THE CHARACTERISTIC METHOD AND THE RELATED FIXED POINT PROBLEM ANALYSIS OF A HAMILTON–JACOBI TYPE EQUATION

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The Cartan–Monge geometric approach to the characteristic method for nonlinear partial differential equations of first and higher orders is analyzed within the modern differential-geometric methods. The structure of characteristic vector fields, related with with classical and generalized solutions of nonlinear Hamilton–Jacobi type partial differential equations is studied, some interesting examples are presented.

1. THE GEOMETRIC CARTAN–MONGE APPROACH TO CLASSICAL CHARACTERISTICS METHOD

1.1. Introduction

The characteristic method [2,8,14,27] proposed in XIX century by A.Cauchy was very nontrivially developed by G.Monge, having introduced the geometric notion of characteristic surface, related with partial differential equations of first order. The latter, being augmented with a very important notion of characteristic vector fields, appeared to be fundamental [13, 15, 23, 27] for the characteristic method, whose main essence consists in bringing about the problem of studying solutions to our partial differential equation to an equivalent one of studying some set of ordinary differential equations. This way of reasoning succeeded later in development of the Hamilton–Jacobi

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theory, making it possible to describe a wide class of solutions to partial differential equations of first order of the form

$$H(x; u, u_x) = 0, \tag{1}$$

where $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n; \mathbb{R})$, $|H_x| \neq 0$, $|\cdot|$ is the standard norm in $\mathbb{R}^n$, is called a Hamiltonian function and $u \in C^2(\mathbb{R}^n; \mathbb{R})$ is unknown function under search. The equation (1) is endowed still with a boundary value condition

$$u|_{\Gamma_\varphi} = u_0, \tag{2}$$

with $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$, defined on some smooth almost everywhere hypersurface

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0, \quad ||\varphi_x|| \neq 0\}, \tag{3}$$

where $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$ is some smooth function on $\mathbb{R}^n$.

Following to the Monge's ideas, let us introduce the characteristic surface $\boldsymbol{S}_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ as

$$S_H := \{(x; u, p) \in \mathbb{R}^{n+1} \times \mathbb{R}^n : H(x; u, p) = 0\}, \tag{4}$$

where we put, by definition, $p := u_x \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. The characteristic surface (4) was effectively described by Monge within his geometric approach by means of the so called Monge cones $K \subset T(\mathbb{R}^{n+1})$ and their duals $K^* \subset T^*(\mathbb{R}^{n+1})$ [12, 20, 23, 27].

The corresponding differential-geometric analysis of this Monge scenario was later done by E.Cartan, who reformulated [5, 6, 12, 27] the geometric picture, drown by Monge, by means of the related compatibility conditions for dual Monge cones and the notion of integral submanifold $\Sigma_H \subset S_H$, naturally assigned to special vector fields on the characteristic surface S_H. In particular, E.Cartan had introduced on S_H the differential 1-form

$$\alpha^{(1)} := du - < p, dx >, \tag{5}$$

where $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$, and demanded its vanishing along the dual Monge cones $K^* \subset T^*(\mathbb{R}^{n+1})$, concerning the corresponding integral submanifold imbedding mapping

$$\pi : \Sigma_H :\longrightarrow S_H. \tag{6}$$

This means that the 1-form

$$\pi^* \alpha_1^{(1)} := du - < p, dx > |_{\Sigma_H} \Rightarrow 0 \tag{7}$$

for all points $(x;u,p) \in \Sigma_H$ of a solution surface Σ_H, defined in such a way that $K^* = T^*(\Sigma_H)$. The obvious corollary from condition (7) is the second Cartan condition

$$d\pi^*\alpha_1^{(1)} = \pi^* d\alpha_1^{(1)} =< dp, \wedge dx > |_{\Sigma_H} \Rightarrow 0. \tag{8}$$

These two Cartan's conditions (7) and (8) should be still augmented with the characteristic surface S_H invariance condition for the differential 1-form $\alpha_2^{(1)} \in \Lambda^1(S_H)$ as

$$\alpha_2^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{9}$$

The conditions (7), (8) and (9), when imposed on the characteristic surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$, make it possible to construct the proper characteristic vector fields on S_H, whose suitable characteristic strips [20, 23, 27] generate the searched solution surface Σ_H.

The reasonings above can be naturally embedded into the classical Cartan theory of integrable ideals in the Grassmann algebra on differentiable manifolds. Within this theory the solution surface $\Sigma_H \subset S_H$ is exactly the maximal integral submanifold of the integrable ideal $I(S_H) \subset \Lambda(S_H)$, generated by the corresponding by one-forms (5), (9) and two-forms $d\alpha_1^{(1)} \in \Lambda(S_H)$. By construction, this ideal is closed, that is $dI(S_H) \subset I(S_H)$, being a criterion [1–3, 6, 12] of its Cartan-Frobenius integrability.

Thereby, having solved the corresponding Cauchy problem related with the boundary value conditions (2) and (3) for these characteristic vector fields, considered as ordinary differential equations on S_H, one can construct a solution to our partial differential equation (1). And what is interesting, this solution in many cases can be represented [8, 21] in exact functional-analytic Hopf–Lax type form. The latter is a natural consequence from the related Hamilton–Jacobi theory, whose main ingredient consists in proving the fact that the solution to our equation (1) is exactly the extremal value of some Lagrangian functional, naturally associated [1, 2, 20, 23] with a given Hamiltonian function.

Below we will construct the proper characteristic vector fields for partial differential equations of first order (1) on the characteristic surface S_H, generating the solution surface Σ_H as suitable characteristic strips related with the boundary conditions (2) and (3), and next generalize the Cartan–Monge geometric approach for partial differential equations of second and higher orders.

1.2. The characteristic vector fields method: differential-geometric aspects

Consider on the surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ a characteristic vector field $K_H : S_H \to T(S_H)$ in the form

$$\left.\begin{array}{l} dx/d\tau = a_H(x;u,p) \\ dp/d\tau = b_H(x;u,p) \\ du/d\tau = c_H(x;u,p) \end{array}\right\} := K_H(x;u,p), \tag{10}$$

where $\tau \in \mathbb{R}$ is a suitable evolution parameter and $(x;u,p) \in S_H$. Since, owing to the Cartan-Monge geometric approach, there hold conditions (7), (8) and (9) along the solution surface Σ_H, we can satisfy them, applying the interior differentiation $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ [1,12,19] to the corresponding differential forms $\alpha_1^{(1)}$ and $d\alpha_1^{(1)}$:

$$i_{K_H}\alpha_1^{(1)} = 0, \quad i_{K_H}d\alpha_1^{(1)} = 0. \tag{11}$$

As a result of simple calculations one finds that

$$\begin{array}{c} c_H =< p, a_H >, \\ \beta^{(1)} :=< b_H, dx > - < a_H, dp > |_{S_H} = 0 \end{array} \tag{12}$$

for all points $(x;u,p) \in S_H$. The obtained 1-form $\beta^{(1)} \in \Lambda^1(S_H)$ must be, evidently, compatible with the defining invariance condition (9) on S_H. This means that there exists a scalar function $\mu \in C^1(S_H;\mathbb{R})$, such that the condition

$$\mu\alpha_2^{(1)} = \beta^{(1)} \tag{13}$$

holds on S_H. This gives rise to such final relationships:

$$a_H = \mu\partial H/\partial p, \quad b_H = -\mu(\partial H/\partial x + p\partial H/\partial u), \tag{14}$$

which together with the first equality of (12) complete the search for the structure of the characteristic vector fields $K_H : S_H \to T(S_H)$:

$$K_H = (\mu\partial H/\partial p; < p, \mu\partial H/\partial p >, -\mu(\partial H/\partial x + p\partial H/\partial u))^{\intercal}. \tag{15}$$

Now we can pose a suitable Cauchy problem for the equivalent set of ordinary differential equations (10) on S_H as follows:

$$\begin{array}{c} dx/d\tau = \mu\partial H/\partial p : x|_{\tau=0} \stackrel{?}{=} x_0(x) \in \Gamma_\varphi, \quad x|_{\tau=t(x)} = x \in \mathbb{R}^n \backslash \Gamma_\varphi; \\ du/d\tau =< p, \mu\partial H/\partial p > \quad : u|_{\tau=0} = u_0(x_0(x)), \quad u|_{\tau=t(x)} \stackrel{?}{=} u(x), \\ dp/d\tau = -\mu(\partial H/\partial x + p\partial H/\partial u) : p|_{\tau=0} = \partial u_0(x_0(x))/\partial x_0, \end{array} \tag{16}$$

point of the corresponding vector field orbit, starting at a fixed point $x \in \mathbb{R}^n \backslash \Gamma_\varphi$, with the boundary hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ at the moment of „time“ $\tau = t(x) \in \mathbb{R}$. As a result of solving the corresponding „inverse“ Cauchy problem (16) one finds the following exact functional-analytic expression for a solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ to the boundary value problem (2) and (3):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \bar{\mathcal{L}}(x; u, p) d\tau, \tag{17}$$

where, by definition,

$$\bar{\mathcal{L}}(x; u, p := < p, \mu \partial H / \partial p > \tag{18}$$

for all $(x; u, p) \in S_H$. If the Hamiltonian function $H : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}$ is nondegenerate, that is $Hess\ H := \det(\partial^2 H / \partial p \partial p) \neq 0$ for all $(x; u, p) \in S_H$, then the first equation of (16) can be solved with respect to the variables

$$p = \psi(x, \dot{x}; u) \tag{19}$$

for all $(x, \dot{x}) \in T(\ \mathbb{R}^n)$, where $\psi : T(\ \mathbb{R}^n) \times \mathbb{R} \quad \to \mathbb{R}^n$ is some smooth mapping. By means of the following classical Lagrangian function expression

$$\mathcal{L}(x, \dot{x}; u) := \bar{\mathcal{L}}(x; u, p)|_{p = \psi(x, \dot{x}; u)} \tag{20}$$

solution (17) takes the form

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \mathcal{L}(x, \dot{x}; u) d\tau, \tag{21}$$

which can be rewritten $[6, 7, 21]$ equivalently as

$$u(x) = \inf_{x_0 \in \mathbb{R}^n} \{u_0(x_0) + \int_0^{t(x)} \mathcal{L}(\tau; x(\tau; x_0), \dot{x}(\tau;_0); u(\tau; x_0)) d\tau\}, \tag{22}$$

The functional-analytic form (22) has the standard inf-type Hopf–Lax representation, being important for finding so called generalized solutions [7,8,13] to the Hamilton–Jacobi equation (1).

1.3. The characteristic vector fields method: application to second order partial differential equations

Assume we are given a second order partial differential equation

$$H(x; u, u_x, u_{xx}) = 0, \tag{23}$$

where solution $u \in C^2(\mathbb{R}^n;\mathbb{R})$ and the generalized „Hamiltonian“ function $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n);\mathbb{R})$. Putting $p^{(1)} := u_x$, $p^{(2)} := u_{xx}$, $x \in \mathbb{R}^n$, one can construct within the Cartan–Monge generalized geometric approach the characteristic surface

$$S_H := \{(x;u,p^{(1)},p^{(2)}) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n) : H(x;u,p^{(1)},p^{(2)}) = 0\} \tag{24}$$

and a suitable Cartan's set of differential one- and two-forms:

$$\begin{gathered}
\alpha_1^{(1)} := du- < p^{(1)}, dx > |_{\Sigma_H} \Rightarrow 0, \\
d\alpha_1^{(1)} :=< dx, \wedge dp^{(1)} > |_{\Sigma_H} \Rightarrow 0, \\
\alpha_2^{(1)} := dp^{(1)}- < p^{(2)}, dx > |_{\Sigma_H} \Rightarrow 0, \\
d\alpha_2^{(1)} :=< dx, \wedge dp^{(2)} > |_{\Sigma_H} \Rightarrow 0,
\end{gathered} \tag{25}$$

vanishing upon the corresponding solution submanifold $\Sigma_H \subset S_H$. The set of differential forms (25) should be augmented with the characteristic surface S_H invariance differential 1-form

$$\alpha_3^{(1)} := dH|_{S_H} \Rightarrow 0, \tag{26}$$

vanishing, respectively, upon the characteristic surface S_H. The solution space $\Sigma_H \subset S_H$ is [5, 6, 12, 16, 19, 20] the maximal integral submanifold of the suitably constructed integrable ideal $I(S_H) \subset \Lambda(S_H)$, generated by the one-forms $\alpha_1^{(1)}, \alpha_3^{(1)} \in \Lambda(S_H)$ and two-forms $d\alpha_1^{(1)} \in \Lambda(S_H)$. This ideal is, by construction, closed, and, thereby, integrable owing to the Cartan criterion [5,6,12,23].

Let the characteristic vector field $K_H : S_H \to T(S_H)$ on S_H is given by expressions

$$\left.\begin{aligned}
dx/d\tau &= a_H(x;u,p^{(1)},p^{(2)}) \\
du/d\tau &= c_H(x;u,p^{(1)},p^{(2)}) \\
dp^{(1)}/d\tau &= b_H^{(1)}(x;u,p^{(1)},p^{(2)}) \\
dp^{(2)}/d\tau &= b_H^{(2)}(x;u,p^{(1)},p^{(2)})
\end{aligned}\right\} := K_H(x;u,p^{(1)},p^{(2)}), \tag{27}$$

for all $(x;u,p^{(1)},p^{(2)}) \in S_H$. To find the vector field (27) it is necessary to satisfy the Cartan compatibility conditions in the following geometric form:

$$\begin{aligned}
&i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0 \pmod{I(S_H)} \\
&i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \pmod{I(S_H)}
\end{aligned} \tag{28}$$

where, as above, $i_{K_H}:\Lambda(S_H)\to\Lambda(S_H)$ is the internal derivative of differential forms along the vector field $K_H: S_H\to T(S_H)$. As a result of conditions (28) one finds that relationships

$$\begin{array}{c} c_H=<p^{(1)},a_H>, \quad b_H^{(1)}=<p^{(2)},a_H>,\\ \beta_1^{(1)}:=<a_H,dp^{(1)}>-<b_H^{(1)},dx>|_{S_H}\Rightarrow 0,\\ \beta_2^{(1)}:=<a_H,dp^{(2)}>-<b_H^{(2)},dx>|_{S_H}\Rightarrow 0,\end{array} \tag{29}$$

are satisfied upon S_H identically. Having solved implications (29), we can find a suitable vector field $K_H: S_H\to T(S_H)$ and, thereby, construct functional-analytic solutions to our partial differential equation of second order (23) via solving the equivalent Cauchy problem for the set of ordinary differential equations (27) on the characteristic surface S_H.

1.4. The characteristic vector fields method: application to partial differential equations of higher orders

Consider a general nonlinear partial differential equation of higher order $m\in Z_+$ as

$$H(x;u,u_x,u_{xx},\dots,u_{mx})=0, \tag{30}$$

where there is assumed that $H\in C^2(\mathbb{R}^{n+1}\times(\mathbb{R}^n)^{\otimes m(m+1)/2};\mathbb{R})$. Within the generalized Cartan-Monge geometric characteristic method we need to construct the related characteristic surface S_H as

$$\begin{array}{c} S_H:=\{(x;u,p^{(1)},p^{(2)},\dots,p^{(m)})\in\mathbb{R}^{n+1}\times(\mathbb{R}^n)^{\otimes m(m+1)/2}:\\ H(x;u,p^{(1)},p^{(2)},\dots,p^{(m)})=0\},\end{array} \tag{31}$$

where we put $p^{(1)}:=u_x\in\mathbb{R}^n$, $p^{(2)}:=u_{xx}\in\mathbb{R}^n\otimes\mathbb{R}^n$, $\dots$, $p^{(m)}\in(\mathbb{R}^n)^{\otimes m}$ for $x\in\mathbb{R}^n$. The corresponding solution manifold $\Sigma_H\subset S_H$ is naturally defined as the maximal integral submanifold of the suitable integrable ideal $I(S_H)\subset\Lambda(S_H)$, generated by the following set of one- and two-forms on S_H :

$$\begin{array}{c} \alpha_1^{(1)}:=du-<p^{(1)},dx>|_{\Sigma_H}\Rightarrow 0,\\ d\alpha_1^{(1)}:=<dx,\wedge dp^{(1)}>|_{\Sigma_H}\Rightarrow 0,\\ \alpha_2^{(1)}:=dp^{(1)}-<p^{(2)},dx>|_{\Sigma_H}\Rightarrow 0,\\ d\alpha_2^{(1)}:=<dx,\wedge dp^{(2)}>|_{\Sigma_H}\Rightarrow 0,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\ \alpha_m^{(1)}:=dp^{(m-1)}-<p^{(m)},dx>|_{\Sigma_H}\Rightarrow 0,\\ d\alpha_m^{(1)}:=<dx,\wedge dp^{(m)}>|_{\Sigma_H}\Rightarrow 0,\end{array} \tag{32}$$

vanishing, by definition, upon Σ_H. The set of differential forms (32) is augmented with the determining characteristic surface S_H invariance condition

$$\alpha_{m+1}^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{33}$$

Proceed now to constructing the characteristic vector field $K_H : S_H \to T(S_H)$ on the hypersurface S_H within the developed above generalized characteristic method. Take the expressions

$$\left.\begin{array}{c} dx/d\tau = a_H(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}) \\ du/d\tau = c_H(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)} \\ dp^{(1)}/d\tau = b_H^{(1)}(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}) \\ dp^{(2)}/d\tau = b_H^{(2)}(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dp^{(m)}/d\tau = b_H^{(m)}(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}), \end{array}\right\} := K_H(x;u,p^{(1)},p^{(2)}), \tag{34}$$

for $(x;u,p^{(1)},p^{(2)},\ldots,p^{(m)}) \in S_H$ and satisfy the corresponding Cartan compatibility conditions in the following geometric form:

$$\begin{array}{c} i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0 \pmod{I(S_H)}, \\ i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0 \pmod{I(S_H)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ i_{K_H}\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0 \pmod{I(S_H)}. \end{array} \tag{35}$$

As a result of suitable calculations in (35) one gets the following expressions:

$$\begin{array}{c} c_H =< p^{(1)}, a_H >, \quad b_H^{(1)} =< p^{(2)}, a_H >, \\ \beta_1^{(1)} :=< a_H, dp^{(1)} > - < b_H^{(1)}, dx > |_{S_H} \Rightarrow 0, \\ \beta_2^{(1)} :=< a_H, dp^{(2)} > - < b_H^{(2)}, dx > |_{S_H} \Rightarrow 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \beta_m^{(1)} :=< a_H, dp^{(m)} > - < b_H^{(m)}, dx > |_{S_H} \Rightarrow 0, \end{array} \tag{36}$$

being identically satisfied upon S_H.

It is now easy to see that all of tensor-valued 1-forms $\beta_j^{(1)} \in \Lambda^1(S_H) \otimes (\mathbb{R}^n)^{\otimes j}$, $j = \overline{1, m-1}$ are vanishing identically upon S_H owing to the relationships (32). As a result, we obtain the only relationship

$$\beta_m^{(1)} :=< a_H, dp^{(m)} > - < b_H^{(m)}, dx > |_{S_H} \Rightarrow 0, \tag{37}$$

which should be compatibly combined with that of (33). Having found from (33), (35) and (37) the suitable vector field $K_H : S_H \to T(S_H)$, we reduce the problem of solving our partial differential equation (30) to solving the set of characteristic equations (34). In many important cases this can be done in a feasible functional-analytic form useful for analyzing its properties.

The resulting set (34) of ordinary differential equations on S_H makes it possible to construct exact solutions to our partial differential equation (30) in a suitable functional-analytic form, being often very useful for analyzing its properties important for applications. On these and related questions we plan to stop in detail elsewhere later.

2. The generalized solutions to a canonical Hamilton–Jacobi equation and their Hopf–Lax type representation

The review article [7] devoted to viscosity solutions of first and second order nonlinear partial differential equations, contains the following Lax formula:

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left\{v(y) + \frac{1}{2t}|x-y|^2\right\} \tag{38}$$

for the solution to the following canonical Hamilton–Jacobi partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 = 0, \quad u|_{t=0} = v, \tag{39}$$

where Cauchy data $v : \mathbb{R}^n \to \mathbb{R}$ are properly convex and semicontinuous from below functions $|\cdot| :=< \cdot,\cdot >$ the standard norm in $\mathbb{R}^n$, $n \in \mathbb{Z}_+$, and $t \in \mathbb{R}_+$ is a positive evolution parameter. They noted also that there is no exact proof of the Lax formula (38) based on general properties of the Hamilton–Jacobi equation (39). Below we give such an exact proof of the Lax formula (38) and present further some other results about solutions to the canonical Hamilton–Jacobi equation both constrained to live on sphere $\mathbb{S}^n$ and perturbed by nonlinear oscillatory terms.

2.1. Hamiltonian dynamics analysis

Consider the following canonical Hamiltonian system, which are naturally associated [2, 3, 8, 14, 27] with (39)

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial x} \tag{40}$$

with „inverse“ Cauchy data

$$x|_{\tau=0} \stackrel{?}{=} x_0, \quad x|_{\tau=t} = x, \quad p|_{\tau=0} = p_0(x_0),$$

for an arbitrary but fixed point $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, where the Hamiltonian function $H_0 \in C^2(T(\mathbb{R}^n);\mathbb{R})$ reads as

$$H_0(x,p) = 1/2|p|^2, \tag{41}$$

and $(x,p) \in T^*(\mathbb{R}^n)$ is any canonical phase space point. The solution to (40) with Cauchy data at $(x_0,p_0) \in T^*(\mathbb{R}^n)$ is given for all $\tau \in \mathbb{R}_+$ as follows:

$$x = x_0 + p_0\tau, \quad p = p_0. \tag{42}$$

Introduce now the „action function“ $u \in C^1(\mathbb{R}^n;\mathbb{R})$ which could be defined locally as

$$du = -H_0(x,p)d\tau + <p, dx>, \tag{43}$$

where in virtue of (42), $p = (x-x_0)/t$ and $u|_{t=0^+} = v(x), \;\; x \in \mathbb{R}^n$. From (1.4) one obtains immediately that

$$\partial u/\partial\tau = -H_0(x,p), \quad \partial u/\partial x = p \tag{44}$$

for all points $(x,p) \in T^*(\mathbb{R}^n)$ and $\tau \in \mathbb{R}_+$. Substituting (44) into (41), one obtains the following lemma.

Lemma 1. *The action function $u \in C^1(\mathbb{R}^{n+1};\mathbb{R})$ satisfies exactly the Hamilton–Jacobi equation* (41).

Now we proceed to computing an expression for the function $u \in C^1(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R})$, defined by (43):

$$\begin{aligned} u(x,t) &= \int_0^t d\tau (du/d\tau)|_{\substack{x=x_0+p_0\tau \\ p=p_0}} + v(x_0)|_{x=x_0+p_0t} = \\ &= \int_0^t d\tau(<p, dx/d\tau> - H_0(x,p))|_{\substack{x=x_0+p_0\tau \\ p=p_0}} + v(x_0)|_{x=x_0+p_0t} = \\ &= \left(\frac{1}{2}|p_0|^2 + v(x_0)\right)|_{x=x_0+p_0t}. \end{aligned} \tag{45}$$

Since owing to (44) $\partial u/\partial x|_{t=0} = p_0 \in \mathbb{R}^n$, from (42) and (45) one arrives at such a formula:

$$u(x,t) = v(x-p_0t) + \frac{t}{2}|p_0|^2, \tag{46}$$

giving an entangled solution to the Hamilton–Jacobi equation (39). The expression (46) one can transform easily into such a useful for further form:

$$u(x,t) = v(\xi) + \frac{1}{2t}|x-\xi|^2, \tag{47}$$

where, by definition

$$\xi(x,t) := x - p_0 t, \quad p_0 := p_0(x) = \partial u/\partial x|_{t=0^+}. \tag{48}$$

For expression (47) to be interpreted more exactly it is useful to remind that Hamiltonian equations (40) are completely equivalent to the following shortened Lagrange minimal action principle:

$$\delta\tilde{u}(x_0; x, t)|_{x_0\in\mathbb{R}^n} = 0, \quad \tilde{u}(x_0; x, t) := \int_0^t d\tau \mathcal{L}_0(x, \dot{x}) + v(x_0), \tag{49}$$

where, by definition, the Lagrange function

$$\mathcal{L}_0(x, \dot{x}) :=< p, \dot{x} > -H_0(x, p)|_{\dot{x}=\partial H(x,p)/\partial p}. \tag{50}$$

Based on (43) and (50), one infers easily that the extremum Hopf–Lax type representation

$$\tilde{u}(x,t) := \inf_{x_0\in\mathbb{R}^n}\left\{v(x_0) + \frac{1}{2}|x - x_0|^2\right\} \Rightarrow v(\tilde{\xi}) + \frac{1}{2}|x - \tilde{\xi}|^2 \tag{51}$$

holds if there assumed that the infimum in the parenthesis $\{\ldots\}$ exists and is attained at some unique point $x_0 = \tilde{\xi}(x,t) \in \mathbb{R}^n$ for fixed $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$. For the above motivation to be validated we shall study in detail the properties of the solution $\tilde{\xi} = \tilde{\xi}(x,t)$ to the extremal problem (51) targeting to prove that $\tilde{\xi} = \tilde{\xi} \Rightarrow x - p_0 t$, $p_0(x) = \partial u/\partial x|_{t=0^+}$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, as it was found before in (48).

2.2. Extremality analysis

Let us consider the problem (51) in case when a function $v : \mathbb{R}^n \to \mathbb{R}$ is convex and semi-continuous from below. Then the following Lemma is true.

Lemma 2. *There exists the unique solution $x_0 = \tilde{\xi}(x,t) \in \mathbb{R}^n$ to the extremum problem* (51) *characterized by the following inequality*

$$\frac{1}{t} < \tilde{\xi} - x, \tilde{\xi} - y >\leq v(y) - v(\tilde{\xi}) \tag{52}$$

for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$.

Now we shall be interested in properties of the solution $\tilde{\xi}(x,t) \in \mathbb{R}^n$ by means of which we shall prove the main equality $\tilde{\xi} = \xi$ assuring that the extremum function $\tilde{u} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ coincides with the solution u :

$\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ to the problem (39). The following Lemma being simply stemmed from the inequality (38), is almost obvious.

Lemma 3. *The mappings* $\tilde{P}_t : \mathbb{R}^n \to \mathbb{R}^n$ *and* $(1 - \tilde{P}_t) : \mathbb{R}^n \to \mathbb{R}^n$, *where by definition,* $\tilde{P}_t x := \tilde{\xi}(x,t)$ *for any* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *are Lipschitzian, that is for any* $x, y \in \mathbb{R}^n$

$$|\tilde{P}_t - \tilde{P}_t y| \leq |x-y|, \quad |(1-\tilde{P}_t)x - (1-\tilde{P}_t)y| \leq |x-y|. \tag{53}$$

Consider now the minimizing function $\tilde{u} \in C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ of the problem (51) which is realized at the unique element $\tilde{\xi} = \tilde{P}_t x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. One can formulate then the following useful Lemma.

Lema 4. *The mapping* $\tilde{u}_t : \mathbb{R}^n \to \mathbb{R}$ *defined as* $\tilde{u}_t(x) = \tilde{u}(x,t)$ *for any* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{R}_+$ *being fixed, is convex and differentiable with respect to* $x \in \mathbb{R}^n$, *that is*

$$\nabla \tilde{u}_t(x) = \frac{1}{t}(x - \tilde{P}_t(x). \tag{54}$$

Moreover, as $t \to 0^+$, for any $x \in \mathbb{R}^n$ there exist limits

$$\lim_{t \to 0^+} \tilde{u}_t(x) = v(x), \quad \lim_{t \to 0^+} \tilde{P}_t(x) = x. \tag{55}$$

As a simple consequence of the expressions (54) and (51) one obtains the following identity:

$$< \nabla v(\tilde{\xi}) - \frac{1}{t}(x - \tilde{\xi}), \delta\tilde{\xi} >= 0, \tag{56}$$

valid for all $t \in \mathbb{R}_+$ and arbitrary bounded variation $\delta\tilde{\xi} \in \mathbb{R}^n$. The latter equality, in virtue of the relationship

$$\nabla v(\tilde{\xi}) = \frac{1}{t}(x - \tilde{\xi}(x,t)) = \nabla \tilde{u}(x,t), \tag{57}$$

holds for all $t \in \mathbb{R}_+$. This leads to a natural identification with relationships (48), making use of the expression

$$\frac{1}{t}(x - \tilde{\xi}(x,t)) = \tilde{p}_0(x) = \partial u / \partial x|_{t=0^+}, \tag{58}$$

valid for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, what proves exactly that at $x_0 = \tilde{\xi}(x,t) \in \mathbb{R}^n$ and $p_0(x) = \tilde{p}_0(x)$ one has $\xi(x,t) = \tilde{\xi}(x,t)$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. Thereby, the following theorem is stated.

Theorem 1. *The solution to the extremum problem* (51) *being achieved at a point* $\tilde{\xi} = \tilde{\xi}(x,t) \in \mathbb{R}^n, t \in \mathbb{R}_+$, *gives rise to the convex semi-continuous* *from below function* $\tilde{u} \in C^1(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, *which coincides with the action function* $u \in C^1(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ (47) *defined implicitly by means of the expression* (48).

As a result of the above Lemmas and Theorem 1, one can formulate the following theorem, characterizing suitable generalized solutions to the Hamilton–Jacobi equation (39).

Theorem 2. *The extremum Lax expression* (38) *does solve the Hamilton–Jacobi equation* (39) *with the Cauchy data, being chosen in the class of convex semi-continuous from below functions from* $\mathbb{R}^n$ *to* $\mathbb{R}$.

3. THE CANONICAL HAMILTON–JACOBI EQUATION ON SPHERE $\mathbb{S}^n$ AND ITS GENERALIZED SOLUTIONS

The approach devised above is generalized here for treating a Hamilton–Jacobi equation constrained to live on the sphere $\mathbb{S}^n$. It is based in part on the theory of completely integrable K.Neumann type dynamical systems studied before in detail in [12, 15, 21]. The following theorem holds.

Theorem 3. *The canonical Hamilton–Jacobi equation* (39) *constrained to live on the sphere* $\mathbb{S}^n$ *possesses generalized solutions in the Hopf–Lax type extremality form*

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \{v(y) + \frac{1}{2t} \arccos^2 < y, x >\} \qquad (59)$$

for semi-continuous from below Cauchy data $v \in BSC(\mathbb{S}^n)$.

3.1. Constrained Hamiltonian analysis

As is well known that the Hamilton–Jacobi equation (39) constrained to live on the sphere $\mathbb{S}^n$ is closely tied [12, 15, 19] within the characteristics method with the finite dimensional Hamiltonian system

$$dx/d\tau = \partial H(x,p)/\partial p, \qquad dp/d\tau = -\partial H(x,p)/\partial x, \qquad (60)$$

where $H(x,p) = 1/2|p|^2|x|^2$, $(x,p) \in T^*(\mathbb{R}^{n+1})$, $\tau \in \mathbb{R}_+$, with such constraints: $(x,p) \in T^*(\mathbb{S}^n) := \{x \in \mathbb{R}^{n+1} : |x|^2 - 1 = 0, < x, p >= 0\}$. From equations (60) one finds that for $\tau \in (0,t]$ on $T^*(\mathbb{R}^{n+1})$

$$dx/d\tau = p|x|^2, \qquad dp/d\tau = -|p|^2 x. \qquad (61)$$

Having taken Cauchy data $x|_{\tau=0^+} = y \in \mathbb{S}^n$, $x|_{\tau=t} = x \in \mathbb{S}^n$, from (61) one easily obtains that for all $t \in \mathbb{R}_+$ the expression

$$|p| = t^{-1} \arccos \langle y, x \rangle , \tag{62}$$

being independent of $\tau \in (0,t]$. Consider now the following infimum expression for the Hamilton–Jacobi equation (39) extended naturally on the whole space $\mathbb{R}^{n+1}$:

$$\begin{aligned} \widetilde{u}(x,t) = & \inf_{\substack{\{x(\tau)\in\mathbb{S}^n : x|_{\tau=0^+}=y\in\mathbb{S}^n,\\ x|_{\tau=t}=x\in\mathbb{S}^n\}\\ \{y\in\mathbb{S}^n\}}} \{v(y) + \\ & + \int_0^t d\tau (< p, dx/dt > -H(x,p))\} = \inf_{y\in\mathbb{S}^n} \{v(y) + \frac{t}{2}|p(t;x,t|y)|^2\}, \end{aligned} \tag{63}$$

where we denoted $x = x(\tau;x,t|y)$, $p = p(\tau;x,t|y)$ for $\tau \in (0,t]$, $x,y \in \mathbb{S}^n$, and made use of the equality $d|p|/d\tau = 0$, following from equations (61). On the other hand for the quantity $|p| \in \mathbb{R}^{n+1}$ one has the expression (62), giving rise together with (63) to the following inf-type Hopf–Lax type representation:

$$\widetilde{u}(x,t) = \inf_{y\in\mathbb{S}^n} \{v(y) + \frac{1}{2t} \arccos^2 < y, x >\}, \tag{64}$$

which we suggest as a candidate solution to the Hamilton–Jacobi equation (39) constrained on the sphere $\mathbb{S}^n$ with Cauchy data $v : \mathbb{S}^n \to \mathbb{R}$. The main wanted equality $\widetilde{u}(x,t) = u(x,t)$ for almost all $x \in \mathbb{S}^n$ and $t \in \mathbb{R}_+$ will follow from an analysis similar to that of [20,21,23,24], giving rise to equality (59). Here we shall stop only on the case, when the Cauchy data $v \in BSC(\mathbb{S}^n)$ are semi-continuous from below functions.

3.2. Extremality problem analysis

Here we shall prove the equality $\widetilde{u}(x,t) = u(x,t)$ all $x \in \mathbb{S}^n$ and $t \in \mathbb{R}_+$, using the functional properties of Cauchy data for (61) and the exact inf-type expression (64). It is easy to state [4,21] that there exists a point $\widetilde{\xi}(x,t) \in \mathbb{S}^n$ such that

$$\widetilde{u}(x,t) = v(\widetilde{\xi}(x,t)) + \frac{1}{2t} \arccos^2 < \widetilde{\xi}(x,t), x > \tag{65}$$

for any $x \in \mathbb{S}^n$ and fixed $t \in \mathbb{R}_+$. At the same time one can show that the exact solution $u : \mathbb{S}^n \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the following differential form expression:

$$du(x,t) =< p(x,t), dx > -1/2|p(x,t)|^2 dt, \tag{66}$$

equivalent completely to the Hamilton–Jacobi equation (39), where $p : \mathbb{S}^n \times \mathbb{R}_+ \to \mathbb{R}^{n+1}$ fulfills (61). Thus, based on (61) and (66) one obtains readily that

$$
\begin{aligned}
u(x,t) &= v(\xi(x,t)) + \int_0^t (du(x(\tau;x,t|y)),t)/d\tau)d\tau \\
&= v(\xi(x,t)) + 1/2\int_0^t d\tau |p(\tau;x,t|y)|^2,
\end{aligned}
$$

being equivalent for all $x \in \mathbb{S}^n$ and fixed $t \in \mathbb{R}_+$ to the expression

$$
u(x,t) = v(\xi(x,t)) + \frac{1}{2t}\arccos^2 < \xi(x,t), x > . \tag{67}
$$

Here $\xi : \mathbb{S}^n \times \mathbb{R}_+ \to \mathbb{S}^n$ is a mapping defined as follows:

$$
(x - \xi < x, \xi >< \xi(x,t), x >= tp_0(\xi)((1- < x, \xi >)^{1/2},
$$

where a function $p_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$satisfies the expression following from (66):

$$
\nabla u(\xi,t)|_{t=0^+} := p_0(\xi), \tag{68}
$$

strongly depending only on the Cauchy data $v \in BSC(\mathbb{S}^n)$. Therefore, it is now sufficient to prove the above equality for all $x \in\mathbb{S}^n$ and $t \in \mathbb{R}_+$, entailing, respectively, the wanted equality $\widetilde{u}(x,t) = u(x,t)$. It is an easy task to state the validity of the following lemma.

Lemma 5. *The expression* (65) *is differentiable for the expression* (65) *is differentiable for each* $t \in \mathbb{R}_+$ *and the following equality holds:*

$$
\nabla \widetilde{u}(\widetilde{\xi},t)|_{t=0^+} = \widetilde{p}_0(\widetilde{\xi}), \tag{69}
$$

where the relationship

$$
(x - \widetilde{\xi} < x, \widetilde{\xi} >< \widetilde{\xi}(x,t), x >= t\widetilde{p}_0(\widetilde{\xi})((1- < x, \widetilde{\xi} >)^{1/2}, \tag{70}
$$

is fulfilled for any $x \in \mathbb{S}^n$ *and* $t \in \mathbb{R}_+$.

As a result of expressions (70) and (68) one infers that at $p_0(x) = \widetilde{p}_0(x)$ for any $x \in \mathbb{S}^n$ the equality $\widetilde{\xi}(x,t) = \xi(x,t)$ holds for all $t \in \mathbb{R}_+$. The latter obviously means, if to take into account (65) and (66), that the inf-type Hopf–Lax expression (64) does solve the Hamilton–Jacobi equation (39), constrained to live on the sphere $\mathbb{S}^n$. This proves our Theorem 3 formulated above.

4. A NONUNIFORM HAMILTON–JACOBI EQUATION AND ITS CLASSICAL AND GENERALIZED SOLUTIONS

It is a very interesting problem of describing a wider class of Hamilton–Jacobi equations generalizing (39), for which one could deliver similar to (38) exact Hopf–Lax type solutions based on the characteristics method considerations. For instance, it is an important for applications problem [12,15,27] to analyze suitable classical and generalized solutions to the following non-canonical Hamilton–Jacobi equation

$$\partial u/\partial t + \frac{1}{2}(|u_x|^2 + \beta u|x|^2) + \frac{1}{2} < \Omega x, x >= 0 \tag{71}$$

with Cauchy data

$$u|_{t=0^+} = v, \tag{72}$$

where $t \in \mathbb{R}_+$, $\beta \in \mathbb{R}$ is arbitrary constant, $v : \mathbb{R}^n \to \mathbb{R}$ is some mapping and $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a positive definite diagonal matrix. If the parameter $\beta = 0$ and the matrix $\Omega = 0$, the equation (71) reduces, obviously, to that analyzed in Sections above. In what to follow we study the classical and generalized solutions to problem (71) and (72), when Cauchy data $v : \mathbb{R}^n \to \mathbb{R}$ are twice absolutely differentiable functions in $\mathbb{R}^n$.

4.1. The Cauchy problem

Consider the Cauchy problem for the following nonuniform Hamilton–Jacobi equation

$$du/dt + \frac{1}{2}(|u_x|^2 + \beta u\,|x|^2) = 0, \qquad u_{t=0} = v, \tag{73}$$

where $v \in H^2(\mathbb{R}^n; \mathbb{R})$ is a given function, $x \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ is some fixed parameter. Within the characteristic method equation (73) can be treated as one on the characteristic surface $S_H \subset \mathbb{R}^{2n+1}$, governed by the characteristic Hamilton equations

$$\frac{d\alpha}{d\tau} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -(\frac{\partial H}{\partial \alpha} + p\frac{\partial H}{\partial u}), \quad \frac{du}{d\tau} =< p, \frac{d\alpha}{d\tau} > -H, \tag{74}$$

where the Hamiltonian function

$$H := H(\alpha; u, p) = \frac{1}{2}(|p|^2 + \beta u\,|\alpha|^2) \tag{75}$$

is defined for all $(\alpha; u, p) \in S_H$. Thus, the set of equations

$$\begin{gathered} \dot{\alpha} := \frac{d\alpha}{d\tau} = p, \quad \dot{p} := \frac{dp}{d\tau} = -(u\alpha + \frac{\beta}{2} p\, |\alpha|^2\, \dot{\alpha}), \\ \dot{u} := \frac{du}{d\tau} = \frac{1}{2}(|\dot{\alpha}| - \beta u\, |\alpha|^2), \end{gathered} \tag{76}$$

when equipped with the related „inverse" Cauchy data

$$a_{\tau=0} = y(x,t),\ \alpha_{\tau=t} = x,\ u_{\tau=0} = v(y(x)), \tag{77}$$

for any reachable point $(x,t) \in \mathbb{R}^{n+1} \cap S_H$, give rise [21] to the following exact analytical solution to equation (73):

$$u(x,t) = v(y) - \frac{1}{2} < y, \dot{\alpha} >|_{\tau=0} - \frac{\beta}{16}(|x|^4 - |y|^4) + \frac{1}{2} < x, \dot{\alpha} >|_{\tau=t}. \tag{78}$$

Here the mapping $y : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is defined through the solution of the „inverse" Cauchy problem (76) and (77). This means, in particular, that the solution $\alpha : \mathbb{R}^{n+1} \to \mathbb{R}^n$ to the system of nonlinear second order ordinary differential equations

$$-\ddot{\alpha} = \beta(u\alpha + \frac{1}{2}\, |\alpha|^2\, \dot{\alpha}),\ \dot{u} = \frac{1}{2}(|\dot{\alpha}|^2 - \beta u\, |\alpha|^2) \tag{79}$$

with the „inverse" Cauchy data (77) on the interval $[0,t] \subset \mathbb{R}_+$ is solvable for all reachable points $(x,t) \in \mathbb{R}^{n+1} \cap S_H$.

4.2. Solution set analysis

Consider the system of equations as that, defined in the Sobolev space $E_1 := H^2(0,t;\mathbb{R}^n) \oplus H^1(0,t;\mathbb{R}^1)$, being rewritten as

$$\hat{a}(\alpha, u) = f_\beta(\alpha, u), \tag{80}$$

where for any $(\alpha, u) \in E_1$ the operator $\hat{a} : E_1 \to E_2$ acts onto the Hilbert space $E_2 := H(0,t;\mathbb{R}^n) \oplus H(0,t;\mathbb{R})$ as

$$\hat{a}(\alpha, u) = (-\ddot{\alpha}, \dot{u}), \tag{81}$$

and the nonlinear mapping $f : E_1 \to E_2$ is naturally defined as

$$f_\beta(\alpha, u) := (u\alpha + \frac{\beta}{2}\, |\alpha|^2\, \dot{\alpha}, \frac{1}{2}(|\dot{\alpha}|^2 - \beta u\, |\alpha|^2)). \tag{82}$$

The corresponding solution set $\mathcal{N}(\hat{a}, f) \subset E_1$ of nonlinear equation (80) one can study by means of a generalized version [18] of the well-known Leray-Schauder fixed point theorem [9]. Namely, the following theorem holds.

Theorem 4. *Let a linear operator $\hat{a} : E_1 \to E_2$ from a Banach space $(E_1, \|\cdot\|_1)$ into a Banach space $(E_2, \|\cdot\|_2)$ be closed and surjective. If a non-linear mapping $f_\beta : E_1 \to E_2$ is $\hat{a}$-compact* [10, 18], *its domain $D(f) = D(\hat{a}) \cap S_r(0)$, where $S_r(0) \subset E_1$ is the ball of radius $r > 0$ centered at zero of E_1, and moreover, the positive value $k_f > k(\hat{a})$, where*

$$\begin{gathered} k_f^{-1} := \sup_{\|(\alpha,u)\|_1 = r} \frac{1}{r} \|f_\beta(\alpha, u)\|_2 , \\ k(\hat{a}) := \sup_{\|w\|_2 = 1} \inf \{\|(\alpha, u)\|_1 : \hat{a}(\alpha, u) = w\} , \end{gathered} \tag{83}$$

then there exists the non-empty set $\mathcal{N}(\hat{a}, f_\beta) \subset E_1$ of solutions $(\alpha, u) \in S_r(0)$ to the non-linear equation (80)*, whose topological dimension* [10]

$$\dim \mathcal{N}(\hat{a}, f_\beta) \geq \dim Ker\hat{a} - 1.$$

It is easy to check now that the linear operator $\hat{a} : E_1 \to E_2$ is, by construction, closed and surjective, and the mapping $f_\beta : E_1 \to E_2$ is $\hat{a}$-compact, being polynomial on E_1, owing to the well know compactness [17, 25, 26, 28] of the embedding $i : E_1 \to E_2$. Thereby, if a parameter $\beta \in \mathbb{R}$ is chosen in such a way that for some $r > 0$ the quantity

$$\sup_{\|(\alpha,u)\|_1 = r} \left\| (u\alpha + \frac{\beta}{2} |\alpha|^2 \dot{\alpha}, \frac{1}{2}(|\dot{\alpha}|^2 - \beta u |\alpha|^2)) \right\|_2 < k^{-1}(\hat{a}), \tag{84}$$

then equation (80) is solvable and its solution set $\mathcal{N}(\hat{a}, f_\beta)$ is evidently, nonempty, being continuously parameterized by arbitrary three constant vectors $\bar{a}$, $\bar{b} \in \mathbb{R}^n$and $\bar{c} \in \mathbb{R}$. Really,

$$Ker\ \hat{a} = \{(\bar{a}\tau + \bar{b}, \bar{c})\} \subset E_1 \tag{85}$$

for any three constant vectors $\bar{a}$, $\bar{b} \in \mathbb{R}^n$ and $\bar{c} \in \mathbb{R}$, and the functional dimension $\dim Ker\hat{a} = 3$.

As we are interested in solutions to the Cauchy problem (77), it is necessary to find the corresponding vectors $\bar{a} := \bar{a}(x, t) \in \mathbb{R}^n$, $\bar{b} := \bar{b}(x, t) \in \mathbb{R}^n$ and $\bar{c} := \bar{c}(x, t) \in \mathbb{R}$ for all $(x, t) \in \mathbb{R}^{n+1} \cap S_H$, satisfying the following constraints: the first one as

$$\alpha(0; \bar{a}, \bar{b}, \bar{c}) = y(x, t),\ \alpha(t; \bar{a}, \bar{b}, \bar{c}) = x,\ u(0; \bar{a}, \bar{b}, \bar{c}) = v(y(x, t)), \tag{86}$$

where we put, by definition,

$$\alpha(\tau) := \alpha(\tau; \bar{a}, \bar{b}, \bar{c}), \ u(\tau) := u(\tau; \bar{a}, \bar{b}, \bar{c}) \tag{87}$$

for all $\tau \in [0, t]$, $(\bar{a}, \bar{b}, \bar{c}) \in \mathbb{R}^{2n+1}$, and the second one as

$$\|(\alpha, u)\|_1 = r := r(\bar{a}, \bar{b}, \bar{c}). \tag{88}$$

Note also that, owing to the construction from [10, 11, 18], the solution (87) depends smoothly on constant parameters $\bar{a}, \bar{b} \in \mathbb{R}^n$ and $\bar{c} \in \mathbb{R}$. It is important now to mention that the mapping (80) satisfies the scaling symmetry condition (84):

$$\sigma : E_1 \times \mathbb{R}^2 \ni (u, \alpha; \tau, \beta) \to (\tilde{u}, \tilde{\alpha}; \tilde{\tau}, \tilde{\beta}) \in E_1 \times \mathbb{R}^2, \tag{89}$$

where, by definition,

$$\tilde{u} := r^{-1}u, \ \tilde{\alpha} := r^{-1}\alpha, \ \tilde{\tau} := r\tau, \ \tilde{\beta} := r^3\beta \tag{90}$$

for any $r \in \mathbb{R}_+$. The symmetry (90) gives rise to the next form of condition (84) :

$$\begin{gathered} \sup_{\|(\tilde{\alpha}, \tilde{u})\|_1 = 1} \left\| f_{\tilde{\beta}}(\tilde{\alpha}, \tilde{u}) \right\|_2 < \tilde{k}^{-1}(\hat{a}), \\ \tilde{k}(\hat{a}) := \sup_{\|w\|_2 = 1} \inf \left\{ \|(\tilde{\alpha}, \tilde{u})\|_1 : \hat{a}(\tilde{\alpha}, \tilde{u}) = \tilde{w} \right\}, \end{gathered} \tag{91}$$

which one can always satisfy choosing, respectively, the value of parameter $\tilde{\beta} \in \mathbb{R}$ small enough. Since the found solution $(\tilde{\alpha}, \tilde{u}) \in E_1$ satisfies additionally the constraint $\|(\tilde{\alpha}, \tilde{u})\|_1 = 1$, one gets easily that it is well determined for only small enough values of the evolution parameter $\tilde{\tau} \in [0, r_\beta t] \subset \mathbb{R}_+$ and the spacial variable $x r_\beta \in \mathbb{R}^3$, where $r_\beta := (\tilde{\beta}/\beta)^{1/3}$. Whence, if the solution $(\tilde{\alpha}, \tilde{u}) \in E_1$ satisfies the conditions

$$\tilde{\alpha}|_{\tilde{\tau}=0} \overset{?}{:=} r_\beta^{-1} y(x), \ \ \tilde{\alpha}|_{\tilde{\tau}=r_\beta t} := r_\beta^{-1} x, \ \ \tilde{u}|_{\tilde{\tau}=0} = r_\beta^{-1} v(y(x)), \tag{92}$$

for some suitable $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \cap S_H$, then, vice versa the constraint $\|(\tilde{\alpha}, \tilde{u})\|_1 = 1$ naturally defines some value $r_\beta > 0$, for which condition (91) is satisfied *a priori*. Note also that the obtained above solution (87) is also smooth with respect to the evolution parameter $\tau \in [0, t] \subset \mathbb{R}_+$, owing to the form of mapping (83), that is $(\alpha, u) \in E_1 \cap C^\infty(0, t; \mathbb{R}^n \times \mathbb{R})$. This property will be important for the Cauchy data analysis below.

4.3. The Cauchy data description

The solution $(\alpha, u) \in E_1$ obtained above, depends via its construction on arbitrary vector parameters $\bar{a}, \bar{b} \in \mathbb{R}^n$ and $\bar{c} \in \mathbb{R}$, which should be determined from the Cauchy data, imposed on system (85). To do this more effectively, rewrite the solution (87) in the following equivalent form:

$$\alpha = \alpha(\tau; y, \eta, \eta'), \;\; u = u(\tau; y, \eta, \eta'), \tag{93}$$

where, by definition, there are imposed the Cauchy data

$$\alpha|_{\tau=0} \stackrel{?}{=} y, \;\; \alpha|_{\tau=t} = x, \;\; \dot{\alpha}|_{\tau=0} = \eta', \;\; u|_{\tau=0} = \eta, \tag{94}$$

for any y, $\eta' \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$ and a fixed vector $x \in \mathbb{R}^n$ at $\tau = t \in \mathbb{R}_+$. Moreover, the constraint $\|(\alpha, u)\|_1 = r_\beta < \infty$ holds, owing to the existence Theorem 4, formulated above.

Take now into account that the following representations

$$\begin{gathered} \alpha(\tau; y, \eta, \eta') = y + \eta'\tau + \int_0^\tau (\tau - s) f_\beta^{(\alpha)}(\alpha, u) ds, \\ u(\tau; y, \eta, \eta') = \eta + \int_0^\tau f_\beta^{(u)}(\alpha, u) ds, \end{gathered} \tag{95}$$

where we denoted $f_\beta := (f_\beta^{(\alpha)}, f_\beta^{(u)})$, hold for all $\tau \in [0, t] \subset \mathbb{R}_+$. Thus, we get at $\tau = t$ the expressions

$$\begin{gathered} x = y + \eta'\tau + \int_0^t (t - s) f_\beta^{(\alpha)}(\alpha, u) ds, \\ u = \eta + \int_0^t f_\beta^{(u)}(\alpha, u) ds, \end{gathered} \tag{96}$$

$$\eta' = \frac{x - y}{t} - \frac{1}{t} \int_0^t (t - s) f_\beta^{(\alpha)}(\alpha, u) ds, \tag{97}$$

and

$$\eta = u(x, t) - \int_0^t f_\beta^{(u)}(\alpha, u) ds, \tag{98}$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \cap S_H$, where the mapping $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, constructed above, solves the Hamilton–Jacobi equation (73) under appropriate Cauchy data. To describe them in more detail, we need to relate with each other two quantities (97) and (88), which, by construction, depend only on the vector $y \in \mathbb{R}^n$. From (97) one gets easily that

$$\eta = v(y), \tag{99}$$

where $y \in \mathbb{R}^n$ and $u(x,t)|_{t=0^+} = v(x)$, $x \in \mathbb{R}^n$, is a suitable Cauchy data for equation (73). Moreover, mapping $\eta' : \mathbb{R}^n \to \mathbb{R}$ can be obtained similarly from (95) as

$$\eta'(y) := \lim_{\tau \to 0^+} \dot{\alpha}(\tau) = \lim_{\tau \to 0^+} \left. \frac{\partial u(\alpha(\tau), \tau)}{\partial \alpha} \right|_{\tau = 0^+}, \tag{100}$$

holding for all $y \in \mathbb{R}^n$. Observe now that the right-hand side of expression (98) is differentiable with respect to the variable $x \in \mathbb{R}^n$, that is the left-hand side should be differentiable too, in particular at $t = 0^+$:

$$\lim_{t \to 0^+} \partial\eta(y)/\partial x = \partial\eta(x)/\partial x = \partial u/\partial x|_{t \to 0^+} \tag{101}$$

for all $x \in \mathbb{R}^n$. Comparing now expressions (100) and (101), derive that for all $x \in \mathbb{R}^n$ there holds the equality $\eta'(x) = \partial\eta(x)/\partial x$, or, taking into account (99),

$$\eta'(x) = \partial v(y)/\partial x. \tag{102}$$

As a result of the reasoning above one can formulate the following theorem.

Theorem 5. *Let Cauchy data $v : \mathbb{R}^n \to \mathbb{R}$ for the Hamilton–Jacobi equation* (73) *be twice continuously differentiable, that is $v \in C^2(\mathbb{R}^n; \mathbb{R})$. Then there exists it a continuously differentiable classical solution $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ (in general, nonunique), which can be represented in the following analytic form:*

$$\begin{aligned} u(x,t) = v(y) + \frac{1}{2} < x - y, \partial v(y)/\partial y > + \\ + \frac{1}{2} < x, \int_0^t f_\beta^{(u)}(\alpha, u) ds > + \frac{1}{16}(|y|^4 - |x|^4), \end{aligned} \tag{103}$$

holding for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \cap S_H$, where the mapping $y : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a compatible smoothly differentiable solution to equations (97), (99) and (102).

If to assume that we are interested in generalized solution to the Hamilton-Jacobi equation (73), satisfying it almost everywhere, it is enough to consider, that equality (98) is differentiable only almost everywhere too.

From this condition, taking into account derived before equalities (99) and (102), one gets easily that the Cauchy data $v : \mathbb{R}^n \to \mathbb{R}$ can be chosen such that its derivative is an absolutely continuous function. Thereby, the following theorem holds.

Theorem 6. *If the Cauchy data for the Hamilton–Jacobi equation* (73) *is chosen to be twice-absolutely differentiable, then this equation possesses an almost everywhere differentiable solution,* $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *(in general, non-unique), which can be representable by means of expression* (103)*, where the compatible mapping* $y : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ *is an almost everywhere differentiable function.*

The analysis of the solution set to the Hamilton–Jacobi equation (73) undertaken above shows that both classical and generalized its solutions can be constructed by means of the modified characteristics method [20, 22–24] and appropriate version of the Leray-Schauder type fixed point theory [9, 18]. Moreover, in many cases, when the so called Hopf–Lax type functional kernels are constructed explicitly, the corresponding both classical and generalized solutions can be effectively enough represented by means of the inf-type extremal problem, which is widely used, for instance, in related optimal control considerations and other applications.

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МЕТОД ХАРАКТЕРИСТИК ТА ПОВ'ЯЗАНИЙ З НИМ АНАЛІЗ ЗАДАЧІ ПРО НЕРУХОМУ ТОЧКУ ДЛЯ РІВНЯННЯ ТИПУ ГАМІЛЬТОНА–ЯКОБІ

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Дано аналіз геометричного підходу Картана–Монжа до методу характеристик для нелінійних рівнянь із частинними похідними першого та вищих порядків. Досліджено структуру характеристичних векторних полів, пов'язаних з класичними та узагальненими розв'язками нелінійних рівнянь із частинними похідними типу Гамільтона–Якобі. Наведено деякі цікаві приклади.