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CONGRUENCES ON THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF $L_n \times_{\mathrm{lex}} \mathbb{Z}$ WITH CO-FINITE DOMAINS AND IMAGES

We study congruences on the semigroup $\mathscr{I}\mathscr{O}_{\infty}(\mathbb{Z}^n_{\mathrm{lex}})$ of monotone injective partial selfmaps of the set of $L_n \times_{\mathrm{lex}} \mathbb{Z}$ having co-finite domains and images, where $L_n \times_{\mathrm{lex}} \mathbb{Z}$ is the lexicographic product of n-elements chain and the set of integers with the usual linear order. The structure of the sublattice of congruences on $\mathscr{I}\mathscr{O}_{\infty}(\mathbb{Z}^n_{\mathrm{lex}})$ which contain in the least group congruence is described.

We follow the terminology [6, 7] and [8]. We shall denote the additive group of integers by $\mathbb{Z}(+)$.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $inv : S \to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

If $\mathfrak{C}$ is an arbitrary congruence on a semigroup S, then we denote by $\Phi_{\mathfrak{C}} : S \to S/\mathfrak{C}$ the natural homomorphisms from S onto the quotient semigroup $S/\mathfrak{C}$. A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruences Δ_S on S, and *group* if the quotient semigroup $S/\mathfrak{C}$ is a group. Every inverse semigroup S admits the least (minimum) group congruence σ:

$$a\sigma b \text{ if and only if there exists } e \in E(S) \text{ such that } ae = be$$

(see [8, Lemma III.5.2].)

If S is a semigroup, then we shall denote the subset of idempotents of S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E.

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [2, Section 2.1]):

$$a\mathscr{R}b \quad \text{if and only if} \quad aS^1 = bS^1,$$

$$a\mathscr{L}b \quad \text{if and only if} \quad S^1a = S^1b,$$

$$a\mathscr{J}b \quad \text{if and only if} \quad S^1aS^1 = S^1bS^1,$$

$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L},$$

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}.$$

A semigroup S is called *simple* if S contains no proper two-sided ideal, i.e., S has a unique $\mathscr{J}$-class, and *bisimple* if S has a unique $\mathscr{D}$-class.

If $\alpha : X \rightharpoonup Y$ is a partial map, then by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ we denote the domain and the range of α, respectively.

Let $\mathscr{I}_\lambda$ denotes the set of all partial one-to-one transformations of an infinite set X of cardinality λ endowed with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [2, Section 1.9]). The symmetric inverse semigroup was introduced by Wagner [1] and it plays a major role in the theory of semigroups. An element $\alpha \in \mathscr{I}_\lambda$ is called *co-finite*, if the sets $\lambda\setminus\operatorname{dom}\alpha$ and $\lambda\setminus\operatorname{ran}\alpha$ are finite.

Let $(X, \leq)$ be a partially ordered set. We shall say that a partial map $\alpha : X \rightharpoonup X$ is *monotone* if $x \leq y$ implies $(x)\alpha \leq (y)\alpha$ for each $x, y \in X$.

Let $\mathbb{Z}$ be the set of integers with the usual linear order «$\leq$». For any positive integer n by L_n we denote the set $\{1, \ldots, n\}$ with the usual linear order «$\leq$». On the Cartesian product $L_n \times \mathbb{Z}$ we define the lexicographic order, i.e.,

$$(i, m) \leq (j, n) \quad \text{if and only if} \quad (i < j) \quad \text{or} \quad (i = j \quad \text{and} \quad m \leq n).$$

Later the set $L_n \times \mathbb{Z}$ with the lexicographic order we denote by $L_n \times_{\text{lex}} \mathbb{Z}$. Also, it is obvious that the set $\mathbb{Z} \times L_n$ with the lexicographic order is order isomorphic to $(\mathbb{Z}, \leq)$.

By $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ we denote a semigroup of injective partial monotone selfmaps of $L_n \times_{\text{lex}} \mathbb{Z}$ with co-finite domains and images. Obviously, $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ is an inverse submonoid of the semigroup $\mathscr{I}_\omega$ and $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ is a countable semigroup. Also, by $\mathscr{IO}_\infty(\mathbb{Z})$ we denote a semigroup of injective partial monotone selfmaps of $\mathbb{Z}$ with co-finite domains and images.

Furthermore, we shall denote the identity of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ by $\mathbb{I}$ and the group of units of $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ by $H(\mathbb{I})$.

Gutik and Repovš in [5] showed that the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ of partial co-finite monotone injective transformations of the set of positive integers $\mathbb{N}$ has algebraic properties similar to those of the bicyclic semigroup: it is bisimple and all of its non-trivial semigroup homomorphisms are either isomorphisms or group homomorphisms.

In [4] Gutik and Repovš studied the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of partial co-finite monotone injective transformations of the set of integers $\mathbb{Z}$ and they showed that $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is bisimple and all of its non-trivial semigroup homomorphisms are either isomorphisms or group homomorphisms.

In the paper [3] we studied the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. There we described Green's relations on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$, showed that the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ is bisimple and established its projective congruences. Also, there we proved that $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ is finitely generated, every automorphism of $\mathscr{IO}_\infty(\mathbb{Z})$ is inner and showed that in the case $n \geq 2$ the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ has non-inner automorphisms. In [3] we proved that for every positive integer n the quotient semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})/\sigma$, where σ is the least group congruence on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$, is isomorphic to the direct power $(\mathbb{Z}(+))^{2n}$.

By Proposition 2.3(iv) [3], the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ is isomorphic to the direct power $(\mathscr{IO}_\infty(\mathbb{Z}))^n$. Fixing this isomorphism further we shall identify elements of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ with elements of the direct product $(\mathscr{IO}_\infty(\mathbb{Z}))^n$, i.e., every element α of $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ we present in the form $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where all α_i belongs to $\mathscr{IO}_\infty(\mathbb{Z})$. Later by α_i° we shall denote the element with the form $(\mathbb{I}_1, \ldots, \mathbb{I}_{i-1}, \alpha_i, \mathbb{I}_{i+1}, \ldots, \mathbb{I}_n)$, where $\mathbb{I}_j$ is the identity of the j-th factor of $(\mathscr{IO}_\infty(\mathbb{Z}))^n$ for all j and $\alpha_i \in (\mathscr{IO}_\infty(\mathbb{Z}))$. It is obvious that for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ we have that $\alpha = \alpha_1^\circ \ldots \alpha_n^\circ$.

For every $i = 1, \ldots, n$ we define a binary relation $\sigma_{[i]}$ on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ in the following way:

$\alpha\sigma_{[i]}\beta$ if and only if there exists an idempotent

$$\varepsilon \in \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}}) \qquad \text{such that} \qquad \alpha\varepsilon_i^\circ = \beta\varepsilon_i^\circ.$$

In [3] we proved that $\sigma_{[i]}$ is a congruence on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ for every $i = 1, \ldots, n$. Also, there is shown that for any subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ of distinct integers, the relation $\sigma_{[i_1, \ldots, i_k]} = \sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]}$ is a congruence on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ and is described the properties of the congruence $\sigma_{[i_1, \ldots, i_k]}$ (see Propositions 2.11-2.13, 2.15 and 2.18 in [3]). Moreover, $\sigma_{[1,2,\ldots,n]}$ is the least group congruence on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$.

For every $i = 1, \ldots, n$ we define a map $\pi^i : \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}}) \to \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ by the formula $(\alpha)\pi^i = \alpha_i^\circ$, i.e., $(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)\pi^i = (\mathbb{I}_1, \ldots, \mathbb{I}_{i-1}, \alpha_i, \mathbb{I}_{i+1}, \ldots, \mathbb{I}_n)$. Simple verifications show that the map $\pi^i : \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}}) \to \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is a homomorphism. Let $\pi^{i\#}$ be the congruence on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ which is generated by the homomorphism π^i.

Let S be an inverse semigroup. For any congruence ρ on S we define a congruence $\rho_{\min}$ on S as follows:

$a\rho_{\min}b$ if and only if $ae = be$ for some $e \in E(S)$ and $e\rho a^{-1}a\rho b^{-1}b$

(see [8, Section III.2]). Then Proposition 2.17 from [3] implies that

$$\pi^{i\#}_{\min} = \sigma_{[1]} \circ \ldots \circ \sigma_{[i-1]} \circ \sigma_{[i+1]} \circ \ldots \circ \sigma_{[n]}$$

for every $i = 1, \ldots, n$.

This paper is a continuation of [3] and we study congruences on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. Here we describe the structure of the sublattice of congruences on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ which contained in the least group congruence.

For arbitrary elements $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ we define:

$$\mathbf{D}_{\alpha,\beta} = \{i \in \{1, \ldots, n\} \mid \alpha_i \neq \beta_i\}.$$

It is obvious that elements $\alpha, \beta \in \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ are equal if and only if $\mathbf{D}_{\alpha,\beta} = \varnothing$.

Lemma 1. *Let $\mathfrak{C}$ be a congruence on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. Let α and β be two distinct $\mathfrak{C}$-equivalent elements of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. Then there exists an element ω in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ such that $\mathbb{I}\mathfrak{C}\omega$ and $\mathbf{D}_{\mathbb{I},\omega} = \mathbf{D}_{\alpha,\beta}$.*

P r o o f. By Proposition 2.3 (*iv*) from [3] the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is isomorphic to the direct power $(\mathscr{IO}_\infty(\mathbb{Z}))^n$. We denote $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$. Then for every $i \in \mathbf{D}_{\alpha,\beta}$ we have that $\alpha_i \neq \beta_i$.

We fix an arbitrary $i \in \mathbf{D}_{\alpha,\beta}$. Then one of the following cases holds:

1) $\alpha_i \mathscr{H} \beta_i$ in $\mathscr{IO}_\infty(\mathbb{Z})$;

2) α_i and β_i are not $\mathscr{H}$-equivalent in $\mathscr{IO}_\infty(\mathbb{Z})$.

Suppose that case ***1***) holds. By Proposition 2.3 from [4] the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ is bisimple and hence by Theorem 2.3 from [2] there exist $\gamma_i, \delta_i \in \mathscr{IO}_\infty(\mathbb{Z})$ such that $\eta_i = \gamma_i\alpha_i\delta_i$ and $\zeta_i = \gamma_i\beta_i\delta_i$ are distinct elements of the group of units of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$. Then we have that $\eta_i^{-1}\eta_i = \eta_i^{-1}\gamma_i\alpha_i\delta_i = \mathbb{I}_i$ is the unit of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ and $\eta_i^{-1}\zeta_i = \eta_i^{-1}\gamma_i\beta_i\delta_i \neq \mathbb{I}_i$. Hence, without loss of generality we can assume that there exist elements γ_i and δ_i of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ such that $\gamma_i\alpha_i\delta_i = \mathbb{I}_i$ is the unit of $\mathscr{IO}_\infty(\mathbb{Z})$ and $\gamma_i\beta_i\delta_i \neq \mathbb{I}_i$.

Suppose that the elements α_i and β_i are not $\mathscr{H}$-equivalent in $\mathscr{IO}_\infty(\mathbb{Z})$. Then by Proposition 2.1 (*vii*) from [4] we have that at least one of the following conditions holds:

$$\operatorname{dom}\alpha_i \neq \operatorname{dom}\beta_i \qquad \text{or} \qquad \operatorname{ran}\alpha_i \neq \operatorname{ran}\beta_i .$$

Since every subset with finite complement in $\mathbb{Z}$ is order isomorphic to $\mathbb{Z}$ we conclude that there exist monotone bijective maps $\gamma_i : \mathbb{Z} \to \operatorname{dom}\alpha_i$ and $\delta_i : \operatorname{ran}\alpha_i \to \mathbb{Z}$. Then we have that $\gamma_i\alpha_i\delta_i$ is an element of the group of units of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$, because $\operatorname{dom}(\gamma_i\alpha_i\delta_i) = \operatorname{ran}(\gamma_i\alpha_i\delta_i) = \mathbb{Z}$.

Suppose we have that $\operatorname{dom}\alpha_i \neq \operatorname{dom}\beta_i$. If there exists an integer $k \in \operatorname{dom}\alpha_i$ such that $k \notin \operatorname{dom}\beta_i$, then $(k)\gamma_i^{-1} \in \operatorname{dom}(\gamma_i\alpha_i\delta_i)$ and $(k)\gamma_i^{-1} \notin \operatorname{dom}(\gamma_i\beta_i\delta_i)$. If there exists an integer $k \in \operatorname{dom}\beta_i$ such that $k \notin \operatorname{dom}\alpha_i$, then $(k)\gamma_i^{-1} \in \operatorname{dom}(\gamma_i\beta_i\delta_i)$ and $(k)\gamma_i^{-1} \notin \operatorname{dom}(\gamma_i\alpha_i\delta_i)$. Therefore, we get that $\operatorname{dom}(\gamma_i\beta_i\delta_i) \neq \operatorname{dom}(\gamma_i\alpha_i\delta_i)$.

Suppose we have that $\operatorname{ran}\alpha_i \neq \operatorname{ran}\beta_i$. If there exists an integer $k \in \operatorname{ran}\alpha_i$ such that $k \notin \operatorname{ran}\beta_i$, then $(k)\delta_i \in \operatorname{ran}(\gamma_i\alpha_i\delta_i)$ and $(k)\delta_i \notin \operatorname{ran}(\gamma_i\beta_i\delta_i)$. If there exists an integer $k \in \operatorname{ran}\beta_i$ such that $k \notin \operatorname{ran}\alpha_i$, then $(k)\delta_i \in \operatorname{ran}(\gamma_i\beta_i\delta_i)$ and $(k)\delta_i \notin \operatorname{ran}(\gamma_i\alpha_i\delta_i)$. This implies that $\operatorname{ran}(\gamma_i\beta_i\delta_i) \neq \operatorname{ran}(\gamma_i\alpha_i\delta_i)$.

Since every translation on an arbitrary element of the group of units of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ is a bijective map of the set of integers $\mathbb{Z}$, without loss of generality we can assume that the element $\gamma_i\alpha_i\delta_i$ is the unit of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$.

Next, we define elements $\gamma = (\gamma_1,\dots,\gamma_n)$ and $\delta = (\delta_1,\dots,\delta_n)$ of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ in the following way. For $i \in \mathbf{D}_{\alpha,\beta}$ we define γ_i and δ_i to be the elements of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ so constructed above. For $i \in \{1,\dots,n\} \setminus \mathbf{D}_{\alpha,\beta}$ we put γ_i and δ_i are the elements of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ such that $\gamma_i\alpha_i\delta_i = \gamma_i\beta_i\delta_i = \mathbb{I}_i$ is the unit of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$.

The existence of so elements γ_i and δ_i in $\mathscr{IO}_\infty(\mathbb{Z})$ follows from Theorem 2.3 from [2] and the fact that the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ is bisimple (see [4, Proposition 2.3]).

Hence we get that

$$\gamma\alpha\delta = \mathbb{I}, \quad \omega = \gamma\beta\delta \neq \mathbb{I} \quad \text{and} \quad \omega\mathfrak{C}\mathbb{I} \quad \text{in} \quad \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}}).$$

Moreover, our construction implies that $\mathbf{D}_{\mathbb{I},\omega} = \mathbf{D}_{\alpha,\beta}$. ♦

Lemma 2. *Let $\mathfrak{C}$ be a congruence on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. Let α and β be two distinct $\mathfrak{C}$-equivalent elements of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. Then there exists an element ψ in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ such that $\mathbb{I}\mathfrak{C}\psi$, $\mathbf{D}_{\mathbb{I},\psi} = \mathbf{D}_{\alpha,\beta}$ and elements $\mathbb{I}$ and ψ are not $\mathscr{H}$-equivalent in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$.*

P r o o f. If α and β are not $\mathscr{H}$-equivalent elements of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$, then by case ***2***) of the proof of Lemma 1 we obtain that $\mathbb{I}\mathfrak{C}\omega = \gamma\beta\delta$ and the elements $\mathbb{I}$ and ω are not $\mathscr{H}$-equivalent in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$.

Next, we suppose that $\alpha\mathscr{H}\beta$ and put $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $\beta = (\beta_1,\ldots,\beta_n)$. Then by Proposition 2.3 from [4] the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ is bisimple and hence by Theorem 2.3 from [2] for every $i = 1,\ldots,n$ there exist $\gamma_i, \delta_i \in \mathscr{IO}_\infty(\mathbb{Z})$ such that $\gamma_i\alpha_i\delta_i = \mathbb{I}_i$ is the unit of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ and $\gamma_i\beta_i\delta_i \neq \mathbb{I}_i$ for each $i \in \mathbf{D}_{\alpha,\beta}$. Since $\alpha\mathscr{H}\beta$ and by Proposition 2.3 (v) from [3] the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is isomorphic to the direct power $(\mathscr{IO}_\infty(\mathbb{Z}))^n$ we conclude that $\gamma_i\beta_i\delta_i$ is an element of the group of units of $\mathscr{IO}_\infty(\mathbb{Z})$ for each $i \in \{1,\ldots,n\}$, and moreover $\gamma_i\beta_i\delta_i = \mathbb{I}_i = \gamma_i\alpha_i\delta_i$ for any $i \in \{1,\ldots,n\} \setminus \mathbf{D}_{\alpha,\beta}$.

We denote $\gamma = (\gamma_1,\ldots,\gamma_n)$ and $\delta = (\delta_1,\ldots,\delta_n)$ and put $æ = (æ_1,\ldots,æ_n) = = \gamma\beta\delta$. Then we have that $\mathbf{D}_{\alpha,\beta} = \mathbf{D}_{\mathbb{I},æ}$. Also the relation $\alpha\mathscr{H}\beta$ implies that $\mathbb{I}\mathscr{H}æ$, and since $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is an inverse semigroup we get that $\mathbb{I}\mathscr{H}æ^m$ for every integer m. By Proposition 2.2 from [4] the group of units of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ is isomorphic to $\mathbb{Z}(+)$. Hence, this implies that without loss of generality we can assume that $(p)æ_i = p + m_i$, where $m_i \neq 0$, for every $i \in \mathbf{D}_{\alpha,\beta}$

Next, for every integer $i = 1,\ldots,n$ we define a partial map $\chi_i : \mathbb{Z} \rightharpoonup \mathbb{Z}$ in the following way:

(***a***) if $i \in \{1,\ldots,n\} \setminus \mathbf{D}_{\alpha,\beta}$, then we define $\chi_i : \mathbb{Z} \to \mathbb{Z}$ be the identity map;

(***b***) if $i \in \mathbf{D}_{\alpha,\beta}$ and $m_i \geq 1$, then we define $\operatorname{dom}\chi_i = \mathbb{Z}$, $\operatorname{ran}\chi_i = \mathbb{Z} \setminus \{1,\ldots,m_i\}$ and

$$(k)\chi_i = \begin{cases} k + m_i, & \text{if} \quad k \geq 1, \\ k, & \text{if} \quad k \leq 0, \end{cases}$$

(***c***) if $i \in \mathbf{D}_{\alpha,\beta}$ and $m_i \leq -1$, then we define $\operatorname{dom}\chi_i = \mathbb{Z}$, $\operatorname{ran}\chi_i = \mathbb{Z} \setminus \{m_i,\ldots,-1\}$ and

$$(k)\chi_i = \begin{cases} k, & \text{if} \quad k \geq 0, \\ k + m_i, & \text{if} \quad k \leq -1. \end{cases}$$

We put $\chi = (\chi_1, \dots, \chi_n)$. The definition of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ implies that χ and its inverse χ^{-1} are elements of $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. Simple verifications show that $\mathbb{I} = \chi\chi^{-1} = \chi\mathbb{I}\chi^{-1}$. Also, since $\mathfrak{C}$ is a congruence on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ we conclude that $\mathbb{I} = \chi\mathbb{I}\chi^{-1}\mathfrak{C}\chi\ae\chi^{-1}$.

Now simple calculations imply that

(*i*) if $m_i > 0$ then

$$(k)\chi_i\ae_i\chi_i^{-1} = \begin{cases} k + m_i, & \text{if} \quad k \geq 1, \\ \text{undefined}, & \text{if} \quad -m_i < k \leq 0, \\ k + m_i, & \text{if} \quad k \leq -m_i, \end{cases}$$

and similarly

(*ii*) if $m_i < 0$ then

$$(k)\chi_i\ae_i\chi_i^{-1} = \begin{cases} k + m_i, & \text{if} \quad k \geq -m_i, \\ \text{undefined}, & \text{if} \quad 0 \leq k < -m_i, \\ k + m_i, & \text{if} \quad k \leq -1. \end{cases}$$

Next we put $\psi = \chi\ae\chi^{-1}$, and hence we obtain that $\mathbb{I}\mathfrak{C}\psi$ but $\operatorname{dom}\psi \neq \mathbb{Z}$. This completes the proof of our lemma. ♦

Remark 1. *The proof of Lemma 2 implies that for element* $\psi = (\psi_1, \dots, \psi_n)$ *the following property holds*:

ψ_i *is not* $\mathscr{H}$*-equivalent to the unit of the semigroup* $\mathscr{IO}_\infty(\mathbb{Z})$ *for every* $i \in \mathbf{D}_{\alpha,\beta}$.

Proposition 1. *Let* $\mathfrak{C}$ *be a congruence on the semigroup* $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. *Let* α *and* β *be two distinct* $\mathfrak{C}$*-equivalent elements of the semigroup* $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. *Then there exists a non-unit idempotent* ε *in* $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ *such that* $\mathbb{I}\mathfrak{C}\varepsilon$ *and* $\mathbf{D}_{\mathbb{I},\varepsilon} = \mathbf{D}_{\alpha,\beta}$.

P r o o f. Lemma 2 implies that there exists an element ψ of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ such that $\psi\mathfrak{C}\mathbb{I}$, $\mathbf{D}_{\mathbb{I},\psi} = \mathbf{D}_{\alpha,\beta}$ and elements $\mathbb{I}$ and ψ are not $\mathscr{H}$-equivalent in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. Also, by Remark 1 for every integer $i \in \mathbf{D}_{\alpha,\beta}$ the element ψ_i is not $\mathscr{H}$-equivalent to the unit $\mathbb{I}_i$ of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$. This implies that for every integer $i \in \mathbf{D}_{\alpha,\beta}$ at least one of the following conditions holds:

$$\psi_i\psi_i^{-1} \neq \mathbb{I}_i \qquad \text{or} \qquad \psi_i^{-1}\psi_i \neq \mathbb{I}_i \quad \text{in} \quad \mathscr{IO}_\infty(\mathbb{Z}).$$

Since $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ is an inverse semigroup we have that $\mathbb{I}\mathfrak{C}\psi^{-1}$. This implies that $\mathbb{I}\mathfrak{C}\psi\psi^{-1}$ and $\mathbb{I}\mathfrak{C}\psi^{-1}\psi$, and hence we get that $\mathbb{I}\mathfrak{C}\varepsilon$, where $\varepsilon = \psi\psi^{-1}\psi^{-1}\psi$. The above arguments show that $\mathbf{D}_{\mathbb{I},\varepsilon} = \mathbf{D}_{\alpha,\beta}$. ♦

Proposition 2. *Let* $\mathfrak{C}$ *be a congruence on the semigroup* $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. *Let* α *and* β *be two distinct* $\mathfrak{C}$*-equivalent elements of the semigroup* $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$. *Then* $\mathbb{I}\mathfrak{C}\varepsilon$ *for any idempotent* ε *in* $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ *such that* $\mathbf{D}_{\mathbb{I},\varepsilon} = \mathbf{D}_{\alpha,\beta}$.

P r o o f. By Proposition 1 there exists an idempotent ε of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ such that $\mathbb{I}\mathfrak{C}\varepsilon$ and $\mathbf{D}_{\mathbb{I},\varepsilon} = \mathbf{D}_{\alpha,\beta}$. We fix an arbitrary non-unit idempotent $\tau \in \mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}})$ such that $\varepsilon \leq \tau$ in $E(\mathscr{IO}_\infty(\mathbb{Z}^n_{\text{lex}}))$. Then we have that

$\tau\mathbb{I} = \tau$ and hence the relation $\mathbb{I}\mathfrak{C}\varepsilon$ implies that $\tau = \tau\mathbb{I}\mathfrak{C}\tau\varepsilon = \varepsilon\mathfrak{C}\mathbb{I}$. Therefore, for every $i \in \mathbf{D}_{\alpha,\beta}$ there exists an idempotent ε_i° such that $\varepsilon_i^\circ\mathfrak{C}\mathbb{I}$ and the set $\mathbb{Z} \setminus \operatorname{dom}\varepsilon_i^\circ$ is singleton. We put $\{m_i\} = \mathbb{Z} \setminus \operatorname{dom}\varepsilon_i^\circ$ for every integer $i \in \mathbf{D}_{\alpha,\beta}$. We fix an arbitrary integer p_i for $i \in \mathbf{D}_{\alpha,\beta}$ and define the map $\rho_i : \mathbb{Z} \to \mathbb{Z}$ by the formula:

$$(j)\rho_i = j - m_i + p_i \qquad \text{for every} \quad j \in \mathbb{Z}.$$

Then ρ_i is an element of the group of units of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ and hence $\rho_i\rho_i^{-1} = \rho_i^{-1}\rho_i = \mathbb{I}_i$ in $\mathscr{IO}_\infty(\mathbb{Z})$. Moreover, it is obvious that $\rho_i^{-1}\varepsilon_i^\circ\rho_i$ is an idempotent of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ such that $\operatorname{dom}(\rho_i^{-1}\varepsilon_i^\circ\rho_i) = \mathbb{Z} \setminus \{p_i\}$. Also, we obtained that $\mathbb{I}_i = \rho_i^{-1}\mathbb{I}_i\rho_i\mathfrak{C}\rho_i^{-1}\varepsilon_i^\circ\rho_i$ in $\mathscr{IO}_\infty(\mathbb{Z})$. Now the definition of the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ implies that $\mathbb{I}\mathfrak{C}\tau_i^\circ$ for any idempotent τ in $\mathscr{IO}_\infty(\mathbb{Z})$, because every idempotent τ in the semigroup $\mathscr{IO}_\infty(\mathbb{Z})$ is equal to a product of finitely many idempotents of the form τ_i°, $i \in \{1,\dots,n\}$, with the property that the set $\mathbb{Z} \setminus \operatorname{dom}\tau_i^\circ$ is singleton. Then for every idempotent ε of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ with the property $\mathbf{D}_{\mathbb{I},\varepsilon} = \mathbf{D}_{\alpha,\beta}$ we have that

$$\varepsilon = \varepsilon_{i_1}^\circ \cdot \ldots \cdot \varepsilon_{i_k}^\circ, \quad \text{where} \quad \{i_1,\dots,i_k\} = \mathbf{D}_{\alpha,\beta},$$

and hence $\mathbb{I}\mathfrak{C}\varepsilon$. This completes the proof of the proposition. ♦

Theorem 1. *Let $\mathfrak{C}$ be a congruence on the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. Then the following statements hold:*

(*i*) *If $\Delta_{\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})} \subseteq \mathfrak{C} \subseteq \sigma_{[i_m]}$, for some $i_m \in \{1,\dots,n\}$, then either $\Delta_{\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})} = \mathfrak{C}$ or $\mathfrak{C} = \sigma_{[i_m]}$.*

(*ii*) *If $\sigma_{[i_1,\dots,i_m]} \subseteq \mathfrak{C} \subseteq \sigma_{[i_1,\dots,i_m,i_{m+1}]}$, for any subset $\{i_1,\dots,i_m,i_{m+1}\} \subseteq \subseteq \{1,\dots,n\}$, then either $\sigma_{[i_1,\dots,i_m]} = \mathfrak{C}$ or $\mathfrak{C} = \sigma_{[i_1,\dots,i_m,i_{m+1}]}$.*

P r o o f. By Proposition 2.15 from [3] we have that for any collection $\{i_1,\dots,i_k\} \subseteq \{1,\dots,n\}$ of distinct indices, $k \le n$, and, hence, $\alpha\sigma_{[i_1,\dots,i_k]}\beta$ in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ if and only if $\alpha\varepsilon_{i_1}^\circ\ldots\varepsilon_{i_k}^\circ = \beta\varepsilon_{i_1}^\circ\ldots\varepsilon_{i_k}^\circ$ for some idempotents $\varepsilon_{i_1}^\circ,\dots,\varepsilon_{i_k}^\circ \in \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$. This implies that $\mathbb{I}\sigma_{[i_1,\dots,i_k]}\varepsilon$ for every idempotent ε of the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ such that $\mathbf{D}_{\mathbb{I},\varepsilon} \subseteq \{i_1,\dots,i_k\}$. Then applying Proposition 1 we get the statement of the theorem. ♦

For any proper subset if indices $\mathrm{I} \subset \{1,\dots,n\}$ we define a map $\pi^{\mathrm{I}} : \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}}) \to \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ by the formula $(\alpha_1,\dots,\alpha_n)\pi_{\mathrm{I}} = (\beta_1,\dots,\beta_n)$, where

$$\beta_i = \begin{cases} \alpha_i, & \text{if} \quad i \in \mathrm{I}, \\ \mathbb{I}_i, & \text{if} \quad i \in \{1,\dots,n\} \setminus \mathrm{I}. \end{cases}$$

Simple verifications show that such defined map $\pi^{\mathrm{I}} : \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}}) \to \mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is a homomorphism. Let $\pi^{\mathrm{I}\#}$ be the congruence on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ which is generated by the homomorphism π^{I}.

Proposition 3. *Let* I *be an arbitrary proper subset of* $\{1,\ldots,n\}$. *Then* $\pi^{\mathrm{I}\#}_{\min} = \sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]}$, *where* $\{i_1,\ldots,i_k\} = \{1,\ldots,n\} \setminus \mathrm{I}$.

P r o o f. Suppose that $\alpha(\sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]})\beta$ in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ for some elements $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $\beta = (\beta_1,\ldots,\beta_n)$. Proposition 2.15 from [3] implies that $\alpha\varepsilon^\circ_{i_1}\ldots\varepsilon^\circ_{i_k} = \beta\varepsilon^\circ_{i_1}\ldots\varepsilon^\circ_{i_k}$ for some idempotent $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$ such that $\varepsilon_i = \mathbb{I}_i$ for all $i \in \mathrm{I}$, i.e., $\alpha\varepsilon = \beta\varepsilon$. Then we have that $\alpha_i = \beta_i$ for all $i \in \mathrm{I}$, and hence $\alpha\varepsilon^* = \beta\varepsilon^*$ for $\varepsilon^* = (\varepsilon^*_1,\ldots,\varepsilon^*_n)$, where

$$\varepsilon^*_i = \begin{cases} \alpha_i^{-1}\alpha_i = \beta_i^{-1}\beta_i, & \text{if} \quad i \in \mathrm{I}, \\ \varepsilon_i, & \text{if} \quad i \in \{1,\ldots,n\} \setminus \mathrm{I}. \end{cases}$$

It is obvious that $\varepsilon^*\pi^{\mathrm{I}\#}\alpha^{-1}\alpha\pi^{\mathrm{I}\#}\beta^{-1}\beta$. This implies the inclusion

$$\sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]} \subseteq \pi^{\mathrm{I}\#}_{\min}.$$

Suppose that $\alpha\pi^{\mathrm{I}\#}_{\min}\beta$ in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ for some elements $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $\beta = (\beta_1,\ldots,\beta_n)$. Then there exists an idempotent $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$ in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ such that $\alpha\varepsilon = \beta\varepsilon$ and $\varepsilon\pi^{I\#}\alpha^{-1}\alpha\pi^{I\#}\beta^{-1}\beta$. The last two equalities imply that $\alpha_i^{-1}\alpha_i = \beta_i^{-1}\beta_i = \varepsilon_i$ for all $i \in \mathrm{I}$. This and the equality $\alpha\varepsilon = \beta\varepsilon$ imply that $\alpha_i\varepsilon_i = \beta_i\varepsilon_i$ for all $i \in \mathrm{I}$ and hence we obtain that $\alpha_i = \alpha_i\alpha_i^{-1}\alpha_i = \alpha_i\varepsilon_i = \beta_i\varepsilon_i = \beta_i\beta_i^{-1}\beta_i = \beta_i$ for all $i \in \mathrm{I}$. Therefore we have that $\alpha\varepsilon^* = \beta\varepsilon^*$, where the idempotent $\varepsilon^* = (\varepsilon^*_1,\ldots,\varepsilon^*_n)$ defined in the following way

$$\varepsilon^*_i = \begin{cases} \alpha_i^{-1}\alpha_i, & \text{if} \quad i \in \mathrm{I}, \\ \varepsilon_i, & \text{if} \quad i \in \{1,\ldots,n\} \setminus \mathrm{I}. \end{cases}$$

This implies that $\alpha\varepsilon^\circ_{i_1}\ldots\varepsilon^\circ_{i_k} = \beta\varepsilon^\circ_{i_1}\ldots\varepsilon^\circ_{i_k}$. By Proposition 2.15 from [3] we get that $\alpha(\sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]})\beta$ in $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$, and hence we get that $\pi^{i\#}_{\min} \subseteq \sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]}$. This completes the proof of equality $\pi^{\mathrm{I}\#}_{\min} = \sigma_{[i_1]} \circ \ldots \circ \sigma_{[i_k]}$. ♦

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КОНГРУЕНЦІЇ НА МОНОЇДІ МОНОТОННИХ ІН'ЄКТИВНИХ ЧАСТКОВИХ ПЕРЕТВОРЕНЬ МНОЖИНИ $L_n \times_{\text{lex}} \mathbb{Z}$ З КО-СКІНЧЕННИМИ ОБЛАСТЯМИ ВИЗНАЧЕННЯ І ЗНАЧЕНЬ

Вивчаються конгруенції напівгрупи $\mathscr{I}\mathscr{O}_\infty(\mathbb{Z}^n_{\text{lex}})$ монотонних ін'єктивних часткових перетворень множини $L_n \times_{\text{lex}} \mathbb{Z}$ з ко-скінченними областями визначення і значень, де $L_n \times_{\text{lex}} \mathbb{Z}$ – лексикографічний добуток n-елементного ланцюга та множини цілих чисел зі звичайним лінійним порядком. Описується структура підгратки конгруенцій на $\mathscr{I}\mathscr{O}_\infty(\mathbb{Z}^n_{\text{lex}})$, які містяться в мінімальній груповій конгруенції.

КОНГРУЭНЦИИ НА МОНОИДЕ МОНОТОННЫХ ИНЪЕКТИВНЫХ ЧАСТИЧНЫХ ПРЕОБРАЗОВАНИЙ МНОЖЕСТВА $L_n \times_{\text{lex}} \mathbb{Z}$ С КО-КОНЕЧНЫМИ ОБЛАСТЯМИ ОПРЕДЕЛЕНИЯ И ЗНАЧЕНИЙ

Изучаются конгруэнции полугруппы $\mathscr{I}\mathscr{O}_\infty(\mathbb{Z}^n_{\text{lex}})$ монотонных инъективных частичных преобразований множества $L_n \times_{\text{lex}} \mathbb{Z}$ с ко-конечными областями определения и значения, где $L_n \times_{\text{lex}} \mathbb{Z}$ – лексикографическое произведение n-элементной цепи и множества целых чисел с обычным линейным порядком. Описана структура подрешётки конгруэнций на $\mathscr{I}\mathscr{O}_\infty(\mathbb{Z}^n_{\text{lex}})$, которые содержатся в минимальной групповой конгруэнции.

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