Математичний Вісник
Наукового Товариства
ім. Тараса Шевченка
2014. — Т.11

Mathematical Bulletin
of Taras Shevchenko
Scientific Society
2014. — V.11

ON DERIVATIONS WITH REGULAR VALUES IN RINGS

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Maria P. Lukashenko, *On derivations with regular values in rings*, Math. Bull. T. Shevchenko Sci. Soc. **11** (2014), 5–11.

If a commutative ring R has a nonzero derivation d such that $d(x) = 0$ or $d(x)$ is regular for every $x \in R$, then the classical ring of quotients Q is a field or $Q = T[X]/(X^2)$, where the characteristic $\operatorname{char} T = 2$, $d(T) = 0$ and $d(X) = 1 + aX$ for some $a \in Z(T)$. We also prove that if a right Goldie ring has a non-identity automorphism φ such that $x - \varphi(x)$ is zero or regular for any $x \in R$, then it is a semiprime ring with the classical right ring of quotients Q which is either

(1) a division ring T, or

(2) the ring direct sum $T \oplus T$, or

(3) the ring $M_2(T)$ of 2×2 matrices over a division ring T.

Лукашенко Марія. *Про диференціювання з регулярними значеннями в кільцях* // Мат. вісн. Наук. тов. ім. Т. Шевченка. — 2014. — Т.11. — С. 5–11.

Якщо комутативне кільце R має ненульове диференціювання d таке, що $d(x) = 0$ або $d(x)$ регулярний для будь-якого $x \in R$, тоді класичне кільце дробів Q є полем або $Q = T[X]/(X^2)$, де характеристика $\operatorname{char} T = 2$, $d(T) = 0$ і $d(X) = 1 + aX$ для деякого $a \in Z(T)$. Також доведено, що якщо праве кільце Голді має неодиничний автоморфізм φ такий, що $x - \varphi(x)$ є нульовим або регулярним для будь-якого $x \in R$, то R – напівпервинне кільце з класичним правим кільцем дробів Q, що є

(1) тілом T, або

(2) кільцевою прямою сумою $T \oplus T$, або

(3) кільцем матриць $M_2(T)$ розміру 2×2 над тілом T.

2010 *Mathematics Subject Classification:* 18B30, 54B30 *УДК:* 515.12+512.58

Key words and phrases: Semiprime ring, derivation, automorphism

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Introduction

Henceforth, R will be an associative ring with the identity element 1. J. Bergen, I. Herstein and C. Lanski [4] have proved that if R has a nonzero derivation d such that $d(x) = 0$ or $d(x)$ is invertible for any $x \in R$, then either R is a division ring or a ring of 2×2 matrices over a division ring T or $R = T[X]/(X^2)$ is a quotient ring of a polynomial ring $T[X]$ by the ideal (X^2) over a division ring T of characteristic 2, $d(T) = 0$ and $d(X) = aX + 1$ for some $a \in Z(T)$. Some time ago J. Bergen and L. Carini [5] have obtained similar results in the case of invertible values on a Lie ideal. Results of these studies are summarized in [14], [9], [8], [12] and [15]. J. Bergen [7] has examined semiprime rings R possessing a nonzero derivation d such that $d(x)$ is nilpotent or invertible for all $x \in R$. Recently I. Kaygorodov and Y. Popov [13] have investigated alternative algebras with a derivation that takes invertible values.

If φ is an automorphism of R, then $1-\varphi$ is its φ-derivation (in the sense of [3, § 1.1]). J. Bergen and I. Herstein [6] have characterized rings R in which $x = \varphi(x)$ or $x - \varphi(x)$ is invertible for every $x \in R$. In this paper we obtain some extensions of results from [4] and [6]. For this, recall that an element $x \in R$ is called *left regular* (respectively *right regular*) in R if, for every $r \in R$, the implication

$$rx = 0 \text{ (respectively } xr = 0) \Rightarrow r = 0$$

is true. If $x \in R$ is both left and right regular in R, then it is *regular*. We say that R *satisfies the condition* $(*)$ if there is a nonzero derivation $d : R \to R$ such that, for every element $x \in R$, $d(x) = 0$ or $d(x)$ is a regular element in R.

We prove the following

Proposition. *Let R be a commutative ring. Then R has a nonzero derivation d satisfying the condition $(*)$ if and only if the classical ring of quotients $Q(R)$ is a field or $Q(R) = T[X]/(X^2)$, where the characteristic $\operatorname{char} T = 2$, $d(T) = 0$ and $d(X) = 1 + aX$ for some $a \in Z(T)$.*

A ring R is called *a right Goldie ring* if it contains no infinite direct sum of right ideals and satisfies the a.c.c. on right annihilators. We say that an automorphism φ of a ring R *satisfies the condition* $(**)$ if, for the φ-derivation $1-\varphi$, the property $(*)$ is true. We obtain an extension of Theorem from [6].

Theorem. *Let R be a right Goldie ring. If R has a non-identity automorphism φ such that $x - \varphi(x)$ is zero or regular for any $x \in R$, then it is a semiprime ring with the classical right ring of quotients Q which is either*

(1) *a division ring T, or*
(2) *is the ring direct sum $T \oplus T$, or*
(3) *the ring $M_2(T)$ of 2×2 matrices over a division ring T.*

By [6], any automorphism $\Phi : Q \to Q$ extending an automorphism $\varphi : R \to R$ with the property $(**)$ has the following propeties:

(i) an automorphism Φ is non-inner if and only if T has a non-inner automorphism ψ such that $\psi^2(x) = u^{-1}xu$ for every $x \in T$, where $\psi(u) = u$ and $u \neq y\psi(u)$ for any $y \in T$,

(ii) an automorphism Φ is inner if and only if T does not contains all quadratic extensions of $Z(T)$.

Any unexplained terminology is standard and follows [11] and [16].

1. Derivation with regular values

Lemma 1.1. *Let R be a ring satisfying the condition $(*)$ and $x \in R$. If $d(x) = 0$, then $x = 0$ or x is a regular element in R.*

Proof. Suppose that $x \neq 0$. Since d is nonzero, we have $d(y) \neq 0$ for some element $y \in R$. By the condition $(*)$, $d(y)$ is a regular element. Then

$$d(xy) = xd(y) \neq 0 \text{ and } d(yx) = d(y)x \neq 0,$$

and hence $xd(y)$ and $d(y)x$ are regular. If $b \in R$ and $bx = 0$ (respectively $xb = 0$), then $b(xd(y)) = (bx)d(y) = 0$ (respectively, $(d(y)x)b = d(y)(xb) = 0$). By the above, $b = 0$ and therefore x is regular in the ring R. □

Lemma 1.2. *Let d be a nonzero derivation of R that satisfies the condition $(*)$. If L is a nonzero left ideal of R, then its image $d(L) \neq 0$ is nonzero.*

Proof. Suppose that $L \neq R$ is a proper left ideal of R. Assume, by contrary, that $d(L) = 0$. If $0 \neq a \in L$, then, by Lemma 1.1, we can conclude that a is regular in R. Since $ra \in L$ for every $r \in R$, we deduce that $0 = d(ra) = d(r)a$. The regularity of $a \in R$ gives that $d(r) = 0$, and so $d = 0$. This contradiction shows that $d(L) \neq 0$. □

The torsion part of a ring R is the set

$$F(R) = \{r \in R \mid r \text{ has a finite order in the additive group } R^+ \text{ of } R\}.$$

If p is a prime, then the *p-component* of R is the set

$$F_p(R) = \{r \in F(R) \mid r \text{ is of order } p^k, \text{where } k \text{ is a non-negative integer}\}.$$

Lemma 1.3. *If R is a ring satisfying the condition $(*)$, then the characteristic* $\operatorname{char} R = p$ *for some prime p or $F(R) = 0$ (and therefore the additive group R^+ is torsion-free).*

Proof. Assume that $F(R) \neq 0$. Then the additive group $F(R)^+$ has the nonzero p-component $F_p(R)$ for some prime p. Let $x \in F_p(R)$ be an element of order p^k. Suppose that $k \geq 2$. Then $p^k d(x) = d(p^k x) = 0$, and therefore $(pd(x))^k = 0$. If $pd(x) \neq 0$, then $pd(x) = d(px)$ is a zero divisor in R, a contradiction with the condition $(*)$. Therefore $d(px) = 0$ and, by Lemma 1.1, px is a regular element in R (and we obtain a contradiction) or $px = 0$. Hence $k = 1$.

Assume that the p-component $F_p(R)$ is proper in $F(R)$. Then there exists a prime q such that $q \neq p$ and $F_q(R)$ is nonzero. By Lemma 1.2, $d(F_q(R)) \neq 0$ and $d(F_p(R)) \neq 0$. As a consequence $d(F_q(R))d(F_p(R)) = 0$, a contradiction with $(*)$. Thus $F(R) = F_p(R)$.

If $F_p(R)$ is proper in R, then pR is nonzero and $F_p(R) \cdot pR = 0$, a contradiction in view of $(*)$ and Lemma 1.2. Hence $F_p(R) = R$. □

A ring without nonzero nilpotent elements is called *reduced*.

Corollary 1.4. *Let d be a nonzero derivation of a ring R satisfying the condition $(*)$ and $e = e^2 \in R$. If R is reduced (respectively commutative), then each idempotent e is trivial (that is $e \in \{0, 1\}$).*

Proof. It is clear that R contains two trivial idempotents 0, 1. Assume, by contrary, that in R there is an idempotent $e \notin \{0, 1\}$. Then $e(1 - e) = 0 = (1 - e)e$, and therefore e is a zero divisor. Since $d(e) = d(e^2) = d(e)e + ed(e)$ and $d(e)e = d(e)e + ed(e)e$, we have $ed(e)e = 0$ and $(d(e)e)^2 = 0$. But R is reduced (respectively commutative) and so $ed(e) = 0 = d(e)e$. By Lemma 1.1, $d(e) \neq 0$ and, by the condition $(*)$, an element $d(e)$ is regular. As a consequence, $e = 0$, a contradiction. □

By $\mathbb{P}(R)$ we denote the prime radical of a ring R that is the intersection of all prime ideals in R.

Lemma 1.5. *If a ring R satisfies the condition $(*)$, then:*

(*i*) $\mathbb{P}(R)^2 = 0$,

(*ii*) *if R^+ is torsion-free (respectively $\operatorname{char} R > 2$), then $\mathbb{P}(R) = 0$ (and consequently the ring R is semiprime).*

Proof. (*i*) If $\mathbb{P}(R)^2 \neq 0$, then $0 \neq d(\mathbb{P}(R)^2)$ by Lemma 1.2. But $d(\mathbb{P}(R)^2) \subseteq \mathbb{P}(R)$ and we obtain a contradiction.

(*ii*) By Proposition 1.3 of [10] (respectively Theorem 8.16 of [2]), we have that $d(\mathbb{P}(R)) \subseteq \mathbb{P}(R)$. Then, in view of $(*)$ and Lemma 1.1, we conclude that $\mathbb{P}(R) = 0$. □

Lemma 1.6. *A semiprime ring R with the condition $(*)$ is prime.*

Proof. Assume that A, B are nonzero ideals of R such that $AB = 0$. Then $BA = 0$ and there exist nonzero elements $a \in A$ and $b \in B$ such that $ab = 0 = ba$, $d(b) \neq 0$ by Lemma 1.2 and $B \ni d(a)b = -ad(b) \in A$, $B \ni d(b)a = -bd(a) \in A$. Since $A \cap B = 0$, we conclude that $ad(b) = 0 = d(b)a$ and this leads to a contradiction with $(*)$. Thus R is a prime ring. □

Corollary 1.7. *Let R be a commutative ring with the condition $(*)$. If the torsion part $F(R) = 0$ is zero (respectively R is of characteristic $n > 0$ and the greatest common divisor* $\mathrm{GCD}(n, 2) = 1$ *is trivial), then R is reduced (and consequently prime).*

Proof. Assume that $x^2 = 0$ for some element $x \in R$. Then $0 = d(x^2) = 2xd(x)$ and therefore $xd(x) = 0$. By the condition $(*)$, $d(x) = 0$ and, by Lemma 1.1, $x = 0$. Hence the ring R is reduced. □

In a commutative ring R, for a set of all its regular elements S, there exist the ring of quotients $Q(R) = RS^{-1}$ (see [1]).

Proof of Proposition. If the ring R is prime (and consequently a domain), then $Q(R)$ is a field. Therefore we assume that R is not a domain. By Lemma 1.5, $\mathbb{P}(R)^2 = 0$ and $\operatorname{char} R = 2$. Let d be a nonzero derivation of R satisfying the property $(*)$. Then we can extended d to a derivation D of $Q(R)$ (see [17]). Thus, by Theorem 1 of [4], $Q(R) = T[X]/(X^2)$, where the characteristic $\operatorname{char} T = 2$, $d(T) = 0$ and $d(X) = 1 + aX$ for some $a \in Z(T)$. □

2. Rings that have a φ-derivation with regular values

Lemma 2.1. *Let R be a ring with a non-identity automorphism φ satisfying the condition $(**)$. If $\varphi(x) = x$ for some $x \in R$, then $x = 0$ or x is regular in R.*

Proof. Since $\varphi(r) - r \neq 0$ for some $r \in R$, $x(\varphi(r) - r) = \varphi(xr) - xr \neq 0$ and $\varphi(r) - r)x = \varphi(rx) - rx \neq 0$. Hence x is regular. □

Corollary 2.2. *Let R be a ring with a non-identity automorphism φ satisfying the condition $(**)$. Then:*

(a) *$\mathbb{P}(R) = 0$ (and so R is semiprime),*

(b) *the additive group R^+ is torsion-free or $pR = 0$ for some prime p.*

Proof. (a) If $0 \neq x \in \mathbb{P}(R)$, then, by Lemma 2.1 and the condition $(**)$,

$$0 \neq \varphi(x) - x \in \mathbb{P}(R)$$

is a regular element of R, a contradiction.

(b) Suppose that there exists a nonzero element $0 \neq x \in F_p(R)$ of order p^k, where k is some positive integer. Then $x - \varphi(x) \in F_p(R)$ and $(p^k \cdot 1)(x - \varphi(x)) = 0$. Lemma 2.1 and the condition $(**)$ imply that $k = 1$ and $pR = 0$. □

If R is a semiprime right Goldie ring, then there exist its classical right ring of quotients $Q = Q(R)$ [11, Theorems 7.2.1–7.2.3]. Every regular element of R is invertible in Q.

Proof of Theorem. Assume that $\varphi \in \operatorname{Aut} R$ satisfies $(**)$ and $\Phi \in \operatorname{Aut} Q$ is its extension on the classical right ring of quotients Q of R. By Corollary 2.2, Q is semiprime. Preliminary we need to prove some properties.

(1°) *If I is a proper left ideal of Q, then $I \cap \Phi(I) = 0$.* If $I \cap \Phi(I) \neq 0$, then $I = Q$, and we obtain a contradiction.

(2°) *Every left ideal $I \neq 0$ of Q is minimal.* Indeed, for a nonzero proper left ideal $I < Q$, the sum $M = I + \Phi(I)$ is also a left ideal in Q and $0 \neq \Phi(I) \leq M$. Therefore $M = Q$ and $Q = I \oplus \Phi(I)$ is a direct sum of left ideals. If S is a nonzero left ideal of Q and $S \leq I$, then, by the same reasons, $Q = S \oplus \Phi(S)$ is a direct sum of left ideals. Therefore, for every $0 \neq l \in I$, we have $l = n + \Phi(m)$ with some elements $n, m \in S$. Hence $\Phi(m) = l - n \in I \cap \Phi(I)$, and this implies that $l = n \in S$, $m = 0$ and $I = S$ is a minimal left ideal of Q.

(3°) *If Q is not simple, then $Q = I_1 \oplus I_2$ is a direct sum of ideals I_1, I_2 such that $I_2 = \Phi(I_1)$ is a division ring.* If I is a nonzero proper ideal of Q, then, by (2°), $Q = I \oplus \Phi(I)$ is a direct sum of ideals. Moreover I is a minimal left ideal of Q. Therefore $I \cong \Phi(I)$ is a division ring.

(4°) *If Q is a simple ring, then Q is a division ring or $Q = M_2(T)$ is a ring of 2×2 matrices over a division ring T.* If we suppose that Q is not a division ring, then, in view of (2°), Q is simple Artinian. It easily hold that $Q = M_2(T)$ over a division ring T.

The rest follows from the Theorem of [6]. □

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Received 16.11.2014