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ON THE FREDHOLM SOLVABILITY FOR A CLASS OF MULTIDIMENSIONAL HYPERBOLIC PROBLEMS

We prove the Fredholm alternative for a class of two-dimensional first-order hyperbolic systems in the Shauder canonic form with periodic Dirichlet boundary conditions. Our approach is based on a regularization via a right parametrix.

1 Introduction. The Fredholm property of linearizations plays the key role in local investigations of nonlinear differential equations via the Implicit Function Theorem and the Lyapunov − Schmidt reduction (see, e.g., [3, 4]). In contrast to ODEs and parabolic PDEs almost nothing is known about the Fredholm property for hyperbolic PDEs.

The Fredholm solvability for ODEs and many classes of parabolic PDEs can be easily derived from the basic fact that Fredholm operators are exactly compact perturbations of bijective operators. The hyperbolic case is much more complicated. It is well known that the inverse of a first-order hyperbolic operator decreases the smoothness. By this reason the Fredholm analysis of hyperbolic problems requires establishing an optimal regularity relation between the spaces of solutions and right-hand sides of the differential equations and finding an appropriate regularization to compensate the loss-of-smoothness effect.

In [7, 8] we presented a quite general approach to proving Fredholmness for first-order one-dimensional hyperbolic PDEs. It is based on the construction of a right regularizer (right parametrix) and using a functional-analytic criterion for Fredholmness in Banach spaces. The techniques of [7, 8] cover the so-called traveling-wave models from laser dynamics. In the present paper we extend this approach (applying completely different techniques) to a class of multidimensional hyperbolic PDEs having the so-called Shauder canonic form (see [1, 2]) and admitting an integral representation. Our results are interesting from the theoretical point of view because the multidimensional case is qualitatively different. We demonstrate a noteworthy effect that a higher dimension requires more regularization for the inverse operator (see Remark 2). Another interesting feature of the hyperbolic systems under consideration is that the «lower order» terms are those terms contributing into the system transversely to characteristic directions (c.f. the definition of the operator D in (11)). Note that well-posedness of initial-boundary problems for multi-dimensional first-order hyperbolic PDEs in Shauder canonic form is studied in [1, 2, 6, 9].

Specifically, we investigate a linear first-order two-dimensional hyperbolic system of the kind

$$\sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t u_j + \partial_x u_j + \beta_i \partial_y u_j + \gamma_i(x,y,t)u_j) + \sum_{j=1}^{n} b_{ij}(x,y,t)u_j =$$

$$= f_i(x,y,t), \qquad i \le n, \quad (x,y,t) \in (0,1) \times \mathbb{R} \times \mathbb{R}, \qquad (1)$$

supplemented with the periodic conditions in y and t

$$u_i(x, y + Y, t + T) = u_i(x,y,t), \quad i \le n, \quad (x,y,t) \in [0,1] \times \mathbb{R} \times \mathbb{R}, \qquad (2)$$

and the Dirichlet boundary conditions in x

$$u_i(0,y,t) = 0, \qquad i \le k, \qquad (y,t) \in \mathbb{R}^2,$$

$$u_i(1,y,t) = 0, \qquad k+1 \le i \le n, \qquad (y,t) \in \mathbb{R}^2. \qquad (3)$$

Here the periods $Y > 0$ and $T > 0$ and the coefficients a_{ij}, $\alpha_i \neq 0$, and $\beta_i \neq \neq 0$ are fixed real constants, the coefficients γ_i, $b_{ij} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the right hand sides $f_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are known functions. Without loss of generality, consider the case $n \geq 3$ and $2 \leq k \leq n - 1$. Fix an arbitrary $\ell \in \mathbb{N}$ in the range $1 \leq \ell \leq k - 1$ and suppose that the matrix $A = (a_{ij})_{i,j=1}^n$ has the following diagonal-block structure

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}, \tag{4}$$

where A_1, A_2, and A_3 are $\ell \times \ell$, $(k - \ell) \times (k - \ell)$, and $(n - k) \times (n - k)$-matrices, respectively, while 0 denotes the null matrices of respective sizes. Moreover, the matrix $B = (b_{ij})_{i,j=1}^n$ is assumed to be one of the following two kinds:

$$B = \begin{pmatrix} 0 & 0 & B_1 \\ B_2 & 0 & 0 \\ 0 & B_3 & 0 \end{pmatrix} \tag{5}$$

or

$$B = \begin{pmatrix} 0 & \tilde{B}_1 & 0 \\ 0 & 0 & \tilde{B}_2 \\ \tilde{B}_3 & 0 & 0 \end{pmatrix}, \tag{6}$$

where $B_1, B_2, B_3, \tilde{B}_1, \tilde{B}_2$, and $\tilde{B}_3$ are, respectively, $\ell \times (n - k)$, $(k - \ell) \times \ell$, $(n - k) \times (k - \ell)$, $\ell \times (k - \ell)$, $(k - \ell) \times (n - k)$, and $(n - k) \times \ell$-matrices. For definiteness, we suppose (5) (the case of (6) is quite similar).

We investigate solvability of the problem (1)−(3) and state our result as a Fredholm alternative. More precisely, we prove that the problem is solvable if the right hand side is orthogonal to all solutions to the homogeneous adjoint system

$$\sum_{j=1}^n a_{ji}(-\alpha_j \partial_t u_j - \partial_x u_j - \beta_j \partial_y u_j + \gamma_j(x,y,t)u_j) + \sum_{j=1}^n b_{ji}(x,y,t)u_j = 0 \,,$$

$$i \leq n, \qquad (x,y,t) \in (0,1) \times \mathbb{R} \times \mathbb{R}, \tag{7}$$

endowed with conditions (2) and (3).

We will work within the algebra $C_{Y,T}([0,1] \times \mathbb{R}^2)$ of continuous functions which are Y-periodic in y and T-periodic in t. Let

$$W \equiv (C_{Y,T}([0,1] \times \mathbb{R}^2))^n \tag{8}$$

denote the space of right-hand sides endowed with the usual max-norm and let

$$V \equiv \Big\{ u \in W : u_i(0,y,t) = 0 \ \text{ for } \ i \leq k, \quad u_i(1,y,t) = 0 \ \text{ for } \ k + 1 \leq i \leq n,$$

$$\Big[\sum_{j=1}^n a_{ij}(\alpha_i \partial_t u_j + \partial_x u_j + \beta_i \partial_y u_j) \Big]_{j=1}^n \in W \ \text{ for } \ i \leq n \Big\} \tag{9}$$

denote the space of solutions. Here $u = (u_1, \ldots, u_n)$ and $\partial_t u_j$, $\partial_x u_j$, and $\partial_y u_j$ are generalized derivatives. The space V is endowed with the norm

$$\|u\|_V \equiv \|u\|_W + \left\| \left[\sum_{i=1}^{n} a_{ij}(\alpha_i \partial_t u_j + \partial_x u_j + \beta_i \partial_y u_j) \right]_{j=1}^{n} \right\|_W. \tag{10}$$

Note that the space V depends on the coefficients of system (1). Notice also the continuous embedding

$$C^1_{Y,T}([0,1] \times \mathbb{R}^2) \rightarrowtail V \rightarrowtail C_{Y,T}([0,1] \times \mathbb{R}^2).$$

To state our result, let us introduce linear operators $C \in \mathcal{L}(V; W)$ and $D \in \mathcal{L}(W)$ by

$$Cu \equiv \left[\sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t u_j + \partial_x u_j + \beta_i \partial_y u_j + \gamma_i(x,y,t)u_j) \right]_{i=1}^{n},$$

$$Du \equiv \left[\sum_{j=1}^{n} b_{ij}(x,y,t)u_j \right]_{i=1}^{n}. \tag{11}$$

The problem (1)−(3) can now be written as

$$Cu + Du = f.$$

In what follows, we also use notation

$$\tilde{\alpha}_i = \begin{cases} 0, & \text{if } b_{ij} \equiv 0 \text{ for all } j \le n, \\ \alpha_i & \text{otherwise,} \end{cases}$$

$$\tilde{\beta}_i = \begin{cases} 0, & \text{if } b_{ij} \equiv 0 \text{ for all } j \le n, \\ \beta_i & \text{otherwise.} \end{cases}$$

Theorem 1. *Suppose that problem* (1)−(3) *satisfies the following assumptions:*

$$\gamma_i \in C((0,1), C^1_{Y,T}(\mathbb{R}^2)), \qquad b_{ij} \in C^1_{Y,T}([0,1] \times \mathbb{R}^2), \tag{12}$$

$$\det(a_{ij})_{i,j=1}^{n} \ne 0 \tag{13}$$

and

$$(\tilde{\beta}_i - \tilde{\beta}_j)(\tilde{\alpha}_j - \tilde{\alpha}_s) - (\tilde{\beta}_j - \tilde{\beta}_s)(\tilde{\alpha}_i - \tilde{\alpha}_j) \ne 0 \tag{14}$$

for all $i,j,s \in \{1,\dots,n\}$ *with* $1 \le i \le \ell$, $\ell+1 \le j \le k$, $k+1 \le s \le n$ *unless* $b_{ij} = 0$ *for all* $j \le n$. *Let* W *and* V *be function spaces defined by* (8), (9), *and* (10). *Let* $C \in \mathcal{L}(V; W)$ *and* $D \in \mathcal{L}(W)$ *be linear operators defined by* (11). *Then the following is true:*
- (***i***) *The operator* C *is an isomorphism from* V *onto* W.
- (***ii***) *The operator* $C + D$ *is a Fredholm operator from* V *into* W *with index zero.*

Part (***i***) of the theorem is a necessary starting point of the Fredholm analysis. It shows that the couple of spaces (V, W) provides the desired optimal regularity relation between the solutions and the right-hand sides of the equations.

Remark 1. *Since the set of Fredholm operators is open, the conclusion of Theorem 1 survives under small perturbations in* $C((0,1), C^1_{Y,T}(\mathbb{R}^2))$ *of the coefficients* γ_i *and in* $C^1_{Y,T}([0,1] \times \mathbb{R}^2)$ *of the entries of* B. *Such perturbations can modify the structure of matrix* (5). *Thus, the structure of* (5) *is not a necessary condition for the conclusion of the theorem* (*though it is essential for our proof*).

In Section 2 we prove a criterion of Fredholmness for linear operators in Banach spaces, which is useful, in particular, for hyperbolic PDEs. Section 3 is devoted to the desired properties of the solution spaces. Our main result, Theorem 1, is proved in Section 4.

2. Fredholmness criterion. Here we prove the following constructive Fredholmness criterion:

Theorem 2. *Let W be a Banach space, I be the identity in W, and $K \in$ $\in \mathcal{L}(W)$ with K^n being compact for some $n \in \mathbb{N}$. B Then $I - K$ is a Fredholm operator of index zero.*

P r o o f. Since

$$I - K^n = (I - K)\sum_{i=0}^{n-1} K^i,$$

the sum $\sum_{i=0}^{n-1} K^i$ is a parametrix for the operator $I - K \in \mathcal{L}(W)$. Therefore, the Fredholmness of $I - K$ follows, i.e. from [11, Proposition 5.7.1] or [10, Theorem 5.5]. Nevertheless, for the reader's convenience here we give an independent, simple, and self-contained proof (of this fact). Our proof extends the argument that was used in [7] in the case $n = 2$. Note first that

$$\dim \ker (I - K) \leq \dim \ker (I - K^n) < \infty. \tag{15}$$

Similarly $\dim \ker (I - K)^* < \infty$, hence $\operatorname{codim} \overline{\operatorname{Im} (I - K)} < \infty$. It remains to show that $\operatorname{Im} (I - K)$ is closed.

Take a sequence $(w_j) \subset W$ and an element $w \in W$ such that

$$(I - K)w_j \to w. \tag{16}$$

We have to show that $w \in \operatorname{Im} (I - K)$.

By (15) there exists a closed subspace V of W such that
$$W = \ker (I - K) \oplus V. \tag{17}$$

Consider the decomposition

$$w_j = u_j + v_j, \quad \text{where} \quad u_j \in \ker (I - K) \quad \text{and} \quad v_j \in V.$$

From (16) we infer that

$$(I - K)v_j \to w. \tag{18}$$

Let us show that the sequence (v_j) is bounded. Suppose this is not true. Without loss of generality we can assume that

$$\lim_{j \to \infty} \|v_j\| = \infty. \tag{19}$$

From (18) and (19) we get

$$(I - K)\frac{v_j}{\|v_j\|} \to 0, \tag{20}$$

hence

$$(I - K^n)\frac{v_j}{\|v_j\|} \to 0. \tag{21}$$

Since K^n is compact, there exist $v \in W$ and a subsequence $(v_{j_k})_{k \in \mathbb{N}}$ such that

$$K^n \frac{v_{j_k}}{\|v_{j_k}\|} \to v. \tag{22}$$

The convergences (21) and (22) immediately imply that

$$\frac{v_{j_k}}{\left\| v_{j_k} \right\|} \to v \in V. \tag{23}$$

Combining (23) with (20), we get $(I - K)v = 0$, i.e. $v \in V \bigcap \ker(I - K)$ and $\|v\| = 1$. This contradicts (17) and proves the boundedness of (v_j).

Now we show that $w \in \operatorname{Im}(I - K)$. As K^n is compact, there exists $v \in W$ and a subsequence (v_{j_k}) such that $K^n v_{j_k} \to v$ as $k \to \infty$. By (18) we also have

$$(I - K^n)v_j = \sum_{i=0}^{n-1} K^i (I - K)v_j \to \sum_{i=0}^{n-1} K^i w.$$

Therefore,

$$\lim_{k \to \infty} v_{j_k} = \sum_{i=0}^{n-1} K^i w + v$$

and

$$w = \lim_{k \to \infty}(I - K)v_{j_k} = (I - K)\left(\sum_{i=0}^{n-1} K^i w + v \right) \in \operatorname{Im}(I - K)$$

as desired. The Fredholm property is thereby proved.

To prove that $I - K$ has index zero, we additionally use a homotopy argument. Let us consider the continuous function

$$s \in \mathbb{R} \mapsto I - sK \in \mathcal{L}(W).$$

Since $K^n \in \mathcal{L}(W)$ is a compact operator, the operators $(sK)^n \in \mathcal{L}(W)$ are compact for each $s \in \mathbb{R}$ and, as we just proved, the operators $I - sK$ are Fredholm. By [11, Proposition 5.8.1], $\operatorname{ind}(I - sK) = \operatorname{const}$ for all $s \in \mathbb{R}$. It remains to note that the identity operator I has index zero. $\blacklozenge$

3. More about solution spaces. We now prove that V is a Banach space.

Lemma 1. *The space V is complete.*

P r o o f. Let $(u^m)_{m \in \mathbb{N}}$ be a fundamental sequence in V. Then

$$(u^m)_{m \in \mathbb{N}} \qquad \text{and} \qquad \left(\left[\sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t u_j^m + \partial_x u_j^m + \beta_i \partial_y u_j^m) \right]_{i=1}^{n} \right)_{m \in \mathbb{N}}$$

are fundamental sequences in W. Due to the completeness of W, there exist $u, v \in W$ such that

$$u^m \to u$$

and

$$\left[\sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t u^m + \partial_x u^m + \beta_i \partial_y u^m) \right]_{i=1}^{n} \to v \quad \text{in} \quad W \quad \text{as} \quad m \to \infty.$$

It remains to show that

$$\left[\sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t u_j + \partial_x u_j + \beta_i \partial_y u_j) \right]_{i=1}^{n} = v$$

in the sense of generalized derivatives. Let $\langle \cdot, \cdot \rangle : \mathcal{D}^* \times \mathcal{D} \to \mathbb{R}$ denote the dual pairing. Then for all $\varphi_1, \ldots, \varphi_n \in \mathcal{D}((0,1) \times (0,Y) \times (0,T))$ we have

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$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t + \partial_x + \beta_i \partial_y) u_j,\ \varphi_i \right\rangle =$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left\langle u_j,\ (\alpha_i \partial_t + \partial_x + \beta_i \partial_y) \varphi_i \right\rangle =$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \lim_{m \to \infty} \left\langle u_j^m,\ (\alpha_i \partial_t + \partial_x + \beta_i \partial_y) \varphi_i \right\rangle =$$

$$= \sum_{i=1}^{n} \lim_{m \to \infty} \left\langle \sum_{j=1}^{n} a_{ij}(\alpha_i \partial_t + \partial_x + \beta_i \partial_y) u_j^m,\ \varphi_i \right\rangle = \sum_{i=1}^{n} \left\langle v_i,\ \varphi_i \right\rangle$$

as desired. $\blacklozenge$

4. Fredholm alternative ($p\,r\,o\,o\,f$ of Theorem 1). To prove part (i) of the theorem, it is sufficient to show that, given $f \in W$, there exists a unique $u \in V$ satisfying the system

$$\sum_{j=1}^{n} a_{ij}\left(\alpha_i \partial_t u_j + \partial_x u_j + \beta_i \partial_y u_j + \gamma_i(x,y,t) u_j\right) = f_i(x,y,t), \qquad i \le n, \quad (24)$$

and the apriori estimate

$$\|u\|_V \le C \|f\|_W \tag{25}$$

with a constant C independent of f and u. Rewrite (24) as

$$\left(\frac{d}{d\xi} + \gamma_i\right)\left[\sum_{j=1}^{n} a_{ij} u_j\left(\xi, y + \beta_i(\xi - x), t + \alpha_i(\xi - x)\right)\right]\Bigg|_{\xi=x} = f_i(x,y,t),$$

$$i \le n.$$

Set

$$A_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Taking into account the structure of matrix A (assumptions (4)) and the non-degenerateness of A (assumption (13)), system (24) has a unique solution in V explicitly given by the formula

$$u_i(x,y,t) = \frac{1}{\det A_0} \sum_{j=1}^{k} (A_{ji}^0)^{\mathrm{ad}} \int_0^x E_j(\xi; x,y,t) f_j(\xi, y +$$

$$+\ \beta_j(\xi - x), t + \alpha_j(\xi - x))\, d\xi, \qquad i \le k,$$

$$u_i(x,y,t) = \frac{1}{\det A_3} \sum_{j=k+1}^{n} (A_{ji}^3)^{\mathrm{ad}} \int_x^1 E_j(\xi; x,y,t) f_j(\xi, y +$$

$$+\ \beta_j(\xi - x), t + \alpha_j(\xi - x))\, d\xi, \qquad k+1 \le i \le n, \qquad (26)$$

where $\{(A_{ij}^s)^{\mathrm{ad}}\}_{i,j}$ stands for the adjoint matrix to A_s and

$$E_i(\xi; x,y,t) \equiv \exp\left\{\int_x^\xi \Gamma_i(\xi_0; x,y,t)\, d\xi_0\right\},$$

$$\Gamma_i(\xi_0; x,y,t) \equiv \gamma_i(\xi_0, y + \beta_i(\xi_0 - x), t + \alpha_i(\xi_0 - x)). \tag{27}$$

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It remains to prove (25). As a straightforward consequence of (26), we have

$$\|u\|_W \le C\|f\|_W\,, \tag{28}$$

where the constant C does not depend on f and u. To derive (25) from (28), it suffices to show that because u defined by (26) satisfies (1) in a distributional sense, then

$$\sum_{j=1}^n a_{ij}(\alpha_i\partial_t u_j + \partial_x u_j + \beta_i\partial_y u_j) = -\sum_{j=1}^n a_{ij}\gamma_i(x,y,t)u_j + f_i(x,y,t)$$

is a known continuous function for each $i \le n$. To this end, notice that the function $\Gamma_i(\xi; x, y, t)$ satisfies the equation $(\alpha_i\partial_t + \partial_x + \beta_i\partial_y)\Gamma_i = 0$ for almost all $\xi \in (0,1)$. Now, fix $i \le k$ (the case $k+1 \le i \le n$ is similar). Take $\varphi \in$ $\in \mathcal{D}((0,1)\times(0,Y)\times(0,T))$ and choose a $C^1([0,1]\times[0,Y]\times[0,T])$-sequence $f_i^m \to$ $\to f_i$ in $C([0,1]\times[0,Y]\times[0,T])$ as $m \to \infty$. We have

$$\left\langle \sum_{j=1}^n a_{ij}(\alpha_i\partial_t + \partial_x + \beta_i\partial_y)u_j, \varphi \right\rangle = -\left\langle \sum_{j=1}^n a_{ij}u_j, (\alpha_i\partial_t + \partial_x + \beta_i\partial_y)\varphi \right\rangle =$$

$$= -\left\langle \int_0^x E_i(\xi; x, y, t)f_i(\xi, y + \beta_i(\xi - x), t + \alpha_i(\xi - x))\,d\xi, \right.$$

$$\left. (\alpha_i\partial_t + \partial_x + \beta_i\partial_y)\varphi \right\rangle =$$

$$= -\lim_{m\to\infty}\left\langle \int_0^x E_i(\xi; x, y, t)f_i^m(\xi, y + \beta_i(\xi - x), t + \alpha_i(\xi - x))\,d\xi, \right.$$

$$\left. (\alpha_i\partial_t + \partial_x + \beta_i\partial_y)\varphi \right\rangle =$$

$$= \lim_{m\to\infty}\left\langle \int_0^x \int_x^\xi (\alpha_i\partial_t + \partial_x + \beta_i\partial_y)\Gamma_i(\xi_0; x, y, t)\,d\xi_0 \times \right.$$

$$\times E_i(\xi; x, y, t)f_i^m(\xi, y + \beta_i(\xi - x), t + \alpha_i(\xi - x))\,d\xi, \varphi \Big\rangle +$$

$$+ \lim_{m\to\infty}\left\langle \int_0^x E_i(\xi; x, y, t)(\alpha_i\partial_t + \partial_x + \beta_i\partial_y) \times \right.$$

$$\times f_i^m(\xi, y + \beta_i(\xi - x), t + \alpha_i(\xi - x))\,d\xi, \varphi \Big\rangle -$$

$$- \left\langle \gamma_i(x, y, t)\int_0^x E_i(\xi; x, y, t) \times \right.$$

$$\times f_i(\xi, y + \beta_i(\xi - x), t + \alpha_i(\xi - x))\,d\xi, \varphi \Big\rangle + \big\langle f_i, \varphi \big\rangle =$$

$$= -\left\langle \sum_{j=1}^n a_{ij}\gamma_i(x, y, t)u_j, \varphi \right\rangle + \big\langle f_i, \varphi \big\rangle,$$

where the last equality holds by (26). The isomorphism property (*i*) is thereby proved.

To prove part (**ii**) of the theorem, note that $C + D \in \mathcal{L}(V, W)$ is Fredholm if $I + DC^{-1} \in \mathcal{L}(W)$ is Fredholm, where I is the identity in W. The proof will be finished by setting $K = -DC^{-1}$ and applying Theorem 2 with $n = 3$. We only need to show that K^3 is compact.

Take a bounded set $N \subset W$ and let M be its image under K^3. To show that M is precompact in W, we use Arzela $-$ Ascoli precompactness criterion in $C([0,1] \times [0,Y] \times [0,T])$. As K^3 is a bounded operator on W, the set M is uniformly bounded in W. It remains to check the equicontinuity property of M in W. Given $\bar{u} \in W$, set $\tilde{u} \equiv DC^{-1}D\bar{u}$. Using the representation (24) for C^{-1}, we get the following equalities:

if $i \le \ell$, then

$$\tilde{u}_i(x,y,t) = \sum_{j=k+1}^{n} b_{ij}(x,y,t) \frac{1}{\det A_3} \sum_{r=k+1}^{n} (A_{rj}^3)^{\mathrm{ad}} \int_x^1 E_r(\xi; x,y,t) \times$$

$$\times \sum_{q=\ell+1}^{k} (b_{rq}\bar{u}_q)(\xi, y + \beta_r(\xi - x), t + \alpha_r(\xi - x)) \, d\xi,$$

if $\ell + 1 \le i \le k$, then

$$\tilde{u}_i(x,y,t) = \sum_{j=1}^{\ell} b_{ij}(x,y,t) \frac{1}{\det A_1} \sum_{r=1}^{\ell} (A_{rj}^1)^{\mathrm{ad}} \int_0^x E_r(\xi; x,y,t) \times$$

$$\times \sum_{q=k+1}^{n} (b_{rq}\bar{u}_q)(\xi, y + \beta_r(\xi - x), t + \alpha_r(\xi - x)) \, d\xi,$$

if $k + 1 \le i \le n$, then

$$\tilde{u}_i(x,y,t) = \sum_{j=\ell+1}^{k} b_{ij}(x,y,t) \frac{1}{\det A_2} \sum_{r=\ell+1}^{k} (A_{rj}^2)^{\mathrm{ad}} \int_0^x E_r(\xi; x,y,t) \times$$

$$\times \sum_{q=1}^{\ell} (b_{rq}\bar{u}_q)(\xi, y + \beta_r(\xi - x), t + \alpha_r(\xi - x)) \, d\xi.$$

Now, given $f \in W$, let $\bar{u} = C^{-1}DC^{-1}f$. Note that for $\tilde{u}$ defined by the formulas above we have $\tilde{u} = (DC^{-1})^3 f$. Furthermore, $\bar{u}$ is explicitely given by:

if $i \le \ell$, then

$$\bar{u}_i(x,y,t) = \frac{1}{\det A_1} \sum_{j=1}^{\ell} (A_{ji}^1)^{\mathrm{ad}} \int_0^x E_j(\xi; x,y,t) \times$$

$$\times \sum_{r=k+1}^{n} b_{jr}(\xi, y + \beta_j(\xi - x), t + \alpha_j(\xi - x)) \, d\xi \times$$

$$\times \frac{1}{\det A_3} \sum_{q=k+1}^{n} (A_{qr}^3)^{\mathrm{ad}} \int_\xi^1 E_q(\xi_1; \xi, y + \beta_j(\xi - x), t + \alpha_j(\xi - x)) \times$$

$$\times f_q(\xi_1, y + \beta_j(\xi - x) + \beta_q(\xi_1 - \xi), t + \alpha_j(\xi - x) + \alpha_q(\xi_1 - \xi)) \, d\xi_1,$$

if $\ell + 1 \le i \le k$, then

$$\overline{u}_i(x,y,t) = \frac{1}{\det A_2} \sum_{j=\ell+1}^{k} (A_{ji}^2)^{\mathrm{ad}} \int_0^x E_j(\xi;x,y,t) \times$$

$$\times \sum_{r=1}^{\ell} b_{jr}(\xi, y + \beta_j(\xi - x), t + \alpha_j(\xi - x)) \, d\xi \, \frac{1}{\det A_1} \sum_{q=1}^{\ell} (A_{qr}^1)^{\mathrm{ad}} \times$$

$$\times \int_0^\xi E_q(\xi_1; \xi, y + \beta_j(\xi - x), t + \alpha_j(\xi - x)) \times$$

$$\times f_q(\xi_1, y + \beta_j(\xi - x) + \beta_q(\xi_1 - \xi), t + \alpha_j(\xi - x) + \alpha_q(\xi_1 - \xi)) \, d\xi_1,$$

if $k + 1 \le i \le n$, then

$$\overline{u}_i(x,y,t) = \frac{1}{\det A_3} \sum_{j=k+1}^{n} (A_{ji}^3)^{\mathrm{ad}} \int_x^1 E_j(\xi;x,y,t) \times$$

$$\times \sum_{r=\ell+1}^{k} b_{jr}(\xi, y + \beta_j(\xi - x), t + \alpha_j(\xi - x)) \, d\xi \, \frac{1}{\det A_2} \sum_{q=\ell+1}^{k} (A_{qr}^2)^{\mathrm{ad}} \times$$

$$\times \int_0^\xi E_q(\xi_1; \xi, y + \beta_j(\xi - x), t + \alpha_j(\xi - x)) \times$$

$$\times f_q(\xi_1, y + \beta_j(\xi - x) + \beta_q(\xi_1 - \xi), t + \alpha_j(\xi - x) + \alpha_q(\xi_1 - \xi)) \, d\xi_1.$$

To prove the desired equicontinuity property, we have to show the existence of a function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ such that $\alpha(p) \to 0$ as $p \to 0$ and

$$\left\| \tilde{u}(x + h_1, y + h_2, t + h_3) - \tilde{u}(x,y,t) \right\| \le \alpha(|h|) \tag{29}$$

uniformly in $\tilde{u} \in M$ and $h = (h_1, h_2, h_3) \in \mathbb{R}^3$. To achieve (29) we transform the expression for $\tilde{u}$ to a convenient form. We make calculations only for one summand contributing into $\tilde{u}$ (similar argument works for all other summands as well), namely,

$$b_{ij}(x,y,t) \int_0^x E_r(\xi;x,y,t) b_{rq}(\xi, y + \beta_r(\xi - x), t + \alpha_r(\xi - x)) \, d\xi \times$$

$$\times \int_\xi^1 E_p(\xi_1; \xi, y + \beta_r(\xi - x), t + \alpha_r(\xi - x)) \times$$

$$\times b_{ps}(\xi_1, y + \beta_r(\xi - x) + \beta_p(\xi_1 - \xi), t + \alpha_r(\xi - x) + \alpha_p(\xi_1 - \xi)) \, d\xi_1 \times$$

$$\times \int_0^{\xi_1} E_m(\xi_2; \xi_1, y + \beta_r(\xi - x) + \beta_p(\xi_1 - \xi), t + \alpha_r(\xi - x) + \alpha_p(\xi_1 - \xi)) \times$$

$$\times f_m(\xi_2, y + \beta_r(\xi - x) + \beta_p(\xi_1 - \xi) + \beta_m(\xi_2 - \xi_1),$$

$$t + \alpha_r(\xi - x) + \alpha_p(\xi_1 - \xi) + \alpha_m(\xi_2 - \xi_1)) \, d\xi_2 \tag{30}$$

(note that this term is considered up to a multiplicative constant).

Changing the order of integration, we have

$$\int\limits_0^x d\xi \int\limits_\xi^1 d\xi_1 \int\limits_0^{\xi_1} d\xi_2 = \int\limits_0^x d\xi \left[\int\limits_0^\xi d\xi_2 \int\limits_\xi^1 d\xi_1 + \int\limits_\xi^1 d\xi_2 \int\limits_{\xi_2}^1 d\xi_1 \right] =$$

$$= \int\limits_0^x d\xi_2 \int\limits_{\xi_2}^x d\xi \int\limits_\xi^1 d\xi_1 + \int\limits_0^x d\xi_2 \int\limits_0^{\xi_2} d\xi \int\limits_{\xi_2}^1 d\xi_1 + \int\limits_x^1 d\xi_2 \int\limits_0^x d\xi \int\limits_\xi^1 d\xi_1. \qquad (31)$$

Furthermore, we introduce new variables μ and η (instead of ξ and ξ_1) by

$$\mu \equiv y - \beta_r x + \xi(\beta_r - \beta_p) + \xi_1(\beta_p - \beta_m),$$

$$\eta \equiv t - \alpha_r x + \xi(\alpha_r - \alpha_p) + \xi_1(\alpha_p - \alpha_m). \qquad (32)$$

Owing to (5), the integers r, p, and m belong to three different sets $\{1, \ldots, \ell\}$, $\{\ell+1, \ldots, k\}$, and $\{k+1, \ldots, n\}$. On the account of (14), the mapping (32) is therefore non-generate. Apply the mapping (32) to the plane (ξ, ξ_1) and let Δ_1 denote the image of the triangle with vertices (ξ_2, ξ_2), $(\xi_2, 1)$, $(x, 1)$. Similarly, let Δ_2 denote the image of the triangle with vertices $(0, 0)$, $(0, 1)$, $(x, 1)$ and Π be the image of the quadrangle with the vertices $(0, \xi_2)$, $(0, 1)$, $(\xi_2, 1)$, (ξ_2, ξ_2). By (31) and (32), the summand (30) transforms to

$$\int\limits_0^x d\xi_2 \iint\limits_{\Delta_1} F(x, y, t, \xi_2, \mu, \eta) f_m(\xi_2, \mu, \eta)\, \partial\mu\, d\eta +$$

$$+ \int\limits_0^x d\xi_2 \iint\limits_{\Pi} F(x, y, t, \xi_2, \mu, \eta) f_m(\xi_2, \mu, \eta)\, \partial\mu\, d\eta +$$

$$+ \int\limits_x^1 d\xi_2 \iint\limits_{\Delta_2} F(x, y, t, \xi_2, \mu, \eta) f_m(\xi_2, \mu, \eta)\, \partial\mu\, d\eta, \qquad (33)$$

where F is a certain function of b_{ij}, b_{rq}, and b_{ps}. By assumption (12), F is continuously differentiable in x, y, t. Our task is therefore reduced to obtaining the estimate (29) with $\tilde{u}$ replaced by (33). The latter is a straightforward consequence of the fact that the lines bounding Δ_1, Δ_2, and Π depend linearly on x, y, t (due to the linearity of (32) in x, y, t). The proof is complete. ◆

Remark 2. *Note that in the course of proving Theorem 1, we applied Theorem 2 with $n = 3$. This choice is essential: a simple analysis of our argument shows that $n = 2$ would not work. This contrasts to the one-dimensional hyperbolic case where $n = 2$ makes the job (see [7, 8]). In general, the structure of the regularizer of the problem depends on the number of independent variables: for m-dimensional hyperbolic PDEs of kind (1) we establish the Fredholm property if we regularize the problem by means of the right regularizer $\sum\limits_{i=0}^{m} (DC^{-1})^i C$ and apply Theorem 2 with $n = m + 1$.*

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ПРО ФРЕДГОЛЬМОВУ РОЗВ'ЯЗНІСТЬ ДЕЯКОГО КЛАСУ БАГАТОВИМІРНИХ ГІПЕРБОЛІЧНИХ ЗАДАЧ

Доведено альтернативу Фредгольма для класу двовимірних гіперболічних систем першого порядку в канонічній формі Шаудера з періодичними граничними умовами Діріхле. В основі підходу лежить регуляризація за допомогою правого параметриксу.

О ФРЕДГОЛЬМОВОЙ РАЗРЕШИМОСТИ НЕКОТОРОГО КЛАССА МНОГОМЕРНЫХ ГИПЕРБОЛИЧЕСКИХ ЗАДАЧ

Доказана альтернатива Фредгольма для класса двумерных гиперболических систем первого порядка в канонической форме Шаудера с периодическими граничными условиями Дирихле. В основе подхода лежит регуляризация с помощью правого параметрикса.