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ON NONPLANARITY OF CUBIC GRAPHS

A cubic graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$*. For a given 2-connected cubic graph* G*, denoted by* $ed(G)$ *the minimal number of edges so that after removal them from* G *the resulting graph becomes planar and* $g(G)$ *the genus of* G*. Moreover, for a given simple graph* G *let* $cr(G)$ *denote the minimal number of crossings of edges needed to draw* G *on the plain (so the minimum is taken over all submersions of* G *in the plane). In this paper, we study relations between the characteristics* $ed(G)$ *and* $g(G)$ *and* $cr(G)$ *for some special classes of graphs and discuss the problems related with their evaluation.*

Introduction. There are known several important measures for the nonplanarity of a graph G. We mention here the most important of them. They are the minimum number of crossings in an embedding of G in the plane, the minimum number of edges needed for removal from G in order to obtain a planar subgraph, the minimum number of decomposition of edge set of the graph into subsets of edges that produce planar subgraphs and the genus. All they take an important role in the topological graph theory. The corresponding decision and optimization problems for these invariants turn out to be NP-complete [2, 3, 9, 11]. We review some relevant results on the complexity of the problems mentioned above in more details.

In this paper we use standard definitions and notations for a graph and its edge and vertex sets. *The nonplanar edge deletion* problem (ED) consists in answering the following question. For a given graph $G=(V,E)$ and a nonnegative integer k, does there exist a subset $E' \subset E$ such that the graph $G = (V, E \setminus E')$ is planar and $|E'| \le k$? If k is the smallest nonnegative integer, such that there exists $E' \subset E$ so that $H=(V, E \setminus E')$ is planar and $|E'| = k$, we shall say that H is a maximum planar subgraph of G. Following [5], the corresponding problem of finding the minimum number of edges so that their removal from $G=(V,E)$ defines a planar graph will be denoted by MINED. The maximization problem of finding the number of edges of a maximum planar subgraph of a given graph $G=(V,E)$ is denoted by MAXPS. In [9] it was proved that ED is NP-complete. Faria et al. [2] proved that ED is NP-complete for cubic graphs. Calinescu et al. [1] proved that MINED and MAXPS are Max SNP-hard and exhibited a polynomial-time $\frac{4}{9}$-approximation algorithm for MAXPS for general graphs. Finally, Faria et al. [2] proved that MINED is Max SNP-hard for cubic graphs. These results have been extended in [4, 5] by the same authors to the nonplanar vertex deletion problem (VD) for the maximum degree 3 graphs. On the other hand, Thomassen [10] proved that the following problem is NP-complete: given a cubic graph G and a natural number g, is it possible to embed G into an orientable closed surface of genus $\le g$?

For a given graph G, denote by $ed(G)$ the minimal number of edges such that after their removal from G the resulting graph becomes planar, and $g(G)$ the genus of G. Moreover for a given simple graph G denote by $cr(G)$ the minimal numbers of crossings of edges needed to draw G on the plain. Let $ed(k)$ and $g(k)$ denote the maximum of the numbers $ed(G)$ and

$g(G)$, respectively, where both the maxima are taken over all 2-connected cubic graphs of (even) order k. In this paper, we study relations between the characteristics $ed(G)$, $g(G)$ and $cr(G)$ of a cubic graph G and speculate about estimation of the functions $ed(k)$ and $g(k)$. Here we only formulate some relevant results and outline their proofs.

1. Comparision. Note that 3-connected cubic planar graphs are dual of triangulations of the sphere.

For any graph G we have obviously the following inequalities: $g(G) \leq \leq ed(G) \leq cr(G)$. When restricting to cubic graphs, we can naturally ask whether the difference between $ed(G)$ and $g(G)$ can growth arbitrarily. The answer to this question is affirmative.

Proposition 1. *For an arbitrary natural number $n > 0$ there is a 3-edge connected cubic graph G of genus 1 with $ed(G) > n$.*

P r o o f. Fix a number n. Consider on the sphere S^2 a triangulation H so that its dual graph G is simple, 3-connected and with $\operatorname{diam}(G)$ being enough big (for example with $\operatorname{diam}(G) \geq 6n$). Consider in G two $(n+1)$-faces, say $s = e_1, \ldots, e_{n+1}$ and $t = f_1, \ldots, f_{n+1}$, so that $d(s,t) \geq 6n - 2$. Pick on each edge e_i (f_i) a new vertex x_i (y_i, respectively), where $i = 1, \ldots, n+1$, and join each pair (x_i, y_i) by an edge. Denote the resulting graph by G'. It is clear that $g(G) = 1$. It is also not difficult to show that when an appropriate choice of edges e and f has been made (they are sufficiently far from each to other) and $n+1$ «parallel» edges are added as before, we shall obtain a cubic graph G with $ed(G) = n+1$. We also use here the fact that the embedding of a 3-connected graph in a sphere S^2 or the plane is unique [6]. ◆

Given a cubic graph of the (even) order n, we have the following obvious upper bound for the number $ed(G)$: $ed(G) \leq \frac{1}{2}n$.

By the girth of a simple graph we shall mean the length of its minimal circle. The following proposition is an immediate consequence of definitions the girth and the genus of a graph and the Euler formula for the cell embedding of a graph in closed surface.

Proposition 2. *Let G be a 2-connected simple cubic graph of order n with the girth k. Then* $g(G) \geq -\frac{3n}{2k} + \frac{1}{4}n + 1$.

To obtain simplest examples of 2-connected cubic graphs of genus $m > 0$ with minimal possible numbers n of vertices, one can use the construction illustrated in Fig. 1.

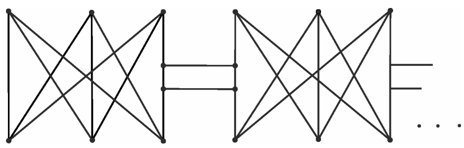

Fig. 1

For a given number r denote by $\lceil r \rceil$ the biggest integer that is less or equal to r.

Let G_m be a graph obtained from m copies of the graph $G = K_{3,3}$ by joining them together as depicted in Fig. 1.

Proposition 3. *For each* m *the graph* G_m *is* 2*-connected and* $g(G_m) = m$.

P r o o f. Since G_m contains m disjoint copies of the graph G homeomorphic to $K_{3,3}$, we have $g(G_m) \geq m$. On the other hand, deleting one edge in each copy G of the graph G_m, we can check that the resulting graph G' does not contain any copy of $K_{3,3}$ and so is planar. It follows that $ed(G_m) = g(G_m) = m$. ♦

The latter example gives a lower bound for the number $ed(k)$: $ed(k) \geq g(k) \geq \lceil \frac{k+4}{10} \rceil$. This estimate is however very rough.

Consider now the graph H_m consisting of m «blocks» H depicted in Fig. 2. These blocks are glued consequently each to other in such a way that the latest block is joined with the first one and all they form a cycle. Each block H contains exactly 27 edges. The total number of edges in the cubic graph H_m is thus $27m$.

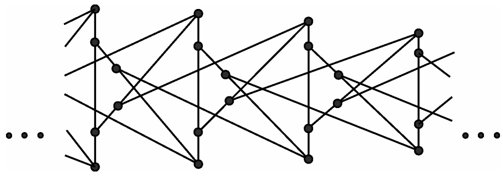

Fig. 2

Proposition 4. *For each* $m \geq 3$ *the cubic graph* H_m *is* 3*-connected. Moreover* $ed(H_m) \geq m$.

P r o o f. Given any two different vertices v and w, we can directly check that there exist exactly three paths in H_m that join these vertices and have only their ends in common. On the other hand, it is not difficult to check that one has to delete at least two edges in each block H of H_m to obtain the graph that contains no subgraph homeomorphic to $K_{3,3}$. ♦

The latter gives the following lower bound for the number $ed(k)$: $ed(k) \geq \lceil \frac{k}{9} \rceil$. However we do not know whether $ed(H_m) = g(H_m)$.

By a snark we shall mean a cyclically 4-edge-connected cubic graph G with girth $g(G) \geq 5$ and with no 3-edge-colouring. An example of such a graph is the well known Petersen graph P. It is known that the $g(P) = 1$ and the Petersen graph does not allow a polyhedral imbedding in any orientable surface but has a polyhedral embedding in the projective plane [7]. In [8], the authors study the Petersen powers P^k of the graph P and their genus. The Petersen power P^k is defined inductively as $P^k = P \cdot P^{k-1}$ via the dot product of two graphs. This operation is denoted by «$\cdot$» and indicated in Fig. 3 (passing from the cubic graph on left picture to the one on the right picture).

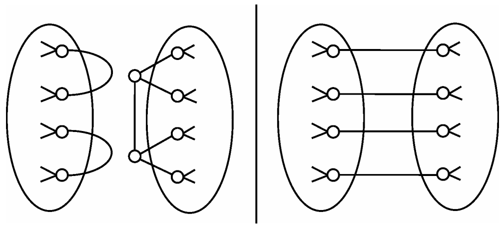

Fig. 3

The dot product $G_1 \cdot G_2$ of the graphs G_1 and G_2 is not defined uniquely, so there are actually many copies of different graphs $G_1 \cdot G_2$. It follows that there are many Petersen powers P^n for each natural $n \geq 2$. However it is known that if G_1 and G_2 are both snarks, then $G_1 \cdot G_2$ is also a snark. In [12], Vodopiviec constructed for each $n \geq 1$ a Petersen power P^n of orientable genus 1. We denote it by V_n. In Fig. 4 we indicate a version of power P^3 introduced by Vodopiviec. This graph is embedded in a torus represented by a rectangular with opposite sides glued together.

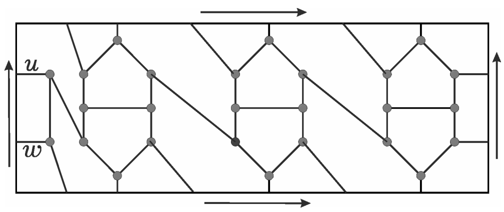

Fig. 4

It is natural to ask what is the number $ed(P^n)$ in the case of Vodopiviec version of Petersen powers P^n.

Proposition 5. *For each* $n \geq 1$ *the following equalities hold*: $ed(V_n) = 2$ *and* $cr(V_n) = 4n - 2$.

P r o o f. A deletion of any edge in V_n contains a subgraph homeomorphic to $K_{3,3}$. On the other hand, one can easy find two edges (for example, the edges u and w in the graph indicated in Fig. 4) so that the removal of them produces a graph H which is actually embedded in a cylinder. It follows that H is planar and the first assertion follows. The second equality follows by direct checking with using the fact that a 3-connected planar graph contains a unique embedding in a plane and some information about the crossing number of torus knots. The details are left to the reader. ♦

It follows that the graphs V_n cannot serve good examples to estimate the number $ed(n)$. On the other hand, in [8], the authors constructed for each pair of natural numbers k and n, where $k \leq n$ and $k, n \geq 1$, the Petersen power P^n with $g(P^n) = k$. Since the order of P^n is $8n + 2$, we have the following lower bound for $g(m)$: $g(m) \geq \lceil \frac{m-2}{8} \rceil$ which is better than the one given before. In this connection, it would be interesting to evaluate the number $ed(P^n)$ for this version of Petersen powers P^n.

The further improvement of the results formulated above will be given by using some topological arguments in the forthcoming paper.

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ПРО НЕПЛАНАРНІСТЬ КУБІЧНИХ ГРАФІВ

Кубічний граф є непланарним тоді й тільки тоді, коли він не містить підграфів, гомеоморфних $K_{3,3}$. *Для заданого 2-зв'язного кубічного графа* G *позначимо через* $ed(G)$ *найменше число ребер в* G, *після викидання яких отримаємо планарний підграф, а через* $g(G)$ – *рід графа* G. *Крім того, через* $cr(G)$ *позначимо мінімальне число (властивих) перетинів ребер графа* G *серед усіх занурень (імерсій) графа в площині. Вивчаються співвідношення між характеристиками* $ed(G)$, $cr(G)$ *і* $g(G)$ *для деяких спеціальних класів графів і розглядається проблема їх обчислення.*

О НЕПЛАНАРНОСТИ КУБИЧЕСКИХ ГРАФОВ

Кубический граф является непланарным тогда и только тогда, когда он не содержит подграфов, гомеоморфных $K_{3,3}$. *Для заданного 2-связного кубического графа* G *обозначим через* $ed(G)$ *наименьшее число ребер в* G, *после выбрасывания которых получаем планарный подграф, а через* $g(G)$ – *род графа* G. *Кроме того, через* $cr(G)$ *обозначено минимальное число (собственных) пересечений ребер графа* G *среди всех погружений (имерсий) графа в плоскости. Изучаются соотношения между характеристиками* $ed(G)$, $cr(G)$ *и* $g(G)$ *для некоторых специальных классов графов и рассматривается проблема их вычисления.*

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