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REMARKS ON TILED TORI

J. S. Birman and W. W. Menasco [3] *introduced and studied a class of embedded tori in closed braid complements which admit a standard tiling. K. Y. Ng showed* [6] *that each essential torus in a closed braid complement which admits standard tiling posseses a staircase pattern. The combinatorial and geometric description of the tori from this class was partially given in* [7]*. In this paper, we continue the study of the decomposition of tiled tori into building blocks and show that they are similar in certain sense. In particular, for the class of geometric tiled tori described by Ng in* [6]*, we show that each such torus consists only of elementary blocks.*

Introduction. In this paper, we continue the study of tiled tori introduced by J. S. Birman and W. W. Menasco in [3]. As combinatorial objects, these tori serve the patterns of geometric tori of standard position in a closed braid complement. In [3], Birman and Menasco study essential tori in closed braid complement via the natural (singular) foliations on them that are induced by the braid fibration and introduced the geometric standard tori of type $k \ge 2$. In [6], K. Y. Ng described a new class $\mathcal{R}$ of geometric tiled tori, the tori that are obtained by making tracks in standard tori of type $k \ge 2$.

In the previous paper [7], we described in combinatorial terms the block decomposition of tiled tori and gave their geometric interpretation. In particular, we showed that each tiled torus can be decomposed into building blocks of width 1, all they are elementary, for exception, may be one which is defective. In this paper, we show that all elementary building blocks of any block decomposition are similar in certain sense. This gives a geometric interpretation of a block decomposition of the corresponding solid tori in a closed braid complement. Moreover, we showed that each torus from the class $\mathcal{R}$ can be decomposed into elementary building blocks.

1. Preliminaries. In this section, we give some preliminaries and review some known facts which will be used later.

Let $L \subset \mathbb{R}^3$ be an oriented closed braid with the axis A and let $\mathcal{H} = \{H_\theta : \theta \in [0, 2\pi]\}$ be the open book decomposition of $\mathbb{R}^3$ by half-planes with boundary on the axis A. Moreover, let S' be a closed orientable incompressible surface in $\mathbb{R}^3 \setminus L$. Assume S' is in general position with respect to $\mathcal{H}$. The intersection of the H_θ's with S' induces a (singular) foliation $\mathcal{F}'$ on the surface S'.

It is known that by using isotopy in $\mathbb{R}^3 \setminus L$, the foliation $\mathcal{F}'$ can be standardize in such a way that the resulting foliated surface S is essential and allows a decomposition into some typical foliated regions ([2], Theorem 1.1). Each such region contains a unique saddle point and admits a canonical embedding in $\mathbb{R}^3$ with respect to the z-axis A [2]. The combinatorial decomposition $\mathcal{S}$ of a foliated surface S is supplied with some additional data (decoration) and serves a combinatorial pattern for the foliated surface S. In a special case, when all regions of such decomposition are the rectangular foliated tiles, the combinatorial pattern $\mathcal{S}$ for the foliated surface S is called a tiled surface.

Birman and Menasco showed [3] that any essential (incompressible and nonperipheral) torus T may be standardized by a sequence of braid isotopies and exchange moves to the one in a special position. The controlled moves used in this process take closed braids to closed braids and preserve link types. Braid isotopy means an isotopy in the complement of the braid axis which preserves braid structure at each stage and the exchange move is a special type of Reidemeister II move. The description of essential tori in a special position falls into three cases [3].

The most interesting case is when the embedded torus T admits a standard tiling (see Fig. 1 for a standard embedding of a tile in 3-dimensional space with respect to the axis z).

Birman and Menasco introduced for each $k \geq 2$ the tori of type k, which form a class of essential tori admitting standard tiling. But the geometric and combinatorial description of such tori was nor completed. More precisely, it was known that the tori of types $k \geq 2$ do not exhaust the class of essential tori which admit standard tiling. In [6], Ng described standard tilings of essential tori in link complements via the so-called staircase tiling patterns P. Such the tori are parameterized by the two parameters d and k, the width and the height of P, and are enhanced with some additional data called the decoration. Ng [6] also showed that every embedded torus T which admits a standard tiling possesses a staircase tiling pattern of even width $2n$ and height $k \geq 2$.

In [7], we gave a combinatorial description of essential tori in closed braid complements which admit standard tiling and bound a solid torus in $\mathbb{R}^3$ in terms of «building blocks». For this, we introduced the notion of a minimal combinatorial meridian on an (embedded) tiled torus and showed that minimal meridians from a suitable collection decompose the torus into building blocks of width 1. Each such building block for a tiled torus is bounded by the two parallel minimal meridians.

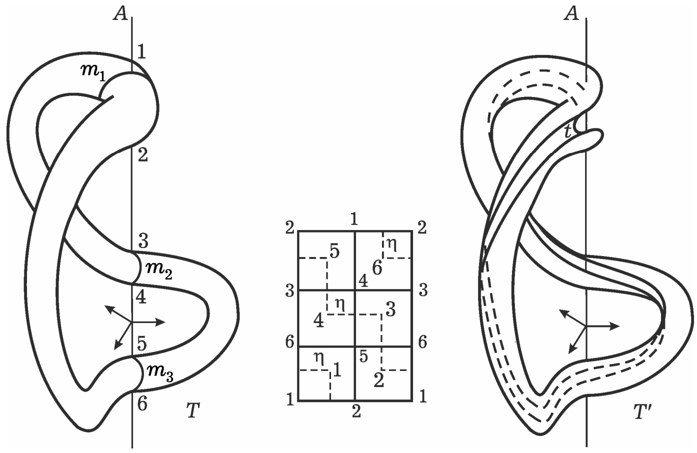
To describe combinatorial and geometric properties of building blocks, we shall introduce in the next section the notion of configuration of a meridian of a torus and study its properties. Note that the geometric position of each building block in $\mathbb{R}^3$ is determined uniquely up to the braid foliation preserving isotopy [1, 2].

2. The combinatorics of tiled tori with standard tiling. Let T be an oriented essential torus in a closed braid complement which admits tiling. We shall say that a tiling of T is *standard* if every its vertex is of valence 4 and for any vertex v the four tiles adjacent to v occur cyclically with signs $+, -, +, -$, when traveling on T around v. It is known [3] that each incompressible torus T in the complement of a closed braid L which admits *tiling* can be performed, via a sequence of braid isotopies, exchange moves (on closed braids) and the isotopies in closed braid complements, to an essential torus T' in the complement of a closed braid L' so that T' admits a *standard tiling*. Note that L' and L have the same link type $\mathcal{L}$ and the embedded torus T' can be chosen to be smooth.

Every torus T which admits a standard tiling can be cut open to a plane tiled fundamental domain. We say that T has the (d, k)-staircase tiling pattern or d by k staircase tiling pattern P if a standard tiling of T has a staircase-tiling fundamental domain P with k rows and d tiles across each row, and its two opposite zig-zag sides are identified on T with a possible shift in the order of vertices, while the top and bottom sides are identified so that the second vertex on the bottom side coincides with the first vertex on the top side. It follows from the definition of standard tiling that $k \geq 2$ and d is even. As discussed in [6], any embedded torus T which admits a standard tiling has essentially two staircase patterns, dual of each other in some sense.

Now, supplying the staircase P with some additional data (which is the cyclic order of vertices, the cyclic order of tiles, the signs of vertices and tiles) we obtain a tiled torus $\mathcal{T}$ represented in a modified form (here the surface T is cut along two cycles). Note that the question of whether a given (decorated) staircase pattern is embeddable can be answered by passing the tests in Proposition 1.1. of [7]. By Proposition 1.2 of [7], if a tiled torus $\mathcal{T}$ is embeddable, the corresponding embedding (in $\mathbb{R}^3$) is unique in a certain sense.

The fundamental domain of any embedded torus T admitting a standard tiling and having a type $k \geq 2$ can be represented by a rectangle of dimension 2 tiles by k tiles, where the opposite edges on the sides with k arcs are identified without any shift in the order [3]. For $k = 3$, see an example indicated in Fig. 1a and Fig. 1b.



a) b) c)

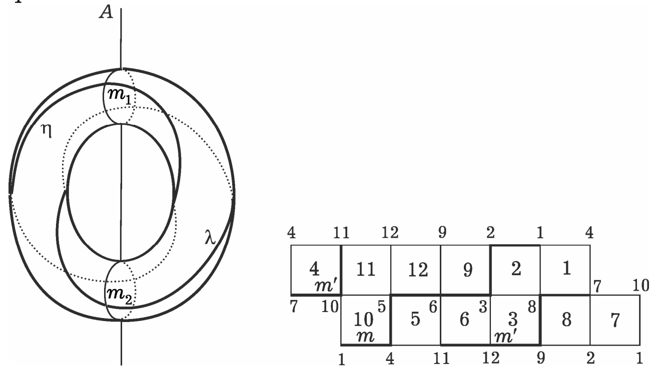
Fig. 1

Now let us recall the definition of the operation of making tracks on tori of type $k \geq 2$.

Let T be a torus of type $k \geq 2$ which is made of k cylinders C_i by consequent gluing them along the corresponding boundary components (see Fig. 1a for $k = 3$ and Fig. 2a for $k = 2$). Each closed curve $m_i = C_i \bigcap C_{i+i}$ is a meridian of the torus T and intersects the axis A exactly at two points, say x_i and y_i, $i = 1,\dots,k$. The points x_i and y_i decompose m_i into two arcs, α_i and β_i. The torus T admits a 2 by k rectangular pattern P (see Fig. 1b in the case $k = 3$). Consider on T a zig-zag longitude η (see Fig. 2a and Fig. 1b). Note that the longitude η intersects on T each meridian m_i, $i = 1,\dots,k$, at a unique point.

Push the surface T along η in the direction of inward pointed normal to the surface at the points of η until each arc α_i (β_i), which intersects η, has been isotoped relatively its endpoints to a new arc α_i' (β_i', respectively) which intersects A in two points more. In this case, we say that the resulting torus T' is obtained from the torus T by making a track along η on it (see Fig. 1c). Let λ be a zig-zag longitude on T, isotopic to η, which is obtained from η by a parallel shift (see Fig. 2a).

In the same way is defined the operation of making s parallel tracks along η and t parallel tracks along λ on the torus T (see also [6]). In this case, each meridian m_i is performed by means of this procedure to a meridian m_i' on T' which intersects A exactly at $2(t + s + 1)$ points. As a result of making $(n-1)$ parallel tracks on the torus T, we shall get a new torus T' which is embedded in S^3 and possesses a $2n$ by k staircase pattern P. Note that T' may be chosen to be smooth. As an example, the pattern for the embedded torus T_1 which is obtained from the one indicated in Fig. 2*a* by making one track along the longitude η and the other track along λ, is shown in Fig. 2*b*. Denote by $\mathcal{R}$ the class of tori obtained from the ones of type $k \geq 2$ by making tracks on them. Ng showed that all tori from $\mathcal{R}$ are essential and addressed the problem of finding a complete set of well-defined moves on the embedded tori of type $k \geq 2$ such that any embedding of a torus which admits standard tiling can be obtained from the one of type $n \geq 2$ by a sequence of these moves.



a) *b*)

Fig. 2

As shown in [7], the operation of making tracks alone on the tori of type $k \geq 2$ is not sufficient to obtain all essential tori which admit standard tiling. However, it is easy to see that after having made some exchange moves on the corresponding link and an isotopy, we shall obtain the torus from the class $\mathcal{R}$ in a new closed braid complement.

It is known [4], that each torus T in 3-dimensional sphere S^3 bounds a solid torus $\mathbf{T}$ at least from one side. Let T be a tiled torus and m its minimal meridian. Fix an edge e on the oriented meridian m. Then m is a perfect curve on T.

It follows from Proposition 2.3 of [7] that, given a minimal combinatorial meridian m on $\mathcal{T}$, we may shift it on $\mathcal{T}$ to obtain a sequence of «parallel» minimal meridians $m = m_1, m_2, \ldots, m_k$, which decompose $\mathcal{T}$ into k cylinders F_i, $i = 1, \ldots, k$. We shall call the cylinders F_i, $i = 1, \ldots, k$, enhanced with the data inherited from $\mathcal{T}$, the combinatorial building blocks for the embeddable tiled torus $\mathcal{T}$. A characteristic feature of a decomposition of $\mathcal{T}$ by minimal combinatorial meridians is that the latter has some geometric sense. Indeed, by definition, each m_i bounds a meridional disc D_i in the solid torus $\mathbf{T}$. Moreover, by application of the standard cut-and-paste technique and general position, we may achieve that discs D_i from the collection $\mathcal{D} = \{D_1, \ldots, D_k\}$ are to be disjoint. The latter gives a decomposition of the solid torus $\mathbf{T}$ into k solid cylinders $\mathcal{C}_i$, where each $\mathcal{C}_i$ is bounded by the embedded cylinder F_i and the meridional discs D_i and $D_{i+1}, i = 1, \ldots, k$. We shall call the solid cylinders $\mathcal{C}_i$ the building blocks for $\mathbf{T}$. The main task is now to study of geometric properties of building blocks F_i for T, embedded with respect to the axis A. Below we describe some properties of them and pose some questions.

It would be nice to standardize the geometric position of embedded building blocks and find among the minimal meridians on T those which are close in some sense to the slice meridional curves. In [7], we showed however that not every embedded torus T which admits a standard tiling and bounds a solid torus in $\mathbb{R}^3$ possess a slice minimal meridian.

We suggest that the minimal meridians bounding a geometric building block are similar in some reasonable sense. To formulate this more accurately, we introduce some needed notions.

Let $b_1 = b(i, j)$ and $b_2 = b(k, \ell)$ be any two oriented edges of the underlying graph H of $\mathcal{T}$ with the orientations defined by the (ordered) pairs (i, j) and (k, ℓ), respectively. We shall write $b_1 \prec b_2$, if the cyclic order of the vertices of b_1 and b_2 on the axis A is either i, j, k, ℓ, or i, ℓ, j, k. The cyclic sequences $b_1, b_2, \dots, b_t$ and $b_1', b_2', \dots, b_t'$ of oriented edges in $\mathcal{T}$ are called alternately A-coherent if for each i there holds $b_i \prec b_i'$ or $b_i' \prec b_i$ and the relations $b_i \prec b_i'$ and $b_k' \prec b_k$ alternate in the cyclic sequence $1, 2, \dots, t$.

Recall that each tile τ of $\mathcal{T}$ contains a unique singularity, s. The sign of s determines the cyclic order of the vertices of τ and so, the orientations of the edges incident to τ. Therefore the edges of the graph H have the natural orientations defined by the decoration of $\mathcal{T}$. Let m and m' be two coherently oriented minimal meridians on the torus T, so that m' is obtained from m by s elementary shifts along a zig-zag longitude ℓ. Then m and m' bound on T a building block [7]. Let V and V' be the sets of vertices on m and m', respectively. Moreover, let $b_1, b_2, \dots, b_t$ and $b_1', b_2', \dots, b_t'$ be the corresponding cyclic sequences of oriented edges in m and m', respectively, with the orientations defined above. There is a natural bijection $h : V \to V'$ defined by a perfect longitude ℓ on $\mathcal{T}$. For instance, let $h(v_i) = v_i'$, so that $h(b_i) = b_i'$, $i = 1, 2, \dots, t$.

We shall say that m and m' are A-equivalent (with respect to h) if the map $h : V \to V'$ acts by a shift in some number on the axis A.

Moreover, m and m' are called A-similar with respect to h if the corresponding sequences $b_1, b_2, \dots, b_t$ and $b_1', b_2', \dots, b_t'$ of oriented edges are alternately A-coherent.

Proposition 1. *Let B be a building block for the torus T bounded by oriented minimal meridional curves C and C', with the orientations being coherent. Let V, V' be the sets of vertices of C and C', respectively. Suppose there is a natural bijection $h : V \to V'$ defined by a pure zig-zag longitude in $\mathcal{T}$. Then the meridians C and C' are A-similar.*

P r o o f. Let T be an essential torus which admits a standard tiling and bounds a solid torus $\mathbf{T}$. Consider a longitude-meridional pattern P for T. Let m_1 and m_2 be the combinatorial meridians on P which represent the parallel perfect meridional curves C and C', respectively. The set E of oriented edges of the underlying graph H of the tiled torus $\mathcal{T}$ can be covered by the two collections $\mathcal{D}$ and $\mathcal{B}$ of the directed paths ℓ_i and ℓ'_i, $i = 1, \dots, q$, respectively, each of which is parallel to a perfect pure zig-zag longitude ℓ (see Fig. 3).

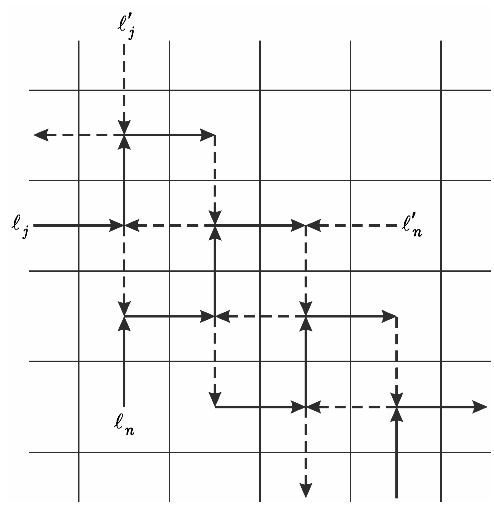


Fig. 3

Let b_k, $b_k = b(s,t)$ be an edge of the meridian m_1 and $b'_k = b(s',t')$ the corresponding (via the bijection h) edge of m_2. We see that b_k and b'_k lie on the same line ℓ_j of some collection. Then the next edges, $b_{k+1} = b(t,u)$ and $b'_{k+1} = b(t',u')$, on the meridians m_1 and m_2, respectively, also lie on the same line ℓ'_j, but from another collection. We have either $b_k \prec b'_k$, or $b'_k \prec$ $\prec b_k$. Suppose the first relation holds. It follows that $b'_{k+1} \prec b_{k+1}$. In the second case, $b'_k \prec b_k$, we have $b_{k+1} \prec b'_{k+1}$, completing the proof. ◊

Recall that to each b-arc $b(i,j)$ on T there corresponds the θ-interval $\theta_{i,j}$ in which this arc exists [1]. Let C be a closed oriented curve on T (on $\mathcal{T}$) and let $b_1, b_2, \dots, b_r$ be the cyclic sequence of its edges induced by the oriented circuit of the cycle C. Then to the sequence of edges $b_1, b_2, \dots, b_r$ there corresponds a sequence of its θ-intervals $\theta_C = (\theta_1, \varphi_1), (\theta_2, \varphi_2), \dots, (\theta_r, \varphi_r)$. Let C and C' be any two oriented perfect meridional curves in the combinatorial torus $\mathcal{T}$ of the length r with the sets of vertices V and V', respectively, and let $\beta = b_1, b_2, \dots, b_r$ and $\beta' = b'_1, b'_2, \dots, b'_r$ be the corresponding cyclic sequences of the edges of C and C'. Let $S = (\theta_1, \varphi_1), (\theta_2, \varphi_2), \dots, (\theta_r, \varphi_r)$ and $S' = (\theta'_1, \varphi'_1), (\theta'_2, \varphi'_2), \dots, (\theta'_r, \varphi'_r)$ be the cyclic sequences of θ-intervals corresponding to the sequences β and β', respectively. Suppose there is a natural map $h : V \to V'$, defined by a perfect (pure zig-zag or straight) longitude ℓ on T. For instance, let $h(v_i) = v'_i$, so $h(b_i) = b'_i$, $i = 1, \dots, r$. We shall say that C and C' are θ-equivalent if there are the value ρ, $\rho \in [0, 2\pi)$, and the sequences $\chi = \chi_1, \dots, \chi_r$ and $\chi' = \chi'_1, \dots, \chi'_r$, where $\chi_i \in (\theta_i, \varphi_i)$ and $\chi'_i \in (\theta'_i, \varphi'_i)$, $i = 1, \dots, r$, so that $\chi'_i = \chi_i + \rho$ modulo 2π for each $i \le r$.

Let (a,b) and (c,d) be two θ-intervals, where the parameter θ is considered modulo 2π. We shall write $(a,b) \preceq (c,d)$ if the values a, b, c, d of the parameter θ appear in the cyclic order (a,b,c,d) or (a,d,b,c). Two sequences of θ-intervals $S = (\theta_1, \psi_1), \dots, (\theta_r, \psi_r)$ and $S' = (\theta'_1, \psi'_1), \dots, (\theta'_r, \psi'_r)$ are called alternately coherent if for each k we have $(\theta_k,\psi_k)\preceq(\theta'_k,\psi'_k)$ or $(\theta'_k,\psi'_k)\preceq(\theta_k,\psi_k)$ and the relations $(\theta_i,\psi_i)\preceq(\theta'_i,\psi'_i)$ and $(\theta'_k,\psi'_k)\preceq(\theta_k,\psi_k)$ alternate in the sequence. Let C and C' be any two perfect meridional curves of length k bounding a building cylinder on the torus T and let $b_1,b_2,\dots,b_r$ and $b'_1,b'_2,\dots,b'_r$ be the corresponding cyclic sequences of the edges of C and C', respectively. Let $S=(\theta_1,\varphi_1),(\theta_2,\varphi_2),\dots,(\theta_r,\varphi_r)$ and $S'=$ $=(\theta'_1,\varphi'_1),(\theta'_2,\varphi'_2),\dots,(\theta'_r,\varphi'_r)$ be the cyclic sequences of θ-intervals corresponding to β and β', respectively. Suppose there is a natural map $g:C\to C'$ defined by a perfect zig-zag longitude on $\mathcal{T}$ and $g(b_i)=b'_i$. We shall say that C and C' are θ-coherent if the sequences S and S' of θ-intervals are alternately coherent.

Proposition 2. *Let B be a building block for the torus T bounded by oriented perfect meridional curves m and m' with the coherent orientations and let V and V' be the sets of vertices of m and m', respectively. Suppose there is a natural map $h:V\to V'$ defined by a perfect pure zig-zag longitude on $\mathcal{T}$. Then m and m' are θ-coherent.*

P r o o f. Let T be an essential torus which admits standard tiling and bounds a solid torus $\mathbf{T}$. Let m be a perfect meridian on the tiled torus $\mathcal{T}$ and m' one of two its neighbours in $\mathcal{T}$. We denote also by m and m' the corresponding meridions on the pattern P (see Fig. 2), where $\ell(m)=\ell(m')=$ $=2q$. Let H be the underlying graph of $\mathcal{T}$, embedded in T, and let G be the dual of the embedded graph H on T (see above). Note that as graphs, G and H are isomorphic. The edges of each tile τ' of the embedded graph G have the natural orientation in accordance to the sign of a unique vertex v contained in τ' and form an oriented cycle. Denote by G' the orgraph obtained from the graph G by the natural orientation of its edges. The set E' of oriented edges of the orgraph G' can be covered by a collection $\mathcal{B}$ of oriented zig-zag paths f_j, $j=1,\dots,2q$, parallel to a pure perfect zig-zag longitude ℓ on P, as indicated in Fig. 3. The direction of an arc e_i in any path f_j indicates the direction of increasing the parameter θ. This means that if e_i starts at a singularity s_r and finishes at s_t and b_i is the edge of the graph H which is dual of e_i, then b_i exists in the θ-interval (θ_r,θ_t) (but not in the (θ_t,θ_r)). Note that for each vertex v of H the vertices v and $h(v)$ have the same sign in $\mathcal{T}$. To each edge b_k in m there corresponds the edge b'_k in m', $k=1,\dots,2q$, via the map h. Now the direct inspection of the relations between b_k and b'_k shows that either $b_k\preceq b'_k$, or $b'_k\preceq b_k$. Suppose $b_k\preceq b'_k$, i.e. the pair of edges (b_k,b'_k) occurs along a zig-zag line f_n from the collection $\mathcal{B}$ in the direction of increasing the parameter θ (see Fig. 3). It follows the pair (b_{k+1},b'_{k+1}) occurs along another zig-zag line f_m, in the direction of decreasing the parameter θ (see Fig. 3), i.e. $b'_{k+1}\preceq b_{k+1}$. This completes the proof of the proposition. $\Diamond$

Therefore the perfect meridional curves m_1 and m_2 bounding a building block on the tiled torus $\mathcal{T}$ or on the embedded torus T have the similar configurations. We do not know however whether m_1 and m_2 are always A-and θ-equivalent in the sense defined above.

It should be noted that the minimal meridians which bound the building blocks for the embeddable tiled tori having the generalized type $k \geq 2$ have the similar configurations in much more strong sense than it has been given by Proposition 1 and Proposition 2. All the examples of embeddable tiled tori considered before have the «regular» structure, i.e. their decorated graphs possess some kind of periodicity. We do not know the examples of embeddable tiled tori with «non-regular» structure.

Proposition 3. *Any embedded tiled torus T from the class $\mathcal{R}$ admits a decomposition into elementary building blocks.*

P r o o f. Let T be a tiled torus embedded in $\mathbb{R}^3$ from the class $\mathcal{R}$. Notice that T has a rectangular pattern of size $2 \times k$. Moreover, let T' be the tiled torus of type $k \geq 2$ in the sense of Birman and Menasco (see above) so that T is obtained from T' by making γ tracks along a zig-zag longitude ℓ and β tracks along another zig-zag longitude, say s, of the torus T' where ℓ and s are parallel in combinatorial sense (see the definition of this operation below). Then T has a staircase pattern of width $2 + 2\beta + 2\gamma$ and height k [6, 7].

By the construction, $w(\ell)$, the winding number of ℓ on T is equal to k. We have also an obvious equality $w(s) = k$. Note that each minimal meridian m' on T' of length 2 is replaced via the procedure of making tracks on T' with minimal meridian m on T which intersects the axis A exactly in $2 + 2\beta + 2\gamma$ points. Assume we have chosen k minimal meridians m_i' on T' of length 2, each on the tube (cylinder) C_i, where the embedded torus T' is obtained by consequent gluing cylinders C_i together $i = 1, \ldots, k$. Since ℓ and s are both the zig-zag longitudes on T' and the procedure of making tracks is homogenies on T' (actually is the same on each tube), we may choose all minimal meridians m_i on T with the same configuration, that is to be parallel. This gives immediately the combinatorial decomposition of T into elementary building blocks with parallel boundary components, so the given decomposition consists only of elementary blocks. This completes the proof. $\Diamond$

An example of standard tiled torus which does not belong to the class $\mathcal{R}$ (that is cannot be obtained from the tori of type $k \geq 2$) by the operation of making was also given in [5].

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ЗАУВАЖЕННЯ ДО ПЛИТКОВИХ ТОРІВ

J. S. Birman, W. W. Menasco [3] *ввели і дослідили клас вкладених торів в доповненні до замкнених сплетень, які допускають стандартне плиткове покриття. K. Y. Ng показала* [6]*, що кожний суттєвий тор, який допускає стандартне плиткове покриття, має східчату модель. Комбінаторний і геометричний опис торів з такого класу частково наведено в* [7]*. Пропоновану роботу можна розглядати як продовження праці* [7]*, в якій вивчаються розбиття плиткових торів на конструктивні блоки і показано, що такі блоки є подібними в природному сенсі. Зокрема, показано, що кожний тор з класу геометричних плиткових торів, описаних K. Y. Ng в* [6]*, складається виключно з елементарних блоків.*

ЗАМЕТКИ О ПЛИТОЧНЫХ ТОРАХ

J. S. Birman, W. W. Menasco [3] *ввели и исследовали класс вложенных торов в дополнении к замкнутым зацеплениям, допускающих стандартное плиточное покрытие. K. Y. Ng показала* [6]*, что каждый существенный тор, который допускает стандартное плиточное покрытие, имеет степеньчатую модель. Комбинаторное и геометрическое описание торов из данного класса частично приводится в* [7]*. Данную работу можна рассматривать как продолжение работы* [7]*, в которой изучаются розбиения плиточных торов на конструктивные блоки и показывается, что такие блоки подобны в естественном смысле. В частности, показано, что кождый тор из класса геометричесих плиточных торов, описанных K. Y. Ng в* [6]*, состоит исключительно из элементарных блоков.*

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