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PROBLEM FOR NONHOMOGENEOUS SECOND ORDER EVOLUTION EQUATION WITH HOMOGENEOUS INTEGRAL CONDITIONS

We propose a method of solving the problem with homogeneous integral conditions for nonhomogeneous evolution equation with abstract operator in Banach space H. The right-hand side of the evolution equation, which for fixed time variable belongs to special subspace $N \subseteq H$, is represented as a Stieltjes integral over a certain measure. The solution of this problem is also represented as a Stieltjes integral over the same measure. We give the examples of applying the method to solving the problem with integral conditions for PDE of second order in time variable and, in general, infinite order in spatial variable.

Introduction. Problems with integral conditions, that are direct generalization of discrete nonlocal conditions, for differential-operator equations and for PDE, have been evoking a great interest of scientists in recent years since they arise when modeling a diffusion of particles in a turbulent medium, processes of heat conduction, moisture transfer in capillary-porous media, the dynamics of population etc.

The problems with integral conditions for heat equation have been studied in the works [1, 2, 15]. Inverse problems for this equation and for general parabolic equations with unknown coefficients and with the usage of integral conditions have been investigated in the work (see [17] and the bibliography therein).

The works [8–14] deal with studying the problems with integral conditions for other types of equations as well as for typeless and for differential-operator equations.

1. Statement of the problem. Let H be a Banach space and A be a given linear operator acting in it $(A : H \to H)$. Arbitrary powers A^j, $j = 2, 3, \ldots$, are also defined in H.

Assume that for $\lambda \in \mathbb{C}$ there exists in H the solution of the equation

$$Ay = \lambda y ,$$

which obviously is the eigenvector $y(\lambda)$ of the operator A, which corresponds to its eigenvalue $\lambda \in \mathbb{C}$.

Consider entire function

$$a(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k ,$$

which is not a constant and is the symbol of linear operator $a(A)$. Since $A^j y(\lambda) = \lambda^j y(\lambda)$, $j = 2, 3, \ldots$, we conclude

$$a(A) y(\lambda) = a(\lambda) y(\lambda), \qquad \lambda \in \mathbb{C}. \tag{1}$$

Note that, for the operator $a\left(\dfrac{d}{dx}\right)$ and the eigenvector $y(\lambda) = e^{\lambda x}$ of the operator $A = \dfrac{d}{dx}$ in the space $H = C^{\infty}(\mathbb{R})$, equality (1) gets the form

$$a\left(\frac{d}{dx}\right) e^{\lambda x} = a(\lambda) e^{\lambda x} . \tag{2}$$

Definition 1. We shall say that for arbitrary $t \in [0, T]$, $T > 0$, vector $f(t)$ from H belongs to $\mathcal{N}$, if on $\Lambda \subseteq \mathbb{C}$ there exist a measure $\mu(\lambda)$ and linear operator $F_f(t, \lambda) : H \to H$ such that $f(t)$ can be represented in the form of Stieltjes integral

$$f(t) = \int_{\Lambda} F_f(t,\lambda) y(\lambda)\, d\mu(\lambda)\,. \tag{3}$$

Besides, we assume that operator $F_f(t,\lambda)$ is analytical on $t \in [0,T]$ and abstract operator $A : H \to H$ commutes with $\frac{d}{dt}$, then the operators $a(A)$ and $\frac{d}{dt}$ also commute.

We consider the following problem:

$$\left[\frac{d}{dt} - a(A)\right]^2 U(t) \equiv \left[\frac{d^2}{dt^2} - 2a(A)\frac{d}{dt} + a^2(A)\right] U(t) = f(t)\,, \tag{4}$$

$$\ell_1 U(t) \equiv \int_0^T U(t)\, dt = \varphi_1\,, \tag{5}$$

$$\ell_2 U(t) \equiv \int_0^T tU(t)\, dt = \varphi_2\,, \tag{6}$$

where $U : [0,T] \to H$ is an unknown vector-function, $f : [0,T] \to H$ is a given vector-function, i. e. can be represented in the form (3), $\varphi_1, \varphi_2 \in H$.

The solution of problem (4)–(6) could be represented as a sum of the solution of equation (4) that satisfies homogeneous integral conditions

$$\ell_1 U(t) \equiv \int_0^T U(t)\, dt = 0\,, \tag{7}$$

$$\ell_2 U(t) \equiv \int_0^T tU(t)\, dt = 0\,, \tag{8}$$

and the solution of homogeneous equation

$$\left[\frac{d^2}{dt^2} - 2a(A)\frac{d}{dt} + a^2(A)\right] U(t) = 0\,, \tag{9}$$

that satisfies nonhomogeneous conditions (5), (6).

In this paper, we will show an approach to solving the abstract problem (4), (7), (8). Problem (9), (5), (6) has been investigated in [18]. Note that this work is the continuation of previous investigations [4, 5, 7].

2. Main results. Consider the entire function

$$I \equiv I(a) \equiv \int_0^T e^{a(\lambda)t}\, dt = \frac{e^{a(\lambda)T} - 1}{a(\lambda)} \tag{10}$$

(note that if $a(\lambda_0) = 0$ then $I = T$) as well as its derivatives in a:

$$I' = \int_0^T t e^{a(\lambda)t}\, dt, \qquad I'' = \int_0^T t^2 e^{a(\lambda)t}\, dt\,.$$

Along the lines of equation (4), according to the differential-symbol method [3, 6], replacing A by $\lambda \in \mathbb{C}$, and $f(t)$ by $e^{\nu t}$, we write down the ODE

$$\left[\frac{d}{dt} - a(\lambda)\right]^2 G = e^{\nu t}\,. \tag{11}$$

Denote by $\Delta(\lambda) = I(a)I''(a) - [I'(a)]^2$. Then

$$\Delta(\lambda) = I^2\left(\frac{d}{da}\right)\left\{\frac{I'}{I}\right\} = I^2\left(\frac{d}{da}\right)^2\{\ln I\} = I^2\left(\frac{d}{da}\right)^2\left\{\ln(e^{aT}-1) - \ln a\right\} =$$

$$= I^2\left(\frac{d}{da}\right)\left\{\frac{Te^{aT}}{e^{aT}-1} - \frac{1}{a}\right\} =$$

$$= I^2\left\{\frac{T^2e^{aT}(e^{aT}-1) - Te^{aT}Te^{aT}}{(e^{aT}-1)^2} + \frac{1}{a^2}\right\} =$$

$$= \left(\frac{e^{aT}-1}{a}\right)^2\frac{(e^{aT}-1)^2 - a^2T^2e^{aT}}{a^2(e^{aT}-1)^2} = \frac{(e^{aT}-1)^2 - a^2T^2e^{aT}}{a^4} =$$

$$= \frac{\left(e^{aT}-1-aTe^{aT/2}\right)\left(e^{aT}-1+aTe^{aT/2}\right)}{a^4} =$$

$$= \frac{e^{aT}\left(e^{aT/2}-e^{-aT/2}-aT\right)\left(e^{aT/2}-e^{-aT/2}+aT\right)}{a^4} =$$

$$= \frac{e^{aT}\left(2\,\mathrm{sh}\,\frac{aT}{2} - aT\right)\left(2\,\mathrm{sh}\,\frac{aT}{2} + aT\right)}{a^4} = \frac{4e^{aT}\left(\mathrm{sh}^2\,\frac{aT}{2} - \frac{a^2T^2}{4}\right)}{a^4}.$$

Thus,

$$\Delta(\lambda) = \frac{4e^{a(\lambda)T}\left(\mathrm{sh}^2\,\frac{a(\lambda)T}{2} - \frac{a^2(\lambda)T^2}{4}\right)}{a^4(\lambda)}.$$

Note that if $a(\lambda_0) = 0$ then $\Delta(\lambda_0) = \frac{T^4}{12}$.

Denote by P the set

$$P = \left\{\lambda \in \mathbb{C} : \mathrm{sh}^2\,\frac{a(\lambda)T}{2} = \frac{a^2(\lambda)T^2}{4}\right\}. \tag{12}$$

Set P is not empty (see [16]). Besides, $P \neq \mathbb{C}$ since $P \cap \mathbb{R} = P \cap (i\mathbb{R}) = \varnothing$, $i^2 = -1$ (see [4]).

We consider the function

$$G \equiv G(t,\nu,\lambda) = \frac{e^{\nu t} - I(\nu)\frac{I''(a) - I'(a)t}{\Delta(\lambda)}e^{at} - I'(\nu)\frac{-I'(a) + I(a)t}{\Delta(\lambda)}e^{at}}{(\nu - a)^2}, \tag{13}$$

where $a = a(\lambda)$, $\Delta(\lambda) = I(a)I''(a) - [I'(a)]^2$.

Lemma 1. *Function* $G(\bullet,\nu,\lambda)$ *satisfies equation* (11) *for* $\nu \in \mathbb{C}$, $\lambda \in \mathbb{C}\setminus P$, *where* P *is set* (12).

P r o o f. In fact,

$$\left[\frac{d}{dt} - a\right]^2 G(t,\nu,\lambda) = \frac{1}{(\nu-a)^2}\left\{\left[\frac{d}{dt} - a\right]^2 e^{\nu t} - \right.$$

$$- \frac{I(\nu)}{\Delta(\lambda)}\left[\frac{d}{dt} - a\right]^2\left((I''(a) - I'(a)t)e^{at}\right) -$$

$$\left. - \frac{I'(\nu)}{\Delta(\lambda)}\left[\frac{d}{dt} - a\right]^2\left((-I'(a) + I(a)t)e^{at}\right)\right\} =$$

$$= \frac{1}{(\nu - a)^2} \left\{ [\nu - a]^2 e^{\nu t} - \frac{I(\nu)}{\Delta(\lambda)} \left[\frac{d}{dt} - a \right] (-I'(a) e^{at}) - \right.$$

$$\left. - \frac{I'(\nu)}{\Delta(\lambda)} \left[\frac{d}{dt} - a \right] (I(a) e^{at}) \right\} =$$

$$= \frac{1}{(\nu - a)^2} \left\{ [\nu - a]^2 e^{\nu t} - \frac{I(\nu)}{\Delta(\lambda)} \cdot 0 - \frac{I'(\nu)}{\Delta(\lambda)} \cdot 0 \right\} = e^{\nu t}.$$

This proves our lemma. ♦

Lemma 2. *Function* (13) *satisfies the homogeneous integral conditions*

$$\int_0^T G(t, \nu, \lambda)\, dt = 0, \qquad \int_0^T tG(t, \nu, \lambda)\, dt = 0, \tag{14}$$

for given $\nu \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus P$, *where* P *is set* (12).

P r o o f. We shall prove the realization of the first integral condition in (14):

$$\int_0^T G\, dt = \frac{1}{(\nu - a)^2} \left\{ \int_0^T e^{\nu t}\, dt - \frac{I(\nu)}{\Delta(\lambda)} I''(a) \int_0^T e^{at}\, dt + \frac{I(\nu)}{\Delta(\lambda)} I'(a) \int_0^T te^{at}\, dt + \right.$$

$$\left. + \frac{I'(\nu)}{\Delta(\lambda)} I'(a) \int_0^T e^{at}\, dt - \frac{I'(\nu)}{\Delta(\lambda)} I(a) \int_0^T te^{at}\, dt \right\} =$$

$$= \frac{1}{(\nu - a)^2} \left\{ I(\nu) - \frac{I(\nu) I''(a) I(a)}{\Delta(\lambda)} + \frac{I(\nu)[I'(a)]^2}{\Delta(\lambda)} + \right.$$

$$\left. + \frac{I'(\nu)}{\Delta(\lambda)} I'(a) I(a) - \frac{I'(\nu)}{\Delta(\lambda)} I(a) I'(a) \right\} =$$

$$= \frac{1}{(\nu - a)^2} \left\{ I(\nu) - I(\nu) \frac{I''(a) I(a) - [I'(a)]^2}{\Delta(\lambda)} \right\} = 0.$$

For the second condition, we have

$$\int_0^T tG\, dt = \frac{1}{(\nu - a)^2} \left\{ \int_0^T te^{\nu t}\, dt - \frac{I(\nu)}{\Delta(\lambda)} I''(a) \int_0^T te^{at}\, dt + \frac{I(\nu)}{\Delta(\lambda)} I'(a) \int_0^T t^2 e^{at}\, dt + \right.$$

$$\left. + \frac{I'(\nu)}{\Delta(\lambda)} I'(a) \int_0^T te^{at}\, dt - \frac{I'(\nu)}{\Delta(\lambda)} I(a) \int_0^T t^2 e^{at}\, dt \right\} =$$

$$= \frac{1}{(\nu - a)^2} \left\{ I'(\nu) - I(\nu) \frac{I''(a) I'(a) - I'(a) I''(a)}{\Delta(\lambda)} + \right.$$

$$\left. + \frac{I'(\nu)}{\Delta(\lambda)} ([I'(a)]^2 - I(a) I''(a)) \right\} =$$

$$= \frac{1}{(\nu - a)^2} \left\{ I'(\nu) - 0 - I'(\nu) \frac{\Delta(\lambda)}{\Delta(\lambda)} \right\} = 0.$$

This proves our lemma. ♦

Lemma 3. *Function* $G(t, \nu, \bullet)$ *is analytical in* $\mathbb{C} \setminus P$ *for all* $t \in [0, T]$, $\nu \in \mathbb{C}$, *where* P *is the set* (12). *Besides,* $G(t, \bullet, \lambda)$ *is entire for* $t \in [0, T]$, $\lambda \in \mathbb{C} \setminus P$.

P r o o f. First we shall prove that function in form (13) contains the denominator whose zeros are removable singular points for G. For this purpose, we shall show that

$$R(t, \nu, \lambda)\big|_{\nu = a} = 0, \qquad R'_\nu(t, \nu, \lambda)\big|_{\nu = a} = 0,$$

where

$$R(t,\nu,\lambda) = e^{\nu t} - I(\nu)\frac{I''(a) - I'(a)t}{\Delta(\lambda)}e^{at} - I'(\nu)\frac{-I'(a) + I(a)t}{\Delta(\lambda)}e^{at}.$$

In fact,

$$R(t,a,\lambda) = e^{at} - I(a)\frac{I''(a) - I'(a)t}{\Delta(\lambda)}e^{at} - I'(a)\frac{-I'(a) + I(a)t}{\Delta(\lambda)}e^{at} =$$

$$= e^{at} - \frac{I(a)I''(a) - [I'(a)]^2}{\Delta(\lambda)}e^{at} = 0,$$

$$R'_\nu(t,a,\lambda) = te^{at} - I'(a)\frac{I''(a) - I'(a)t}{\Delta(\lambda)}e^{at} - I''(a)\frac{-I'(a) + I(a)t}{\Delta(\lambda)}e^{at} =$$

$$= te^{at} - \frac{I''(a)I(a) - [I'(a)]^2}{\Delta(\lambda)}te^{at} = 0.$$

Since $e^{\nu t}$, $I(\nu)$, $I'(\nu)$ are entire functions with respect to ν, we conclude that $G(t,\bullet,\lambda)$ is also entire. The zeros of the denominator $\Delta(\lambda)$ in function $G(t,\nu,\lambda)$ are not removable singular points, hence $G(t,\nu,\bullet)$ is analytical on $\mathbb{C}\setminus P$, where P is set (12), that completes our proof. ♦

Lemma 4. *On the set* $[0,T]\times\mathbb{C}\times(\mathbb{C}\setminus P)$ *there holds the following identity*

$$\left[\frac{d}{dt} - a(A)\right]^2 \{G(t,\nu,\lambda)y(\lambda)\} \equiv e^{\nu t}y(\lambda),$$

where $y(\lambda)$ *is the eigenvector of operator* A *corresponding to the eigenvalue* $\lambda\in\mathbb{C}\setminus P$.

P r o o f. From equality (1) and Lemma 1, for arbitrary $(t,\nu,\lambda)\in$ $\in(0,T)\times\mathbb{C}\times(\mathbb{C}\setminus P)$ we have:

$$\left[\frac{d}{dt} - a(A)\right]^2 \{G(t,\nu,\lambda)y(\lambda)\} = \frac{d^2}{dt^2}\{G(t,\nu,\lambda)y(\lambda)\} -$$

$$- 2\frac{d}{dt}a(A)\{G(t,\nu,\lambda)y(\lambda)\} + a^2(A)\{G(t,\nu,\lambda)y(\lambda)\} =$$

$$= \left\{\frac{d^2}{dt^2}G(t,\nu,\lambda)\right\}y(\lambda) - 2a(\lambda)\left\{\frac{d}{dt}G(t,\nu,\lambda)\right\}y(\lambda) +$$

$$+ a^2(\lambda)\{G(t,\nu,\lambda)\}y(\lambda) =$$

$$= \left\{\left[\frac{d}{dt} - a(\lambda)\right]^2 G(t,\nu,\lambda)\right\}y(\lambda) = e^{\nu t}y(\lambda).$$ ♦

Theorem 1. *Let the vector-function* $f(t)$ *in equation* (4) *for arbitrary* $t\in[0,T]$ *belongs to* $\mathcal{N}$, *i. e. could be represented in the form* (3), *and* $G(t,\nu,\lambda)$ *be function* (13), $\Lambda = \mathbb{C}\setminus P$, *where* P *is set* (12). *Then the formula*

$$U(t) = \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda) \quad (15)$$

defines a formal solution of problem (4), (7), (8).

P r o o f. We shall show that vector-function (15) satisfies equation (4). In fact, using Lemma 1 and equality (3), we obtain:

$$\left[\frac{d}{dt} - a(A)\right]^2 U(t) = \left[\frac{d}{dt} - a(A)\right]^2 \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\left[\frac{d}{dt}-a(A)\right]^2 \{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{e^{\nu t}y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda F_f(t,\lambda)y(\lambda)\,d\mu(\lambda) = f(t).$$

Using Lemma 2, we prove the realization of condition (7):

$$\int_0^T U(t)\,dt = \int_0^T \left[\int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}|_{\nu=0}\, d\mu(\lambda)\right]dt =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\left[\int_0^T G(t,\nu,\lambda)\,dt\right]y(\lambda)\Big|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)[0]y(\lambda)|_{\nu=0}\, d\mu(\lambda) = 0.$$

Similarly, using Lemma 2, we show that vector-function (15) satisfies condition (8):

$$\int_0^T tU(t)\,dt = \int_0^T t\left[\int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda)\right]dt =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\left[\int_0^T tG(t,\nu,\lambda)\,dt\right]y(\lambda)\Big|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)[0]y(\lambda)\Big|_{\nu=0} d\mu(\lambda) = 0.$$

This completes our proof. ♦

Remark 1. *The Stieltjes integral in formula* (15) *is taken only on* $\Lambda = \mathbb{C} \setminus P$ *since function* $G(t,\nu,\lambda)$ *is analytical with respect to* λ *only in* Λ *(by Lemma* 3*).*

Remark 2. *Vector-function* (15) *defines a formal solution of problem* (4), (7), (8), *since we assume the following equalities to hold:*

$$\left[\frac{d}{dt}-a(A)\right]^2 \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda) =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\left[\frac{d}{dt}-a(A)\right]^2 \{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda),$$

$$\int_0^T \left[\int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda)\right]dt =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\left[\int_0^T G(t,\nu,\lambda)\,dt\right]y(\lambda)\Big|_{\nu=0} d\mu(\lambda),$$

$$\int_0^T t\left[\int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda)\right]dt =$$

$$= \int_\Lambda F_f\left(\frac{d}{d\nu},\lambda\right)\left[\int_0^T tG(t,\nu,\lambda)\,dt\right]y(\lambda)\Big|_{\nu=0} d\mu(\lambda),$$

as well as the integrals in those formulas to exist.

Remark 3. *If* $a(\lambda) \equiv a = \text{const}$ *and* $\text{sh}\frac{aT}{2} \neq \pm\frac{aT}{2}$ *(e. g.* $a \in \mathbb{R}$ *or* $a \in i\mathbb{R}$*) then* $P = \varnothing$. *Then in Theorem 1* $\Lambda = \mathbb{C}$. *If for* $a \in \mathbb{C}$ *there holds the condition* $\text{sh}\frac{aT}{2} = \pm\frac{aT}{2}$, *then* $P = \mathbb{C}$. *In this case, the solution of problem* (4), (7), (8) *does not exist for* $f(t) \not\equiv 0$, *and problem* (4), (7), (8) *has nonzero solutions if* $f(t) \equiv 0$.

Remark 4. *The solution of problem* (4)–(6) *is a sum of a solution of problem* (4), (7), (8) *and a solution of problem* (9), (5), (6), *i. e. it could be represented in the form*

$$U(t) = \int_{\Lambda} F_f\left(\frac{d}{d\nu}, \lambda\right)\{G(t,\nu,\lambda)y(\lambda)\}\Big|_{\nu=0} d\mu(\lambda) +$$

$$+ \sum_{k=1}^{2} \int_{\Lambda} R_{\varphi_k}(\lambda)\{M_k(t,\lambda)y(\lambda)\}\, d\mu(\lambda),$$

where $\{M_1(t,\lambda), M_2(t,\lambda)\}$ *is the system of solutions of the ODE*

$$\left[\frac{d}{dt} - a(\lambda)\right]^2 M = 0,$$

which satisfies the conditions

$$\int_0^T M_1(t,\lambda)\, dt = 1, \qquad \int_0^T tM_1(t,\lambda)\, dt = 0,$$

$$\int_0^T M_2(t,\lambda)\, dt = 0, \qquad \int_0^T tM_2(t,\lambda)\, dt = 1,$$

and the vectors φ_1, φ_2 *are represented in the form*

$$\varphi_k = \int_{\Lambda} R_{\varphi_k}(\lambda) y(\lambda)\, d\mu(\lambda), \qquad k \in \{1,2\},$$

where $R_{\varphi_1}(\lambda)$, $R_{\varphi_2}(\lambda)$ *are certain linear operators in* H, $\lambda \in \Lambda = \mathbb{C} \setminus P$ *(see* [18]).

3. Problem for nonhomogeneous PDE with homogeneous integral conditions. In this section, we shall describe the usage of the proposed abstract approach for solving a similar problem with integral conditions for PDE of second order in time and generally infinite order in a spatial variable:

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]^2 U(t,x) = f(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}, \tag{16}$$

$$\int_0^T U(t,x)\, dt = 0, \qquad \int_0^T tU(t,x)\, dt = 0, \qquad x \in \mathbb{R}, \tag{17}$$

where $a\left(\frac{\partial}{\partial x}\right)$ is a differential expression with entire symbol $a(\lambda) \neq \text{const}$.

We can treat problem (16), (17) as a particular case of the problem (4), (7), (8) with operator $A = \frac{\partial}{\partial x}$, and its eigenvector $e^{\lambda x}$ acting in a Banach space H of entire functions $U(t,x)$.

As a class $\mathcal{N}$ we take the class of entire functions $f(t,x)$, which for fixed $t \in [0,T]$ are quasipolynomials of the form

$$f(t,x) = \sum_{j=1}^{m} Q_j(t,x) e^{\alpha_j x}, \tag{18}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{C} \setminus P$, P is the set (12), $Q_1(t,x), \ldots, Q_m(t,x)$ are entire functions which for fixed $t \in [0,T]$ are polynomials, $m \in \mathbb{N}$.

For $f \in \mathcal{N}$, as a measure $\mu(\lambda)$ on $\Lambda = \mathbb{C} \setminus P$ take the Dirac measure on Λ, for which $\int_\Lambda g(\lambda)\, d\mu(\lambda)$ equals $g(0)$ (under assumption $0 \in \Lambda$), as analytical operator F_f take the operator defined as follows:

$$F_f(t,\lambda)e^{\lambda x} \equiv f\left(t, \frac{\partial}{\partial \xi}\right) e^{\xi x}\Big|_{\xi=\lambda}.$$

We shall show that for $f \in \mathcal{N}$ there holds the equality (analogue of equality (3)):

$$f(t,x) = \int_\Lambda F_f(t,\lambda) e^{\lambda x}\, d\mu(\lambda).$$

In fact,

$$\int_\Lambda F_f(t,\lambda)e^{\lambda x}\, d\mu(\lambda) = \int_\Lambda f\left(t, \frac{\partial}{\partial \xi}\right) e^{\xi x}\Big|_{\xi=\lambda} \delta(\lambda)\, d\lambda =$$

$$= \int_\Lambda f(t,x)e^{\lambda x}\, \delta(\lambda)\, d\lambda = f(t,x)e^{\lambda x}\big|_{\lambda=0} = f(t,x).$$

From solution (15) of problem (4), (7), (8), for problem (16), (17) we have

$$U(t,x) = \int_\Lambda f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\nu=0} d\mu(\lambda) =$$

$$= f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}}.$$

Theorem 2. *Let* $f(t,x)$ *in equation* (16) *be entire function of the form* (18) *and belongs to* $\mathcal{N}$. *Then in class* $\mathcal{N}$ *there exists a unique solution of the problem* (16), (17) *which can be represented in the form*

$$U(t,x) = f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}}, \tag{19}$$

where $G(t,\nu,\lambda)$ *is the function* (13).

P r o o f. Function $G(t,\nu,\lambda)$ by Lemma 4 is entire with respect to ν and analytical in $\mathbb{C} \setminus P$ with respect to λ. For $f \in \mathcal{N}$, we define the infinite order differential expression $f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)$ as a formal substitution of t by $\frac{\partial}{\partial \nu}$ and x by $\frac{\partial}{\partial \lambda}$ in Maclaurin series of $f(t,x)$. Then expression (19) is a series that defines entire function $U(t,x)$, which is entire with respect to t ($G(t,\nu,\lambda)$ is a first order entire function with respect to t) and for each $t \in [0,T]$ is a quasipolynomial of the form (18), that follows from the equality

$$\left[\sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right) e^{\alpha_j \partial/\partial\lambda}\right]\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}} =$$

$$= \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=\alpha_j\\ \nu=0}},$$

i. e. $U(t,x)$ belongs to $\mathcal{N}$.

Let's show that function (19) satisfies equation (16):

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]^2 U(t,x) = \left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]^2 \left(f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}} \right) =$$

$$= f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\left\{ \left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]^2 (G(t,\nu,\lambda)e^{\lambda x}) \right\}\Bigg|_{\substack{\lambda=0\\ \nu=0}} =$$

$$= f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\left\{ e^{\lambda x}\left[\frac{d}{dt} - a(\lambda)\right]^2 G(t,\nu,\lambda) \right\}\Bigg|_{\substack{\lambda=0\\ \nu=0}} =$$

$$= f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{e^{\lambda x+\nu t}\}\Big|_{\substack{\lambda=0\\ \nu=0}} = f(t,x)\{e^{\lambda x+\nu t}\}\big|_{\substack{\lambda=0\\ \nu=0}} = f(t,x).$$

Now let's show that integral conditions (17) hold for function (19):

$$\int_0^T U(t,x)\,dt = \int_0^T f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}} dt =$$

$$= f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\left\{ e^{\lambda x}\int_0^T G(t,\nu,\lambda)\,dt \right\}\Bigg|_{\substack{\lambda=0\\ \nu=0}} = 0,$$

$$\int_0^T tU(t,x)\,dt = \int_0^T tf\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}} dt =$$

$$= f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\left\{ e^{\lambda x}\int_0^T tG(t,\nu,\lambda)\,dt \right\}\Bigg|_{\substack{\lambda=0\\ \nu=0}} = 0.$$

Now we shall prove the uniqueness of the solution of problem (16), (17) in the class $\mathcal{N}$. Assume that for $f \in \mathcal{N}$ in the class $\mathcal{N}$ there exist two solutions $U_1(t,x)$, $U_2(t,x)$ of problem (16), (17) in the strip $[0,T]\times\mathbb{R}$. Then they could be represented in the form:

$$U_1(t,x) = f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G_1(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}},$$

$$U_2(t,x) = f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{G_2(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}},$$

where $G_1(t,\nu,\lambda)$, $G_2(t,\nu,\lambda)$ are solutions of equation (11).

Function $U(t,x) = U_1(t,x) - U_2(t,x)$ in the strip $[0,T]\times\mathbb{R}$ satisfies the homogeneous equation

$$\left[\frac{\partial}{\partial t} - a\left(\frac{\partial}{\partial x}\right)\right]^2 U(t,x) = 0$$

and the homogeneous conditions (17). Thus, we obtain

$$f\left(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \lambda}\right)\{(G_1(t,\nu,\lambda) - G_1(t,\nu,\lambda))e^{\lambda x}\}\Big|_{\substack{\lambda=0\\ \nu=0}} \equiv 0.$$

The left-hand side of the last equation is a quasipolynomial with respect to λ. It contains the values of the function $G_1(t,\nu,\lambda) - G_2(t,\nu,\lambda)$ and its derivatives with respect to ν and λ at the points $(t,\nu,\lambda) \in [0,T]\times\mathbb{C}\times(\mathbb{C}\setminus P)$. Since $G_1(t,\nu,\lambda) - G_2(t,\nu,\lambda)$ satisfies the homogeneous ODE

$$\left[\frac{d}{dt} - a(\lambda)\right]^2 G = 0$$

and the homogeneous integral conditions (14) as well as $\lambda \in \mathbb{C} \setminus P$, i. e. $\Delta(\lambda) \neq 0$, we conclude that $G_1(t,\nu,\lambda) - G_2(t,\nu,\lambda) \equiv 0$ on $[0,T] \times \mathbb{C} \times (\mathbb{C} \setminus P)$. Hence $U_1(t,x) = U_2(t,x)$. ♦

4. Examples of applications.

Example 1. Find the solution of the problem for bicalorical equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^2 U(t,x) = e^{x+t}$$

with homogeneous integral conditions

$$\int_0^1 U(t,x)\,dt = \int_0^1 tU(t,x)\,dt = 0 .$$

► For the given problem, we have $a(\lambda) = \lambda^2$, $T = 1$, $f(t,x) = e^{x+t}$,

$$\Delta(\lambda) = \frac{4e^{\lambda^2}\left(\operatorname{sh}^2 \frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right)}{\lambda^4},$$

$$P = \left\{\lambda \in \mathbb{C}\setminus\{0\} : \operatorname{sh}\frac{\lambda^2}{2} = \pm\frac{\lambda^2}{2}\right\}, \qquad 1 \notin P,$$

$$G(t,\nu,\lambda) = \frac{e^{\nu t} - I(\nu)\frac{I''(a) - I'(a)t}{\Delta(\lambda)} e^{\lambda^2 t} - I'(\nu)\frac{-I'(a) + I(a)t}{\Delta(\lambda)} e^{\lambda^2 t}}{[\nu - \lambda^2]^2},$$

$$I(a) = \frac{e^{\lambda^2} - 1}{\lambda^2}.$$

Using formula (19), we obtain

$$U(t,x) = e^{\partial/\partial\lambda + \partial/\partial\nu}\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\nu=0\\ \lambda=0}} = \{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\nu=1\\ \lambda=1}}.$$

Since the denominator of the function $G(t,\nu,\lambda)$ vanishes at $\nu = 1$, $\lambda = 1$, we treat the substitution in the last expression as the calculation of a limit:

$$U(t,x) = \lim_{\substack{\nu\to 1\\ \lambda\to 1}} \{G(t,\nu,\lambda)e^{\lambda x}\}.$$

After simplification, we obtain

$$U(t,x) = \frac{e^{x+t}((e^2 - 3e + 1)t^2 + (2e^2 - 7e + 4)t - e^2 + 2e + 2)}{2(e^2 - 3e + 1)}.$$

This solution is unique in the class of entire functions $\mathcal{N}$. ◄

Example 2. Find the solution of the differential-functional equation

$$\frac{\partial^2}{\partial t^2} U(t,x) - 2\frac{\partial}{\partial t} U(t,x+1) + U(t,x+2) = te^{t+x}, \tag{20}$$

which satisfies homogeneous integral conditions

$$\int_0^1 U(t,x)\,dt = \int_0^1 tU(t,x)\,dt = 0. \tag{21}$$

► It's easy to see that equation (20) could be represented in the form (16), where $a\left(\frac{\partial}{\partial x}\right) = e^{\partial/\partial x}$, $f(t,x) = te^{t+x}$. Function $G(t,\nu,\lambda)$ has the form

$$G(t,\nu,\lambda) = \frac{e^{\nu t} - I(\nu)\dfrac{I''(a) - I'(a)t}{\Delta(\lambda)} e^{e^{\lambda}t} - I'(\nu)\dfrac{-I'(a) + I(a)t}{\Delta(\lambda)} e^{e^{\lambda}t}}{[\nu - e^{\lambda}]^2},$$

where $a(\lambda) = e^{\lambda}$, $I(a) = \dfrac{e^{e^{\lambda}} - 1}{e^{\lambda}}$.

Set (12) for problem (20), (21) achieves the form

$$P = \left\{ \lambda \in \mathbb{C} : \Delta(\lambda) \equiv \frac{4e^{e^{\lambda}}\left(\operatorname{sh}^2 \dfrac{e^{\lambda}}{2} - \dfrac{e^{2\lambda}}{4}\right)}{e^{4\lambda}} = 0 \right\},$$

and $1 \notin P$ at that.

By formula (19), we compute the solution of the problem (20), (21):

$$U(t,x) = f\left(\frac{\partial}{\partial\nu}, \frac{\partial}{\partial\lambda}\right)\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\nu=0\\ \lambda=0}} =$$

$$= e^{\partial/\partial\nu + \partial/\partial\lambda} \frac{\partial}{\partial\nu}\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\nu=0\\ \lambda=0}} =$$

$$= \frac{\partial}{\partial\nu}\{G(t,\nu,\lambda)e^{\lambda x}\}\Big|_{\substack{\nu=1\\ \lambda=1}} = G'_{\nu}(t,1,1)e^{x}.$$

After simplification, we obtain:

$$U(t,x) = \frac{e^{x}}{(e-1)^3(e^{e+2} - e^{2e} + 2e^{e} - 1)}\big[(t-2)e^{t} - te^{t+1} + (-2t+4)e^{t+e} +$$

$$+ 2te^{t+e+1} + (-t+2)e^{t+e+2} + te^{t+e+3} + (t-2)e^{t+2e} -$$

$$- te^{t+2e+1} + 6e^{et+1} + (3t-10)e^{et+2} + (-7t+3)e^{et+3} +$$

$$+ (3t-1)e^{et+4} - te^{et+5} - 6e^{et+e+1} + (-3t+16)e^{et+e+2} +$$

$$+ (10t-16)e^{et+e+3} + (-6t+7)e^{et+e+4} + (t-1)e^{et+e+5}\big].$$

The found solution of problem (20), (21) is unique in the class of entire functions $\mathcal{N}$. ◄

Conclusions. We propose an approach to solving a problem for nonhomogeneous differential-operator equation of second order with homogeneous integral conditions.

Provided that the right-hand side of the equation is expressed in the form of Stieltjes integral over a certain measure, the solution of the problem is also expressed in such a form.

We also extend the proposed method to the case of the problem with homogeneous time integral conditions for PDE of second order in time and, in general, infinite order in spatial variable.

In future research, it would be interesting to study similar problems with integral conditions in the form of moments for differential-operator equation of any finite order.

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ЗАДАЧА ДЛЯ НЕОДНОРІДНОГО ЕВОЛЮЦІЙНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ З ОДНОРІДНИМИ ІНТЕГРАЛЬНИМИ УМОВАМИ

Запропоновано метод розв'язування задачі з однорідними інтегральними умовами для неоднорідного еволюційного рівняння з абстрактним оператором у лінійному просторі H *. Права частина еволюційного рівняння, що для фіксованої часової змінної належить до спеціального підпростору* $N \subseteq H$ *, зображається інтегралом Стілтьєса за деякою мірою. Розв'язок задачі зображено також у вигляді інтеграла Стілтьєса за цією ж мірою. Подано приклади застосування методу до розв'язування задачі з інтегральними умовами для рівняння із частинними похідними другого порядку за часовою змінною і в загальному випадку – нескінченного порядку за просторовою змінною.*

ЗАДАЧА ДЛЯ НЕОДНОРОДНОГО ЭВОЛЮЦИОННОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ОДНОРОДНЫМИ ИНТЕГРАЛЬНЫМИ УСЛОВИЯМИ

Предложен метод решения задачи с однородными интегральными условиями для неоднородного эволюционного уравнения с абстрактным оператором в линейном пространстве H *. Правая часть эволюционного уравнения, которая для фиксированной временной переменной принадлежит специальному подпространству* $N \subseteq H$ *, представляется интегралом Стилтьеса по некоторой мере. Решение задачи представлено также в виде интеграла Стилтьеса по этой же мере. Приведены примеры использования метода к решению задачи с интегральными условиями для уравнения с частными производными второго порядка по временной переменной и в общем случае – бесконечного порядка по пространственной переменной.*

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