UDK 517.956.4

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ESTIMATES OF WEAK SOLUTIONS TO NONDIAGONAL PARABOLIC SYSTEM OF TWO EQUATIONS

Estimates of L^∞-norms of weak solutions has been obtained for a model nondiagonal parabolic system of nonlinear differential equations with matrix of coefficients satisfying special structure conditions. A technique based on estimating the certain function of unknowns is employed to this end.

1. Introduction. In the present paper we study the boundedness of weak solutions to the nonlinear nondiagonal parabolic system of two equations in divergence form under special assumptions upon its structure.

It is well-known that the De Giorgi – Nash – Moser estimates are no longer valid in general for an elliptic system, the latter can be regarded as a special case of the parabolic version. An example of an unbounded solution to the linear elliptic system with bounded coefficients was built up by E. De Giorgi in [4]. There is yet another example due to J. Nečas and J. Souček of a nonlinear elliptic system with the coefficients sufficiently smooth, but the weak solution not belonging to $W^{2,2}$.

These two and many other examples prove that the regularity problem for elliptic systems proves to be far more complicated then that for second order elliptic equations.

Concerning systems of differential equations until now a priori estimates of De Giorgi type has been extended only to a special class of parabolic systems of equations, the so-called weakly coupled systems.

Therefore there constitutes an interest the question of finding strongly-coupled systems, whose solutions exhibit certain regularity.

The technique we are utilizing has been employed earlier in [6] for semilinear systems (see also [3, 7] and [5]), and consists in switching to new function, for which the estimate is established in a conventional way, whence the final conclusion about each component of the vector function solution follows. This technique allows to tackle nonlinear nondiagonal systems.

The main idea of our approach is as follows: instead of trying to establish estimates for each component of solution (u,v) rather to introduce some new function of components of the solution $H(u,v)$ from whose estimate we shall be able to derive the estimates for the components of solution (u,v).

In the present paper, although restricting ourselves to systems of second order equations in divergence form possessing special structure, we demonstrate boundedness of solution to nonlinear parabolic systems of equations in which coupling occurs in the leading derivatives and whose leading coefficients depend on x, u, and v.

2. Basic notations and hypotheses. Here and onward we accept the following notations: $Q = \Omega \times (0,T]$; $S = \partial\Omega \times (0,T]$; $\partial Q = \{\Omega \times \{0\}\} \cup \{\partial\Omega \times (0,T]\}$; Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary; $x \in \Omega$; $T > 0$; $t \in (0,T]$; $n \ge 2$; $i = 1,\ldots,n$; $j = 1,2$; and summation convention over repeated indices is assumed; $W_0^{1,2}(\Omega)$ is a space of functions in $W^{1,2}(\Omega)$ vanishing on $\partial\Omega$ in the sense of traces for a.e. $t \in (0,T]$.

We shall be concerned with a system of two equations of the form:

$$u_t - \operatorname{div}\left(a_1(x,u,v)\nabla u + b_1(x,u,v)\nabla v\right) = f_1(x,t)\frac{1}{\sqrt{1+|u|+|v|}},$$

$$v_t - \operatorname{div}\left(a_2(x,u,v)\nabla u + b_2(x,u,v)\nabla v\right) = f_2(x,t)\frac{1}{\sqrt{1+|u|+|v|}}, \quad (x,t)\in Q, \tag{1}$$

$$f_j(x,t)\in L^\tau(Q), \qquad \tau > (n+2)/2. \tag{2}$$

About the coefficients of the model system we suppose that there is a function of two variables $\tilde{H}(u,v)$ such that $\forall\ x,u,v\in\mathbb{R}$

$$C_1(u^2+v^2)\le \tilde{H}(u,v)\le C_2(u^2+v^2), \tag{3}$$

$$0\le \left|\tilde{H}_u(u,v)\right|, \left|\tilde{H}_v(u,v)\right| \le C_2(|u|+|v|), \tag{4}$$

$$0\le \left|\tilde{H}_{uu}(u,v)\right|, \left|\tilde{H}_{uv}(u,v)\right|, \left|\tilde{H}_{vv}(u,v)\right| \le C_2, \tag{5}$$

where $C_1>0$, $C_2<\infty$ are constants; and there holds the following hypotheses

$$a_1(x,u,v)\tilde{H}_u(u,v) + a_2(x,u,v)\tilde{H}_v(u,v) = \Lambda(x,u,v)\tilde{H}_u(u,v),$$

$$b_1(x,u,v)\tilde{H}_u(u,v) + b_2(x,u,v)\tilde{H}_v(u,v) = \Lambda(x,u,v)\tilde{H}_v(u,v), \tag{6}$$

and

$$a_1\tilde{H}_{uu}(u,v) + a_2\tilde{H}_{uv}(u,v)\ge 0, \tag{7}$$

$$\begin{Vmatrix} 2(a_1\tilde{H}_{uu}+a_2\tilde{H}_{uv}) & (a_1+b_2)\tilde{H}_{uv}+b_1\tilde{H}_{uu}+a_2\tilde{H}_{vv} \\ (a_1+b_2)\tilde{H}_{uv}+b_1\tilde{H}_{uu}+a_2\tilde{H}_{vv} & 2(b_1\tilde{H}_{uv}+b_2\tilde{H}_{vv}) \end{Vmatrix} \ge 0, \tag{8}$$

where Λ is a measurable $\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ function such that

$$0<\Lambda_1\le\Lambda(x,u,v)\le\Lambda_2 \qquad \forall x,u,v\in\mathbb{R}, \tag{9}$$

$\Lambda_{1,2}$ are numbers.

By parabolicity of system (1) it is meant that the part without derivatives with respect to time is elliptic. The notion of ellipticity of a system of differential equations is understood in the sense introduced in [1]. We assume that the coefficients a_1, a_2, b_1, b_2 are such that the system is parabolic.

Example. Here is the example of a parabolic model system satisfying our hypotheses:

$$a_1(u,v)=\Lambda(u,v)-\frac{a_2(u,v)}{\alpha}, \qquad b_2(u,v)=\Lambda(u,v)-b_1(u,v)\alpha,$$

$$\alpha=\frac{K_u}{K_v}, \qquad K=u^2+v^2+\varepsilon uv,$$

$$C_1\le\Lambda(u,v)\le C_2, \qquad |a_2|\le\frac{C_3|\alpha|}{(1+|\alpha|)},$$

$$|b_1|\le\frac{C_3}{(1+|\alpha|)}, \qquad \varepsilon<\frac{1}{10}, \qquad C_1\ge 5, \qquad C_3>0. \qquad ◄$$

The boundary conditions of the Dirichlet type are assigned:

$$(u-g_1, v-g_2)(x,t)\in W_0^{1,2}(\Omega), \qquad t\in(0,T),$$

$$(u,v)(x,0)=(u_0,v_0)(x). \tag{10}$$

A solution to system (1) with Dirichlet data (10) is understood in the weak sense, as in [2].

Definition. A measurable vector function $(u^1,u^2)=(u,v)$ is called a *weak solution* of problem (1)–(10) if

$$u^j\in C(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$$

and for all $t \in (0,T]$

$$\int_\Omega u^j \varphi_j(x,t)\,dx + \iint_{\Omega\times(0,T]} \{-u^j \varphi_{j\,t} + a_j u^1_{x_i} \varphi_{j\,x_i} + b_j u^2_{x_i}\varphi_{j\,x_i}\}\,dx\,d\tau =$$

$$= \int_\Omega u_0^j \varphi_j(x,0)\,dx + \iint_{\Omega\times(0,T]} f_j \varphi_j \frac{1}{\sqrt{1+|u^1|+|u^2|}}\,dx\,d\tau$$

for all testing functions

$$\varphi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)).$$

The boundary condition in (10) is meant in the weak sense.

About the coefficients of the system (1) it is additionally assumed that they are measurable $\Omega\times\mathbb{R}^2 \to \mathbb{R}$ Caratheodory functions that satisfy the ellipticity condition and are subject to the growth conditions:

$$\exists \Lambda_2 > 0 \quad \forall r \in \mathbb{R}^2, \quad x \in \mathbb{R}^n, \qquad |a_j(x,r)|, \qquad |b_j(x,r)| \le \Lambda_2. \tag{11}$$

On the functions $g_j(x,t)$, $(u_0, v_0)(x)$ in boundary data (10) we assume to be fulfilled the following assumptions:

$$g_j(x,t) \in L^\infty(S), \qquad (u_0,v_0)(x) \in L^\infty(\bar{\Omega}\times\{0\}).$$

3. Estimates of L^∞-norms. Let us now turn our attention to the question of boundedness of weak solutions to a system with whose coefficients satisfy assumptions (6)–(8). Our main result is the following.

Theorem 1. *Let* (u,v) *be a solution to system* (1). *For the function* $\tilde{H}$ *defined by* (3)–(5) *the following estimate holds*

$$\|\tilde{H}\|_{L^\infty(Q)} \le C,$$

hence it is easily seen that the same estimates take place for the components of the solution themselves:

$$\|u\|_{L^\infty(Q)} \le C, \qquad \|v\|_{L^\infty(Q)} \le C,$$

where constant C *depends only on the data*: n, f_j, $\Lambda_{1,2}$, $\operatorname{mes} Q$, $\|g_1\|_{L^\infty(S)}$, $\|g_2\|_{L^\infty(S)}$, $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$, *constants in the embedding theorems*, *constants* $C_{1,2}$ *in hypotheses* (3)–(5), *and is independent of* u *and* v.

To prove the Theorem we need the well-known Stampacchia's lemma.

Lemma 1. *Let* $\psi(y)$ *be a nonnegative nonincreasing function defined on* $[k_0,\infty)$ *which satisfies*

$$\psi(m) \le \frac{C}{(m-k)^\vartheta}\{\psi(k)\}^\delta \qquad \text{for} \qquad m > k \ge k_0$$

with $\vartheta > 0$ *and* $\delta > 1$. *Then* $\psi(k_0 + d) = 0$, *where* $d = C^{1/\vartheta}\{\psi(k_0)\}^{(\delta-1)/\vartheta} 2^{\delta/(\delta-1)}$.

For p r o o f see lemma 4.1 [1, p. 8].

We make also use of the following lemma (see Prop. 3.1 [2, p. 7]).

Lemma 2. *If* $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$, *then there holds the inequality*

$$\int_Q u^q \le \Big(\int_Q |\nabla u|^2\Big)\Big(\operatorname{ess\,sup}_{0<t<T}\int_\Omega u^2\Big)^{2/n}$$

with $q = 2(n+2)/n$ *and constant* C *depending only on* n.

P r o o f of Theorem 1. Multiply the first equation of (1) by H_u and add the second one multiplied by H_v (the H is to be defined later). Choose $(H-k)_+$ as a testing function with $k \ge k_0 = \max\{\|H(g_1,g_2)\|_{L^\infty(S)}, \|H(u_0,v_0)\|_{L^\infty(\Omega)}\}$ it is easy to check that this choice of testing function is admissible. After integration in τ from 0 to t, $t \le T$, and in x over the domain Ω, this results in

$$\frac{1}{2}\int_\Omega (H-k)^2\chi_{A(k)}(t) + \int_0^t\int_\Omega \{\langle a_1\nabla u + b_1\nabla v,\ H_{uu}(H-k)\nabla u + H_{uv}(H-k)\nabla v +$$

$$+ H_u^2\nabla u + H_u H_v \nabla v\rangle + \langle a_2\nabla u + b_2\nabla v, H_{uu}(H-k)\nabla u +$$

$$+ H_{uv}(H-k)\nabla v + H_u^2\nabla u + H_u H_v\nabla v\rangle\}\chi_{A(k)} =$$

$$= \int_0^t\int_\Omega (f_1 H_u + f_2 H_v)\frac{(H-k)\chi_{A(k)}}{\sqrt{1+|u|+|v|}},$$

$\chi_{A(k)}$ is a characteristic function of the domain $A(k,t) = \{x \in \Omega \mid H-k \ge 0\}$. We have

$$\langle a_1\nabla u + b_1\nabla v,\ H_{uu}(H-k)\nabla u + H_{uv}(H-k)\nabla v + H_u^2\nabla u + H_u H_v\nabla v\rangle +$$

$$+\langle a_2\nabla u + b_2\nabla v,\ H_{uu}(H-k)\nabla u + H_{uv}(H-k)\nabla v + H_u^2\nabla u +$$

$$+ H_u H_v\nabla v\rangle = \{[a_1 H_u^2 + a_2 H_u H_v]|\nabla u|^2 + [(a_1+b_2)H_u H_v +$$

$$+ b_1 H_u^2 + a_2 H_v^2](\nabla u\nabla v) + [b_1 H_u H_v + b_2 H_v^2]|\nabla v|^2\} +$$

$$+\{[a_1 H_{uu} + a_2 H_{uv}]|\nabla u|^2 + [(a_1+b_2)H_{uv} + b_1 H_{uu} +$$

$$+ a_2 H_{vv}](\nabla u\nabla v) + [b_1 H_{uv} + b_2 H_{vv}]|\nabla v|^2\}(H-k).$$

Making the substitution

$$F(x) = \sqrt{x}, \qquad H = F(\tilde H), \qquad H_u = F'\tilde H_u, \qquad H_v = F'\tilde H_v,$$

$$H_{uu} = F''\tilde H_u^2 + F'\tilde H_{uu}, \qquad H_{uv} = F''\tilde H_u\tilde H_v + F'\tilde H_{uv},$$

$$H_{vv} = F''\tilde H_v^2 + F'\tilde H_{vv},$$

according to hypothesis (6) the first group of terms in curly brackets gives

$$\{\ldots\} = \Lambda H_u^2|\nabla u|^2 + \Lambda H_u H_v(\nabla u\nabla v) + \Lambda H_v^2|\nabla v|^2 =$$

$$= \Lambda|\nabla H|^2 = \Lambda F'^2|\nabla\tilde H|^2.$$

In virtue of hypothesis (7), (8) for the second group of terms in curly brackets we have

$$\{\ldots\}(H-k) = \Lambda F''|\nabla\tilde H|^2(H-k) + \{[a_1\tilde H_{uu} + a_2\tilde H_{uv}]|\nabla u|^2 +$$

$$+[(a_1+b_2)\tilde H_{uv} + b_1\tilde H_{uu} + a_2\tilde H_{vv}](\nabla u\nabla v) + [b_1\tilde H_{uv} +$$

$$+ b_2\tilde H_{vv}]|\nabla v|^2\}F'(H-k) \ge \Lambda F''|\nabla\tilde H|^2(F(\tilde H)-k).$$

Hence, making use of hypothesis (4), we get

$$\frac{1}{2}\int_\Omega (F(\tilde H)-k)^2\chi_{A(k)}(t) + \int_0^t\int_\Omega \Lambda\frac{k}{4\tilde H^{3/2}}|\nabla\tilde H|^2\chi_{A(k)} \le$$

$$\le \int_0^t\int_\Omega C|f|F'(F(\tilde H)-k)\chi_{A(k)},$$

where it is denoted $|f| = |f_1| + |f_2|$. Recalling the definition of $\tilde H$ and making some transformations we can rewrite this as

$$\sqrt{k}\int_\Omega (\sqrt[4]{\tilde H}-\sqrt{k})^2\chi_{A(\sqrt{k})}(t)+k\Lambda_1\int_0^t\int_\Omega \left|\nabla(\sqrt[4]{\tilde H}-\sqrt{k})\right|^2\chi_{A(\sqrt{k})}\le$$

$$\le\int_0^t\int_\Omega C|f|(\sqrt[4]{\tilde H}-\sqrt{k})\chi_{A(\sqrt{k})},$$

where $\chi_{A(\sqrt{k})}$ is a characteristic function of the set $A(\sqrt{k})=\{x\in\Omega \mid \sqrt[4]{\tilde H}-\sqrt{k}\ge 0\}$. Since $t\in(0,T]$ is arbitrary, then taking the supremum we have:

$$\sqrt{k}\sup_{0<t<T}\int_\Omega (\sqrt[4]{\tilde H}-\sqrt{k})^2\chi_{A(\sqrt{k})}(t)+k\Lambda_1\int_0^T\int_\Omega \left|\nabla(\sqrt[4]{\tilde H}-\sqrt{k})\right|^2\chi_{A(\sqrt{k})}\le$$

$$\le\int_0^T\int_\Omega C|f|(\sqrt[4]{\tilde H}-\sqrt{k})\chi_{A(\sqrt{k})}. \tag{12}$$

Applying generalized Hölder's inequality to the right of (12) we obtain

$$\sqrt{k}\sup_{0<t<T}\int_\Omega w^2+k\Lambda_1\int_0^T\int_\Omega|\nabla w|^2\le C\|w\|_{q,Q}\|f\|_{r,Q}\left(\int_0^T\int_\Omega\chi_{A(\sqrt{k})}\right)^{1-1/q-1/r},$$

where $w=(\sqrt[4]{\tilde H}-\sqrt{k})_+$, and r has been selected such that

$$\tau>r>4(2+n)/(n+8),$$

since it is not difficult to check that the later inequality holds. From Lemma 2 it follows that:

$$\|w\|_{q,Q}\le\left(\sup_{0<t<T}\int_\Omega w^2+\int_0^T\int_\Omega|\nabla w|^2\right)^{1/2}. \tag{13}$$

Since without loss of generality we may assume $k\ge1$, on the strength of this inequality we get:

$$\|w\|_{q,Q}^2\le C(k_0,\Lambda_1)\|w\|_{q,Q}\|f\|_{r,Q}\{\psi(\sqrt{k})\}^{1-1/q-1/r}, \tag{14}$$

here we've denoted:

$$\psi(\sqrt{k})=\int_0^T \text{mes}\,A(\sqrt{k},t)\,dt.$$

Applying Young's inequality to the right-hand side of (14) gives

$$\|w\|_{q,Q}\le C\{\psi(\sqrt{k})\}^{1-1/q-1/r}. \tag{15}$$

Let us estimate:

$$(\sqrt{m}-\sqrt{k})\{\psi(\sqrt{m})\}^{1/q}=$$

$$=(\sqrt{m}-\sqrt{k})\left(\int_0^T\int_\Omega\chi_{A(\sqrt{m})}\right)^{1/q}<\left(\int_0^T\int_\Omega w^q\chi_{A(\sqrt{m})}\right)^{1/q}<\|w\|_{q,Q},$$

where $m>k\ge k_0$. Substituting this into (15) we come down to

$$(\sqrt{m}-\sqrt{k})^q\psi(\sqrt{m})\le C\{\psi(\sqrt{k})\}^{q(1-1/q-1/r)}=C\{\psi(\sqrt{k})\}^{\delta}. \tag{16}$$

From the hypotheses on f_j and by the choice of r

$$\tau>r>\frac{(n+2)}{2},$$

hence $2\dfrac{(n+2)}{n}\left(1-\dfrac{n}{2(n+2)}-\dfrac{1}{r}\right)>1$ and thus $\delta>1$. On the strength of Lemma 1 from relation (16) we can conclude that

$$\psi(\sqrt{k_0}+d)=0,$$

for some d sufficiently large, but finite, depending only on the data: n, f_j, Λ_1, $\|g_1\|_{L^\infty(S)}$, $\|g_2\|_{L^\infty(S)}$, $\|u_0\|_{L^\infty(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$, constants in the embedding theorems and is independent of u and v. And thus

$$\|\tilde{H}\|_{L^\infty(Q)} \le C .$$

It is not difficult to see that due to the Young inequality the same estimates hold for the components (u, v) of solution themselves. Namely,

$$\|u\|_{L^\infty(Q)} \le C_1, \qquad \|u\|_{L^\infty(Q)} \le C_2 .$$

4. Conclusions. In the present paper we have established boundedness of weak solution to strongly coupled semilinear parabolic system of second order partial differential equations. The smooth properties of solutions to the systems of this kind are determined not just by smoothness of coefficients, right-hand sides and boundary data, but strongly depend upon the structure of the matrix of coefficients. We have demonstrated that in order for the solutions of such systems to exhibit certain amount of regularity additional, besides ellipticity, hypotheses upon the coefficients, like hypotheses (3)–(9), are needed. We have shown that there are strongly coupled nonlinear systems, in our case system (1), whose weak solutions are bounded. The L^∞-norms of these solutions depend not just on norms of right-hand sides of equations, norms of functions in boundary data (10), constants in the ellipticity condition and the domain Q, but also on the constants C_1, C_2, Λ_1, Λ_2 from structure hypotheses (3)–(5) and (9).

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ОЦІНКИ СЛАБКИХ РОЗВ'ЯЗКІВ НЕДІАГОНАЛЬНОЇ ПАРАБОЛІЧНОЇ СИСТЕМИ ДВОХ РІВНЯНЬ

Оцінки L^∞-норм слабких розв'язків встановлено для модельної недіагональної параболічної системи нелінійних диференціальних рівнянь з матрицею коефіцієнтів, що задовольняє спеціальні структурні умови. Застосовується техніка, що базується на оцінці певної функції від невідомих.

ОЦЕНКИ СЛАБЫХ РЕШЕНИЙ НЕДИАГОНАЛЬНОЙ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ ДВУХ УРАВНЕНИЙ

Оценки L^∞-норм слабых решений получены для модельной недиагональной параболической системы нелинейных дифференциальных уравнений с матрицей коэффициентов, удовлетворяющей специальным структурным условиям. Применяется техника, основывающаяся на оценке определённой функции от неизвестных.

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Received
25.02.08