hat{\tau}_{B(S_i)})$ is a topological Brandt λ_i^0-extension of a Hausdorff pseudocompact semitopological monoid $T_i = S\sqcup 0_{T_i}$ with isolated zero 0_{T_i}. By τ_i we denote the topology of the space T_i. Let $\tau_B^{T_i}$ be the topology on the Brandt λ^0-extension $(B^0_{\lambda_i}(T_i),\tau_B^{T_i})$ of (T_i,τ_i) defined in *Example 1*. Next we algebraically identify the semigroup $B^0_{\lambda_i}(T_i)$ with the Brandt λ_i-extension $B_{\lambda_i}(S_i)$ and the topology $\tau_B^{T_i}$ on $B_{\lambda_i}(S_i)$ we shall denote by $\tau_B^{S_i}$.

2°) Let $i\in\mathcal{I}$ be an index such that λ_i is a finite cardinal. We put $T_i = S\sqcup 0_{T_i}$ with isolated zero 0_{T_i}. It is obvious that the semitopological semigroup T_i is pseudocompact if and only if so is the space S_i. Then by Theorem 7 from [18] there exists the unique topological Brandt λ_i^0-extension $(B^0_{\lambda_i}(T_i),\hat{\tau}_{B(T_i)})$ of the semitopological monoid T_i in the class of semitopological semigroups. Also, Theorem 7 from [18] implies that the topological space $(B^0_{\lambda_i}(T_i),\hat{\tau}_{B(T_i)})$ is homeomorphic to the topological sum of topological copies of the space S_i and isolated zero, and hence we obtain that the space $(B^0_{\lambda_i}(T_i),\hat{\tau}_{B(T_i)})$ is pseudocompact if and only if so is the space S_i. Next we algebraically identify the semigroup $B^0_{\lambda_i}(T_i)$ with the Brandt λ_i-extension $B_{\lambda_i}(S_i)$ and the topology $\hat{\tau}_{B(T_i)}$ on $B_{\lambda_i}(S_i)$ we shall denote by $\hat{\tau}_{B(S_i)}$. Also in this case (when λ_i is a finite cardinal) we put $\tau_B^{S_i}=\hat{\tau}_{B(S_i)}$.

Then the definitions of topologies $\hat{\tau}_{B(S_i)}$ and $\tau_B^{S_i}$ on $B_{\lambda_i}(S_i)$ imply that for every index $i \in \mathcal{I}$ the identity map $\widehat{\mathrm{id}}_i : (B_{\lambda_i}(S_i), \hat{\tau}_{B(S_i)}) \to (B_{\lambda_i}(S_i), \tau_B^{S_i})$ is continuous. Let $\tau_{B(S_i)}^R$ be the regularization of the topology $\hat{\tau}_{B(S_i)}$ on $B_{\lambda_i}(S_i)$. Then the definition of the topology $\tau_B^{S_i}$ on $B_{\lambda_i}(S_i)$ implies that the identity map $\mathrm{id}_i^R : (B_{\lambda_i}(S_i), \tau_B^{S_i}) \to (B_{\lambda_i}(S_i), \tau_{B(S_i)}^R)$ is continuous. Since the pseudocompactness is preserved by continuous maps we obtain that $(B_{\lambda_i}(S_i), \tau_{B(S_i)}^R)$ is a semiregular pseudocompact space (which is not necessarily a semitopological semigroup). Also, repeating the proof of Theorem 8 for our case, we get that the Tychonoff product $\prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \tau_B^{S_i})$ is a pseudocompact space. Then the Cartesian products $\prod_{i\in\mathcal{I}} \widehat{\mathrm{id}}_i : \prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \hat{\tau}_{B(S_i)}) \to \prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \tau_B^{S_i})$ and $\prod_{i\in\mathcal{I}} \mathrm{id}_i^R : \prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \tau_B^{S_i}) \to \prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \tau_{B(S_i)}^R)$ are continuous maps. This implies that $\prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \tau_{B(S_i)}^R)$ is a pseudocompact space. Then by Lemma 20 of [26] the regularization of the product $\prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \hat{\tau}_{B(S_i)})$ coincides with $\prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \tau_{B(S_i)}^R)$ and hence by Lemma 3 of [26] we have that the space $\prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \hat{\tau}_{B(S_i)})$ is pseudocompact.

Let $\pi_{B_S^i} : B_{\lambda_i}(S_i) \to B_{\lambda_i}^0(S_i) = B_{\lambda_i}(S_i)/\mathcal{J}$ be the natural homomorphism, where $\mathcal{J} = \{0_i\} \cup \{(\alpha, 0_{S_i}, \beta) : 0_{S_i}$ is zero of $S_i\}$ is an ideal of the semigroup $B_{\lambda_i}(S_i)$. Then the natural homomorphism $\pi_{B_S^i} : (B_{\lambda_i}(S_i), \hat{\tau}_{B(S_i)}) \to (B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0)$ is a continuous map. This implies that the product $\prod_{i\in\mathcal{I}} \pi_{B_S^i} : \prod_{i\in\mathcal{I}}(B_{\lambda_i}(S_i), \hat{\tau}_{B(S_i)}) \to$ $\to \prod_{i\in\mathcal{I}}(B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0)$ is a continuous map, and hence we get that the Tychonoff product $\prod_{i\in\mathcal{I}}(B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0)$ is a pseudocompact space. ♦

Proposition 4. *Each H-closed space is pseudocompact.*

P r o o f. Let X be an H-closed space. Assume that the space X is not pseudocompact. Then there exists an infinite locally finite family $\mathcal{U}$ of non-empty open subsets of the space X. Since the family $\mathcal{U}$ is locally finite, each point $x \in X$ has an open neighborhood U_x intersecting only finitely many members of the family $\mathcal{U}$. Since the space X is H-closed and $\{U_x : x \in X\}$ is an open cover of the space X, by Exercise 3.12.5(4) from [12] (also see [3, Chapt. III, Theorem 4]) there exists a finite subset F of the space X such that $X = \bigcup\{\mathrm{cl}_X(U_x) : x \in F\}$. But then the set X, as the union of the finite family $\{\mathrm{cl}_X(U_x) : x \in F\}$ intersects only finitely many members of the family $\mathcal{U}$, a contradiction. ♦

Let λ be any cardinal ≥ 1 and S be any semigroup. We shall say that a subset $\Phi \subset B_\lambda^0(S)$ has the λ*-finite property* in $B_\lambda^0(S)$, if $\Phi \cap S_{\alpha,\beta}^*$ is finite for all $\alpha, \beta \in \lambda$ and $\Phi \not\ni 0$, where 0 is zero of $B_\lambda^0(S)$.

Example 3. Let λ be an infinite cardinal and $\mathbb{T}$ be the unit circle with the usual multiplication of complex numbers and the usual topology $\tau_{\mathbb{T}}$. It is obvious that $(\mathbb{T}, \tau_{\mathbb{T}})$ is a topological group. The base of the topology τ_B^{fin} on the Brandt semigroup $B_\lambda(\mathbb{T})$ we define as follows:

- for every non-zero element (α, x, β) of the semigroup $B_\lambda(\mathbb{T})$ the family
  $\mathscr{B}_{(\alpha,x,\beta)} = \{(\alpha, U(x), \beta) : U(x) \in \mathscr{B}_{\mathbb{T}}(x)\}$,
  where $\mathscr{B}_{\mathbb{T}}(x)$ is a base of the topology $\tau_{\mathbb{T}}$ at the point $x \in \mathbb{T}$, is the base of the topology τ_B^{fin} at $(\alpha, x, \beta) \in B_\lambda(\mathbb{T})$;
- the family $\mathscr{B}_0 = \{U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; F) : \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \lambda, n \in \mathbb{N},\ F$ has the λ-finite property in $B_\lambda^0(S)\}$ where $U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; F) = B_\lambda(\mathbb{T}) \setminus (\mathbb{T}_{\alpha_1,\beta_1} \cup \ldots \cup \mathbb{T}_{\alpha_n,\beta_n} \cup F)$, is the base of the topology τ_B^{fin} at zero $0 \in B_\lambda(\mathbb{T})$.

Simple verifications show that $(B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})$ is a non-semiregular Hausdorff pseudocompact topological space for every infinite cardinal λ. Next we shall show that the semigroup operation on $(B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})$ is separately continuous. The proof of the separate continuity of the semigroup operation in the cases $0 \cdot 0$ and $(\alpha, x, \beta) \cdot (\gamma, y, \delta)$, where $\alpha, \beta, \gamma, \delta \in \lambda$ and $x, y \in \mathbb{T}$, is trivial, and hence we only consider the following cases:

$$(\alpha, x, \beta) \cdot 0 \qquad \text{and} \qquad 0 \cdot (\alpha, x, \beta).$$

For arbitrary $\alpha, \beta \in \lambda$ and $\Phi \subset B_\lambda(\mathbb{T})$ we denote $\Phi^{\alpha,\beta} = \Phi \cap \mathbb{T}^{\alpha,\beta}$ and put $\Phi_{\mathbb{T}}(\alpha, \beta)$ is a subset of $\mathbb{T}$ such that $(\Phi_{\mathbb{T}}(\alpha, \beta))_{\alpha,\beta} = \Phi \cap \mathbb{T}_{\alpha,\beta}$.

Fix an arbitrary non-zero element $(\alpha, x, \beta) \in B_\lambda(\mathbb{T})$. Let $\Phi \subset B_\lambda^0(S)$ be an arbitrary subset with the λ-finite property in $B_\lambda^0(S)$. Since $\mathbb{T}$ is a group, there exist subsets $\Upsilon, \Psi \subset B_\lambda^0(S)$ with the λ-finite property in $B_\lambda^0(S)$ such that

$$(x \cdot \Upsilon_{\mathbb{T}}(\beta, \gamma))_{\alpha,\gamma} = \Phi \cap \mathbb{T}_{\alpha,\gamma} \text{ and } (\Psi_{\mathbb{T}}(\gamma, \alpha) \cdot x)_{\gamma,\beta} = \Phi \cap \mathbb{T}_{\gamma,\beta}.$$

Then we have that

$$\begin{aligned}
&(\alpha, x, \beta) \cdot U(\beta, \beta_1; \ldots; \beta, \beta_n; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Upsilon) \subseteq \\
&\qquad \subseteq \{0\} \cup \bigcup \{\mathbb{T}_{\alpha,\gamma} \setminus ((\alpha, x, \beta) \cdot \Upsilon^{\beta,\gamma}) : \gamma \in \lambda \setminus \{\beta_1, \ldots, \beta_n\}\} \subseteq \\
&\qquad \subseteq \{0\} \cup \bigcup \{\mathbb{T}_{\alpha,\gamma} \setminus (x \cdot \Upsilon_{\mathbb{T}}(\beta, \gamma))_{\alpha,\gamma} : \gamma \in \lambda \setminus \{\beta_1, \ldots, \beta_n\}\} \subseteq \\
&\qquad \subseteq U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Phi)
\end{aligned}$$

and similarly

$$\begin{aligned}
&U(\alpha_1, \alpha; \ldots; \alpha_n, \alpha; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Psi) \cdot (\alpha, x, \beta) \subseteq \\
&\qquad \subseteq \{0\} \cup \bigcup \{\mathbb{T}_{\gamma,\beta} \setminus (\Psi^{\gamma,\alpha} \cdot (\alpha, x, \beta)) : \gamma \in \lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\} \subseteq \\
&\qquad \subseteq \{0\} \cup \bigcup \{\mathbb{T}_{\gamma,\beta} \setminus (\Psi_{\mathbb{T}}(\gamma, \alpha) \cdot x)_{\gamma,\beta} : \gamma \in \lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\} \subseteq \\
&\qquad \subseteq U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Phi),
\end{aligned}$$

for every $U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Phi) \in \mathscr{B}_0$. This completes the proof of separate continuity of the semigroup operation in $(B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})$.

Next we shall show that the space $(B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})$ is not countably pracompact. Suppose to the contrary: there exists a dense subset A in $(B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})$ such that $(B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})$ is countably compact at A. Then the definition of the topology τ_B^{fin} implies that $A \cap \mathbb{T}_{\alpha,\beta}$ is a dense subset in $\mathbb{T}_{\alpha,\beta}$ for all $\alpha,\beta \in \lambda$. We construct a subset $\Phi \subset B_\lambda(\mathbb{T})$ in the following way. For all $\alpha,\beta \in \lambda$ we fix an arbitrary point $(\alpha, x^A_{\alpha,\beta}, \beta) \in A \cap \mathbb{T}_{\alpha,\beta}$ and put $\Phi = \{(\alpha, x^A_{\alpha,\beta}, \beta) : \alpha,\beta \in \lambda\}$. Then Φ is the subset with the λ-finite property in $B^0_\lambda(S)$, and the definition of the topology τ_B^{fin} on $B_\lambda(\mathbb{T})$ implies that Φ has no an accumulation point x in $(B_\lambda(\mathbb{T}), \tau_B^{\mathrm{fin}})$, a contradiction.

Example 3 shows that there exists a Hausdorff non-semiregular pseudocompact topological Brandt λ^0-extension of a Hausdorff compact topological group with adjoined isolated zero which is not a countably pracompact space. Also, Example 18 from [18] shows that there exists a Hausdorff non-semiregular pseudocompact topological Brandt λ^0-extension of a countable Hausdorff compact topological monoid with adjoined isolated zero which is not a countably compact space. But, as a counterpart for the H-closed case or the sequentially pseudocompact case we have the following.

Proposition 5. *Let S be semitopological monoid with zero which is an H-closed (respectively, a sequentially pseudocompact) space. Then every Hausdorff pseudocompact topological Brandt λ^0-extension $B^0_\lambda(S)$ of S in the class of Hausdorff semitopological semigroup is an H-closed (respectively, a sequentially pseudocompact) space.*

P r o o f. First we consider the case when S is an H-closed space. Suppose to the contrary that there exists a Hausdorff pseudocompact topological Brandt λ^0-extension $(B^0_\lambda(S), \tau_B)$ of S in the class of Hausdorff semitopological semigroup such that $(B^0_\lambda(S), \tau_B)$ is not an H-closed space. Then there exists a Hausdorff topological space X which contains the topological space $(B^0_\lambda(S), \tau_B)$ as a non-closed subspace. Without loss of generality we may assume that $(B^0_\lambda(S), \tau_B)$ is a dense subspace of X such that $X \setminus B^0_\lambda(S) \neq \varnothing$. Fix an arbitrary point $x \in X \setminus B^0_\lambda(S)$. Then we have that $U(x) \cap B^0_\lambda(S) \neq \varnothing$ for any open neighborhood $U(x)$ of the point x in X. Now, the Hausdorffness of X implies that there exist open neighborhoods $V(x)$ and $V(0)$ of x and zero 0 of the semigroup $B^0_\lambda(S)$ such that $V(x) \cap V(0) = \varnothing$. Then by Lemma 4 we obtain that there exist at most finitely many pairs of indices $(\alpha_1,\beta_1), \ldots, (\alpha_n,\beta_n) \in \lambda \times \lambda$ such that $S^*_{\alpha_i,\beta_i} \nsubseteq \mathrm{cl}_{B^0_\lambda(S)}(V(0))$ for any $i = 1, \ldots, n$. Hence by Corollary 1.1.2 of [12], the neighborhood $V(x)$ intersects at most finitely many subsets $S_{\alpha,\beta}$, $\alpha,\beta \in \lambda$. Then by Lemma 2 of [18] we get that $S_{\alpha,\beta}$ is a closed subset of X for all $\alpha,\beta \in \lambda$, and hence $B^0_\lambda(S)$ is a closed subspace of X, a contradiction.

Next we suppose that S is a sequentially pseudocompact space. Let $\{U_n : n \in \mathbb{N}\}$ be any sequence of non-empty open subsets of the space $B^0_\lambda(S)$. If there exists finitely many pairs of indices $(\alpha_1,\beta_1), \ldots, (\alpha_n,\beta_n) \in \lambda \times \lambda$ such that $\bigcup\{U_n : n \in \mathbb{N}\} \subseteq S_{\alpha_1,\beta_1} \cup \ldots \cup S_{\alpha_n,\beta_n}$ the sequential pseudocompactness of S and Lemma 2 from [18] imply that there exist a point $x \in S_{\alpha_1,\beta_1} \cup \ldots \cup S_{\alpha_n,\beta_n}$ and an infinite set $S \subset \mathbb{N}$ such that for each neighborhood $U(x)$ of the point x the set $\{n \in S : U_n \cap U(x) = \varnothing\}$ is finite. In the other case by Lemma 4 we get that there exists an infinite set $S \subset \mathbb{N}$ such that for each neighborhood $U(0)$ of zero 0 of the semigroup $B_{\lambda}^{0}(S)$ the set $\{n \in S : U_n \bigcap U(0) = \varnothing\}$ is finite. This completes the proof of our lemma. ♦

Since by Theorem 3 from [8] (see also Problem 3.12.5(d) in [12]) the Tychonoff product of the non-empty family non-empty H-topological spaces is H-closed, and by Proposition 2.2 from [21], the Tychonoff product of a non-empty family of non-empty sequentially pseudocompact spaces is sequentially pseudocompact Proposition 5 implies the following

Corollary 3. *Let* $\{(B_{\lambda_i}^{0}(S_i), \tau_{B(S_i)}^{0}) : i \in \mathcal{I}\}$ *be a non-empty family of Hausdorff pseudocompact topological Brandt* λ_i^0*-extensions of Hausdorff* H*-closed* (*respectively, a sequentially pseudocompact*) *semitopological monoids with zero. Then the direct product* $\prod\{(B_{\lambda_i}^{0}(S_i), \tau_{B(S_i)}^{0}) : i \in \mathcal{I}\}$ *with the Tychonoff topology is a Hausdorff* H*-closed* (*respectively, a sequentially pseudocompact*) *semitopological semigroup.*

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ПСЕВДОКОМПАКТНІСТЬ, ДОБУТКИ ТА ТОПОЛОГІЧНІ λ^0-РОЗШИРЕННЯ БРАНДТА НАПІВТОПОЛОГІЧНИХ МОНОЇДІВ

Вивчається збереження псевдокомпактності (відповідно, зліченної компактності, секвенціальної компактності, ω-обмеженості, цілком зліченної компактності, зліченної пракомпактності, секвенціальної псевдокомпактності) тихоновськими добутками псевдокомпактних (і зліченно компактних) топологічних λ_i^0-розширень Брандта напівтопологічних моноїдів з нулем. Зокрема, показано, що, якщо $\{(B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0) : i \in \mathcal{I}\}$ – сім'я гаусдорфових псевдокомпактних топологічних λ_i^0-розширень Брандта псевдокомпактних напівтопологічних моноїдів з нулем таких, що тиховновський добуток $\prod\{S_i : i \in \mathcal{I}\}$ є псевдокомпактним простором, то прямий добуток $\prod\{(B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0) : i \in \mathcal{I}\}$ з тихоновською топологією є гаусдорфовою псевдокомпактною напівтопологічною напівгрупою.

ПСЕВДОКОМПАКТНОСТЬ, ПРОИЗВЕДЕНИЯ И ТОПОЛОГИЧЕСКИЕ λ^0-РАСШИРЕНИЯ БРАНДТА ПОЛУТОПОЛОГИЧЕСКИХ МОНОИДОВ

Изучается сохранение псевдокомпактности (соответственно, счетной компактности, секвенциальной компактности, ω-ограничености, вполне счетной компактности, счетной пракомпактности, секвенциальной псевдокомпактности) тихоновскими произведениями псевдокомпактных (и счетно компактных) топологических λ_i^0-расширений Брандта полутопологических моноидов с нулем. В частности, показано, что, если $\{(B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0) : i \in \mathcal{I}\}$ – семья хаусдорфовых псевдокомпактных топологических λ_i^0-расширений Брандта псевдокомпактных полутопологических моноидов с нулем таких, что тихоновское произведение $\prod\{S_i : i \in \mathcal{I}\}$ является псевдокомпактным пространством, то прямое произведение $\prod\{(B_{\lambda_i}^0(S_i), \tau_{B(S_i)}^0) : i \in \mathcal{I}\}$ с тихоновской топологией является хаусдорфовой псевдокомпактной полутопологической полугруппой.

[1] Ivan Franko Nat. Univ. of L'viv, L'viv,
[2] Pidstryhach Inst. for Appl. Problems of Mech. and Math. NASU, L'viv

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