UDC 511.3

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ON CONVERGENTS OF CERTAIN VALUES OF TASOEV CONTINUED FRACTIONS ASSOCIATED WITH DIOPHANTINE EQUATIONS

Let a real number ξ yield the Tasoev continued fraction. For example, $\xi = [0; a, a^2, a^3, \ldots]$ for an integer $a \ge 2$. Let $\eta = |h(\xi)|$, where $h(t)$ is a non-constant rational function with algebraic coefficients. We compute upper and lower bounds for the approximation of the values η derived from the Tasoev continued fractions by rationals x/y such that x and y satisfy Diophantine equations. We show that there are infinitely many coprime integers x and y such that $|y\eta - x| \ll y^{\frac{1}{3} - \sqrt{\frac{2\log a}{3\log y}}}$ and a Diophantine equation holds simultaneously relating x and y and some integer z. Conversely, all positive integers x and y with $y \ge c_0$ solving the Diophantine equation satisfy $|y\eta - x| \gg y^{-1-\sqrt{\frac{2\log a}{\log y}}}$.

Introduction. Let $\alpha = [a_0; a_1, a_2, \ldots]$ denotes the regular (or simple) continued fraction expansion of a real α, where

$$\alpha = a_0 + \theta_0, \qquad a_0 = \lfloor \alpha \rfloor,$$

$$1/\theta_{n-1} = a_n + \theta_n, \qquad a_n = \lfloor 1/\theta_{n-1} \rfloor, \qquad n \ge 1.$$

Assume that the continued fraction expansion of a real ξ is quasi-periodic of the form

$$[a_0; a_1, \ldots, a_n, \overline{g_1(k), \ldots, g_s(k)}]_{k=1}^{\infty} = \\ = [a_0; a_1, \ldots, a_n, g_1(1), \ldots, g_s(1), g_1(2), \ldots, g_s(2), g_1(3), \ldots], \qquad (1)$$

where a_0 is an integer; a_1, …, a_n are positive integers; g_1, …, g_s are positive integer-valued functions for $k = 1, 2, \ldots$. If every $g_i(k)$, $i = 1, 2, \ldots, s$, is polynomial and at least one of them is not constant, (1) is called *Hurwitz continued fraction* [14]. If every $g_i(k)$, $i = 1, 2, \ldots, s$, is exponential and at least one of them is not constant, (1) is called *Tasoev continued fraction.*

Tasoev continued fractions [7–9, 11, 12, 16] are systematic but have hardly been known before. In [8], the author found some general Tasoev continued fractions. Namely,

$$\xi_1 := [0; \overline{ua^k}]_{k=1}^{\infty} = \frac{\displaystyle\sum_{n=0}^{\infty} u^{-2n-1} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\displaystyle\sum_{n=0}^{\infty} u^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}, \qquad (2)$$

$$\xi_2 := [0; ua - 1, \overline{1, ua^{k+1} - 2}]_{k=1}^{\infty} = \\ = \frac{\displaystyle\sum_{n=0}^{\infty} (-1)^n u^{-2n-1} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\displaystyle\sum_{n=0}^{\infty} (-1)^n u^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}, \qquad (3)$$

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$$\xi_3 := [0; \overline{ua^k, va^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^i - 1)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^i - 1)^{-1}} \tag{4}$$

and

$$\xi_4 = [0; ua-1, 1, va-2, \overline{1, ua^{k+1}-2, 1, va^{k+1}-2}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^i - 1)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^i - 1)^{-1}}. \tag{5}$$

We can safely say that Tasoev continued fractions are geometric and Hurwitz continued fractions are arithmetic [9]. The Tasoev continued fractions corresponding to e-type Hurwitz continued fractions were also derived in [11]:

$$\xi_5 = [0; \overline{ua^k - 1, 1, v-1}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} u^{-2n-1} v^{-2n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty} ((uv)^{-2n} a^{-n^2} - (uv)^{-2n-1} a^{-(n+1)^2}) \prod_{i=1}^{n} (a^{2i} - 1)^{-1}} \tag{6}$$

and

$$\xi_6 := [0; \overline{v-1, 1, ua^k - 1}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} (u^{-2n} v^{-2n-1} a^{-n^2} + u^{-2n-1} v^{-2n-2} a^{-(n+1)^2}) \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty} (uv)^{-2n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - 1)^{-1}}. \tag{7}$$

The different types of Tasoev continued fractions with period 3 shown in [9] are

$$\xi_7 := [0; \overline{ua^{2k-1} - 1, 1, va^{2k} - 1}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - (-1)^i)^{-1}}, \tag{8}$$

$$\xi_8 := [0; ua, \overline{va^{2k} - 1, 1, ua^{2k+1} - 1}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n u^{-n-1} v^{-n} a^{-(n+1)^2} \prod_{i=1}^{n} (a^{2i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n^2} \prod_{i=1}^{n} (a^{2i} - (-1)^i)^{-1}}, \tag{9}$$

$$\xi_9 := [0; \overline{ua^k - 1, 1, va^k - 1}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^i - (-1)^i)^{-1}}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^i - (-1)^i)^{-1}} \tag{10}$$

and

$$\xi_{10} := [0; ua, \overline{va^k - 1, 1, ua^{k+1} - 1}]_{k=1}^{\infty} =$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n u^{-n-1} v^{-n} a^{-(n+1)(n+2)/2} \prod_{i=1}^{n} (a^i - (-1)^i)^{-1}}{\sum_{n=0}^{\infty} u^{-n} v^{-n} a^{-n(n+1)/2} \prod_{i=1}^{n} (a^i - (-1)^i)^{-1}}. \tag{11}$$

Proposition 1. *Let* s *be a positive integer and* x *and* y (≥ 3) *relatively prime integers with* $y \equiv 0 \pmod 2$ *such that* $x^2 + y^2 = z^2$. *Then*

$$\left| y \sinh\left(\frac{1}{s}\right) - x \right| > C_{33} \frac{\log\log y}{\log y}.$$

On the other hand, there are infinitely many pairs x, y *as just described satisfying*

$$\left| y \cdot \sinh\left(\frac{1}{s}\right) - x \right| < C_{34} \frac{\log\log y}{\log y}.$$

A similar paper [1] appeared without giving bounds. In [5] we continued to consider upper and lower bounds for the approximation of the values η derived from the Tasoev continued fractions by rationals x/y such that x and y satisfy Diophantine equations found in e.g. [2, 3, 6]. Diophantine approximations of Tasoev continued fractions without restrictions have been investigated by e.g. [10, 13, 15]. In this paper we compute upper and lower bounds for the approximation of the values η derived from the Tasoev continued fractions by rationals x/y such that x and y satisfy Diophantine equations.

2. Fundamental Lemma. Let $h : \overline{\mathbb{Q}} \to \mathbb{R}$ be a non-constant rational function. If ξ is transcendental, then $h(\xi) \neq 0$, $h'(\xi) \neq 0$, and so there exists an interval $I = [\xi - \delta, \xi + \delta]$, where δ may depend on ξ, h, such that $h(t)$, $h'(t)$ do not vanish on I. In particular, we choose a rational function h such that for each $p, q\ (> 0) \in \mathbb{Z}$ the function h takes the form

$$h\left(\frac{p}{q}\right) = \frac{h_1(p,q)}{h_2(p,q)}$$

where $h_1, h_2 \in \mathbb{Z}[p,q]$. Assume that there is a polynomial P, whose coefficients are in $\mathbb{Z}$, such that

$$P(h_1, h_2, h_3(h_1, h_2)) = 0. \tag{12}$$

Suppose that ξ is any one of ξ_5, ξ_6, ξ_7 and ξ_8, which yield Tasoev continued fractions (6), (7), (8) and (9), respectively.

Lemma 1. *There exist real constants* C_3 *and* C_4 *such that*

$$C_3 Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}} \le \left| h(\xi) - h\left(\frac{P_n}{Q_n}\right) \right| \le C_4 Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}}.$$

P r o o f. If $\xi = \xi_5 = [0; \overline{ua^k - 1, 1, v - 1}]_{k=1}^{\infty}$, then we have

$$\frac{1}{(a_{k+1}+2)q_k^2} < \left| \xi - \frac{p_k}{q_k} \right| < \frac{1}{a_{k+1} q_k^2}.$$

Choose $P_n = p_{3n}$ and $Q_n = q_{3n}$. Then by $a_{3n+1} = ua^{n+1} - 1$

$$\frac{1}{(ua^{n+1}+1)Q_n^2} < \left| \xi - \frac{P_n}{Q_n} \right| < \frac{1}{(ua^{n+1}-1)Q_n^2}.$$

Hence,

$$\begin{aligned} \log Q_n &= \log q_{3n} = \log a_1 a_2 \dots a_{3n} + \mathcal{O}(1) = \\ &= \log (ua-1)(ua^2-1)\dots(ua^n-1)(v-1)^n + \mathcal{O}(1) \sim \\ &\sim \log ua \cdot ua^2 \dots ua^n + n \log (v-1) = \\ &= \frac{n(n+1)}{2} \log a + n \log u(v-1) = \\ &= \frac{\log a}{2} \left(n + \frac{1}{2} + \frac{\log u(v-1)}{\log a} \right)^2 - \\ &- \frac{1}{8} \log a - \frac{1}{2} \log u(v-1) - \frac{1}{2} \frac{(\log u(v-1))^2}{\log a} \sim \\ &\sim \frac{\log a}{2} \left(n + \frac{1}{2} + \frac{\log u(v-1)}{\log a} \right)^2. \end{aligned}$$

Thus,

$$n \sim \sqrt{\frac{2\log Q_n}{\log a}} - \frac{1}{2} - \frac{\log u(v-1)}{\log a}.$$

Therefore,

$$\begin{aligned} \frac{1}{ua^{n+1}Q_n^2} &= Q_n^{-2-\frac{\log ua}{\log Q_n} - n\frac{\log a}{\log Q_n}} \sim \\ &\sim Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}} + \frac{\log((v-1)/\sqrt{a})}{\log Q_n}} = \\ &= \frac{v-1}{\sqrt{a}} Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}}. \end{aligned}$$

We have for positive constants D_3 and D_4

$$D_3 Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}} < \left| \xi - \frac{P_n}{Q_n} \right| < D_4 Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}}.$$

Since

$$t_1 := \frac{uv^2a^2}{u^2v^2a^3 - uva^2 + 1} = \frac{p_6}{q_6} = \frac{P_2}{Q_2} < \xi < \frac{P_1}{Q_1} = \frac{p_3}{q_3} = \frac{v}{uva-1} := t_2,$$

by the hypotheses on the function h the positive constants

$$D_5 = \min_{t_1 \le t \le t_2} |h'(t)| \qquad \text{and} \qquad D_6 = \max_{t_1 \le t \le t_2} |h'(t)|$$

exist. By the mean value theorem,

$$D_5 \left| \xi - \frac{P_n}{Q_n} \right| \le \left| h(\xi) - h\left(\frac{P_n}{Q_n}\right) \right| \le D_6 \left| \xi - \frac{P_n}{Q_n} \right|.$$

Putting $C_3 = D_3 D_5$ and $C_4 = D_4 D_6$, we get the desired inequalities.

The result is similar for

$$\xi = \xi_6 = [0; \overline{v-1, 1, ua^k - 1}]_{k=1}^{\infty}.$$

If

$$\xi = \xi_7 = [0; \overline{ua^{2k-1} - 1, 1, va^{2k} - 1}]_{k=1}^{\infty}$$

or

$$\xi_8 = [0; ua, \overline{va^{2k} - 1, 1, ua^{2k+1} - 1}]_{k=1}^{\infty},$$

then we get

$$\frac{1}{ua^{n+1} Q_n^2} \sim C^* Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}},$$

where

$$C^* = \begin{cases} \sqrt{v/(ua)}, & u \ge v, \\ \sqrt{u/(va)}, & u < v. \end{cases}$$

The rest of the part is similar and omitted. ♦

2. Main Theorem.

Theorem 1. *Let* ξ *be one of real numbers* ξ_5, ξ_6, ξ_7 *and* ξ_8, *and* $\eta = (\xi^{-1} - \xi)/2$. *Then there are infinitely many triples* (x, y, z) *of integers satisfying simultaneously*

$$\left| \eta - \frac{x}{y} \right| < C_4 y^{-\frac{2}{3} - \sqrt{\frac{2\log a}{3\log y}}} \qquad \text{and} \qquad x^2 + y^2 = z^2.$$

Conversely, for given integers x, y *with* $x^2 + y^2 = z^2$, *we have the inequality*

$$\left| \eta - \frac{x}{y} \right| > C_5 y^{-2-\sqrt{\frac{2\log a}{\log y}}}.$$

P r o o f. Let

$$h(t) = \frac{1}{2}\left(t - \frac{1}{t} \right).$$

So, $h(t)$ is monotonously increasing for $0 < t < 1$ and $h \in C^{(1)}(0,1)$. Then

$$h'(t) = \frac{1}{2} + \frac{1}{2t^2}, \qquad 1 \le h'(t) \le \frac{1}{2}\left(1 + \frac{1}{t_1^2} \right), \qquad t_1 \le t \le t_2 < 1,$$

where t_1 and t_2 are some fixed real numbers in $I = [\xi - \delta, \xi + \delta]$. We apply Lemma 1 by putting $x_n = P_n^2 - Q_n^2 (< 0)$ and $y_n = 2P_n Q_n$. We know that

$$h\left(\frac{P_n}{Q_n} \right) = \frac{P_n^2 - Q_n^2}{2P_n Q_n} = \frac{x_n}{y_n},$$

so $x_n^2 + y_n^2 = (P_n^2 + Q_n^2)^2$. Since

$$0 < t_1 \le \frac{P_n}{Q_n} \le t_2 < 1,$$

we have

$$t_1 Q_n^2 \le P_n Q_n = \frac{y_n}{2} \le t_2 Q_n^2.$$

So, $y_n \le 2t_2 Q_n^2 \le Q_n^3$. Since the function $x^{-2-\sqrt{(2\log a)/(\log x)}}$ is monotonously decreasing for $x > 1$, by Lemma 1

$$\left|-\eta + \frac{x}{y}\right| = \left|h(\xi) - h\left(\frac{P_n}{Q_n}\right)\right| \le C_4 Q_n^{-2-\sqrt{\frac{2\log a}{\log Q_n}}} < C_4 y_n^{\frac{1}{3}(-2-\sqrt{\frac{6\log a}{\log y_n}})}.$$

Conversely, there exists a closed interval $I = [\xi - \delta, \xi + \delta] \subset [0,1]$ such that for any positive integers p, $q(\ge 3)$, $p/q \in I$ the inequality

$$\left|h(\xi) - h\left(\frac{p}{q}\right)\right| > Cq^{-2-\sqrt{\frac{2\log a}{\log q}}}$$

holds, where C is a positive constant depending only on a, u, v and the function h (*Cf*. [5, Lemma 2.1]). Let positive integers x, y (≥ 3), z be given such that $x^2 + y^2 = z^2$. Since $h((0,1)) = \mathbb{R}_{<0}$, x/y takes every positive rational number. We have

$$x = q^2 - p^2 (> 0), \qquad y = 2pq, \qquad z = p^2 + q^2$$

and $h(p/q) = -x/y$. If $p/q = h^{-1}(-x/y) \in I$, then

$$\left|\eta - \frac{x}{y}\right| > Cq^{-2-\sqrt{\frac{2\log a}{\log q}}}.$$

Since $y = 2pq > q$,

$$q^{-2-\sqrt{\frac{2\log a}{\log q}}} > y^{-2-\sqrt{\frac{2\log a}{\log y}}}, \qquad y \ge 3.$$ ♦

Similarly to Theorem 2 and Theorem 3 in [4], if we consider the numbers $\eta = (\xi + \xi^{-1})/2$ and $(1-\xi^2)/(1+\xi^2)$ instead of $\eta = (\xi^{-1} - \xi)/2$, we have the same inequalities where $x^2 + y^2 = z^2$ is replaced by $x^2 - y^2 = z^2$ and $y^2 - x^2 = z^2$, respectively. In such cases we use $h(t)$ as

$$h(t) = \frac{1}{2}\left(t + \frac{1}{t}\right) \qquad \text{and} \qquad h(t) = \frac{t^2 - 1}{t^2 + 1},$$

respectively.

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ПРО НАБЛИЖЕННЯ ДЕЯКИХ ЗНАЧЕНЬ НЕПЕРЕРВНИХ ДРОБІВ ТАСОЄВА, ПОВ'ЯЗАНИХ З ДІОФАНТОВИМИ РІВНЯННЯМИ

Нехай дійсне число ξ *породжено неперервним дробом Тасоєва. Наприклад,* $\xi = [0; a, a^2, a^3, \ldots]$ *для цілого числа* $a \ge 2$. *Нехай* $\eta = |h(\xi)|$, *де* $h(t)$ *– відмінна від сталої раціональна функція з алгебраїчними коефіцієнтами. Обчислені верхня и нижня межі наближення значень* η, *отриманих з неперервних дробів Тасоєва, раціональними* x/y *такими, що* x *і* y *задовольняють діофантові рівняння. Показано, що існує нескінченно багато взаємно простих цілих чисел* x *і* y *таких, що* $|y\eta - x| \ll y^{\frac{1}{3} - \sqrt{\frac{2\log a}{3\log y}}}$, *і діофантове рівняння задовольняє одночасно і пов'язане з* x *і* y *деяке ціле число* z. *Навпаки, всі натуральні числа* x *і* y *з* $y \ge c_0$ *– розв'язки діофантового рівняння, задовольняють* $|y\eta - x| \gg y^{-1-\sqrt{\frac{2\log a}{\log y}}}$.

О ПРИБЛИЖЕНИЯХ НЕКОТОРЫХ ЗНАЧЕНИЙ НЕПРЕРЫВНЫХ ДРОБЕЙ ТАСОЕВА, СВЯЗАННЫХ С ДИОФАНТОВЫМИ УРАВНЕНИЯМИ

Пусть действительное число ξ *порождено непрерывной дробью Тасоева. Например,* $\xi = [0; a, a^2, a^3, \ldots]$ *для целого числа* $a \ge 2$. *Пусть* $\eta = |h(\xi)|$, *где* $h(t)$ *– отличная от постоянной рациональная функция с алгебраическими коэффициентами. Вычислены верхняя и нижняя границы приближения значений* η, *полученных из непрерывных дробей Тасоева, рациональными* x/y *такими, что* x *и* y *удовлетворяют диофантовым уравнениям. Показано, что существует бесконечно много взаимно простых целых чисел* x *и* y *таких, что* $|y\eta - x| \ll y^{\frac{1}{3} - \sqrt{\frac{2\log a}{3\log y}}}$, *и диофантовому уравнению удовлетворяет одновременно и связанное с* x *и* y *некоторое целое число* z. *Наоборот, все натуральные целые числа* x *и* y *с* $y \ge c_0$ *– решения диофантового уравнения, удовлетворяют* $|y\eta - x| \gg y^{-1-\sqrt{\frac{2\log a}{\log y}}}$.

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