STABILITY OF NONLINEAR DIFFERENCE FUNCTIONAL EQUATIONS ON UNBOUNDED DOMAINS

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Received August 14, 2007.

We give a theorem on the error estimate of approximate solutions for difference functional equations of the Volterra type with an unknown function of several variables. The error is estimated by a solution of an initial problem for nonlinear differential equation.

We apply this general result to the investigation of the stability of difference schemes generated by initial problems for hyperbolic functional differential equations. We assume nonlinear estimates of the Perron type with respect to functional variable for given operators. Numerical examples are presented.

1. INTRODUCTION

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E = [0,a] \times \mathbb{R}^n, \quad E_0 = [-d_0, 0] \times \mathbb{R}^n, \quad D = [-d_0, 0] \times [-d, d]$$

where $a > 0$, $d_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$ and $d = (d_1, \dots, d_n) \in \mathbb{R}^n_+$. For a function $z : E_0 \cup E \to \mathbb{R}$ and for a point $(t,x) \in E$ we define a function $z_{(t,x)} : D \to \mathbb{R}$ as follows: $z_{(t,x)}(\tau, y) = z(t+\tau, x+y)$, $(\tau, y) \in D$. Then $z_{(t,x)}$ is the restriction of z to the set $[t-d_0, t] \times [x-d, x+d]$ and this restriction is shifted to the set D. The maximum norm in the space $C(D,\mathbb{R})$ is denoted

UDC 519.41/47; MSC 2000: 20E34, 20F18

by $\|\cdot\|_D$. Write $\Sigma = E \times C(D,\mathbb{R}) \times \mathbb{R}^n$ and suppose that $f : \Sigma \to \mathbb{R}$ and $\varphi : E_0 \to \mathbb{R}$ are given functions. We consider the functional differential equation

$$\partial_t z(t,x) = f(t,x,z_{(t,x)},\partial_x z(t,x)) \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \tag{2}$$

where $x = (x_1,\ldots,x_n)$ and $\partial_x z = (\partial_{x_1} z,\ldots,\partial_{x_n} z)$. We consider classical solutions of (1), (2). We are interested in establishing a method of approximation of solutions to problem (1), (2) by means of solutions of associated difference functional problems and in estimating of the difference between the exact and approximate solutions.

In this time numerous papers were published concerning difference methods for initial or initial-boundary value problems related to first order partial differential functional equations [1, 4, 6, 10, 11, 14, 15]. All these problems have the following property: the main question in the investigation of numerical methods is to find a difference functional equation generated by the original problem which is stable. The method of difference inequalities or theorems on nonlinear recurrent inequalities are used in the investigation of the stability of nonlinear difference schemes. It is important in these considerations that solutions of differential functional problems and solutions of corresponding difference schemes are defined on bounded domains. The results presented in the above mentioned papers are not applicable to (1), (2). We prove that there is a class of difference methods for (1), (2) which are convergent. The stability of the methods is investigated by a comparison technique with nonlinear estimates of the Perron type for given functions with respect to the functional variable.

Differential equations with deviated variables and differential integral equations can be obtained as particular cases of (1) by suitable definitions of the operator f. Existence and uniqueness results for classical or generalized solutions for (1), (2) are given in [2, 3, 5], [12, Chapter 5].

The paper is organized as follows. In Section 2 we propose a general method for the investigation of the stability of difference schemes generated by initial problems for nonlinear functional differential equations. We prove a theorem on error estimates of approximate solutions to functional difference equations of the Volterra type with unknown function of several variables. The error of an approximate solution is estimated by a solution of an initial problem for a nonlinear differential equation. In Section 3 we apply the above general idea to the investigation of the convergence of difference schemes for

(1), (2). A generalized Euler method for (1), (2) is presented in Section 4. Numerical examples are given in the last part of the paper.

We use in the paper general ideas for finite difference equations which were introduced in [9, 12, 13, 16].

2. APPROXIMATE SOLUTIONS OF FUNCTIONAL DIFFERENCE EQUATIONS

For any two sets U and W we denote by $\mathbb{F}(U,W)$ the class of all functions defined on U and taking values in W. If $\alpha : U \to W$ and $\Omega \subset U$ then $\alpha\,|_{\Omega}$ is the restriction of α to the set Ω. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. We define a mesh on $E_0 \cup E$ in the following way. Suppose that $(h_0, h') = h$, $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. For $(r,m) \in \mathbb{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = (m_1 h_1, \ldots, m_n h_n).$$

Let us denote by Δ the set of all h such that there are $K_0 \in \mathbb{Z}$ and $K = (K_1, \ldots, K_n) \in \mathbb{Z}^n$ with the properties: $K_0 h_0 = d_0$ and $(K_1 h_1, \ldots, K_n h_n) = d$. Set

$$R_h^{1+n} = \{\, (t^{(r)}, x^{(m)}) : \ (r,m) \in \mathbb{Z}^{1+n} \,\}$$

and

$$E_{0.h} = E_0 \cap \mathbb{R}_h^{1+n}, \quad E_h = E \cap R_h^{1+n}, \quad D_h = D \cap \mathbb{R}_{1+n}.$$

Let $N_0 \in \mathbb{N}$ be defined by the relations: $N_0 h_0 \le a < (N_0+1)h_0$ and

$$E'_h = \{\, (t^{(r)}, x^{(m)}) \in E_h : \ 0 \le r \le N_0 - 1 \,\}.$$

Write $L = (L_1, \ldots, L_n) \in \mathbb{N}^n$ where $L_i = \max\{\, 1, \ L_i \,\}$ for $1 \le i \le n$ and

$$\Omega_h = \{\, (t^{(r)}, x^{(m)}) : \ \ -K_0 \le r \le 0, \ \ -L \le m \le L \,\}.$$

Let X be a linear space with the norm $\|\cdot\|_X$. For functions $z : E_{0.h} \cup E_h \to X$ and $w : \Omega_h \to X$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ on $E_{0.h} \cup E_h$ and $w^{(r,m)} = w(t^{(r)}, x^{(m)})$ on Ω_h. If $z : E_{0.h} \cup E_h \to X$ and $(t^{(r)}, x^{(m)}) \in E_h$ then the function $z_{\langle r,m\rangle} : \Omega_h \to X$ is given by

$$z_{\langle r,m\rangle}(\tau, y) = z(t^{(r)} + \tau, x^{(m)} + y), \quad (\tau, y) \in \Omega_h.$$

Suppose that the operator $F_h : E'_h \times \mathbb{F}(\Omega_h, X) \to X$ is given. For $(t^{(r)}, x^{(m)}, w) \in E'_h \times \mathbb{F}(\Omega_h, X)$ we write $F_h[\, w\,]^{(r,m)} = F_h(t^{(r)}, x^{(m)}, w)$. Given $\varphi_h \in \mathbb{F}(E_{0.h}, X)$, we consider the functional difference equation

$$z^{(r+1,m)} = F_h[\, z_{\langle r,m\rangle}\,]^{(r,m)} \tag{3}$$

with the initial condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h}. \tag{4}$$

It is clear that there exists exactly one solution $z_h : E_{0.h} \cup E_h \to X$ of (3), (4).

Let $Y_h \subset \mathbb{F}(\Omega_h, X)$ be a fixed subset. Suppose that the functions $v_h : E_{0.h} \cup E_h \to X$ and $\tilde{\alpha}$, $\tilde{\gamma} : \Delta \to \mathbb{R}_+$ satisfy the conditions

$$\|v_h^{(r+1,m)} - F_h[\,(v_h)_{\langle r,m\rangle}\,]^{(r,m)}\|_X \leq \tilde{\gamma}(h) \ \text{ on } \ E'_h$$

$$\|(v_h - v_h)^{(r,m)}\|_X \leq \tilde{\alpha}(h) \ \text{ on } \ E_{0.h}, \quad \lim_{h\to 0} \tilde{\alpha}(h) = 0, \ \lim_{h\to 0} \tilde{\gamma}(h) = 0,$$

and

$$(v_h)_{\langle r,m\rangle} \in Y_h \ \text{ for } \ (t^{(r)}, x^{(m)}) \in E_h.$$

The function v_h satisfying the above relations is considered as an approximate solution of (3), (4).

We look for approximate solutions of (3), (4) such that $(v_h)_{\langle r,m\rangle} \in Y_h$ for $(t^{(r)}, x^{(m)}) \in E_h$. We give a theorem on the estimate of the difference between the exact and approximate solutions of (3), (4). Write

$$A_h = \{\,(t^{(r)}x^{(m)}) \in \Omega_h : \ \ r = 0, \ \ -1 \leq m_i \leq 1 \ \text{ for } \ 1 \leq i \leq n\,\}.$$

For a function $w : \Omega_h \to X$ we put

$$\|w|_{A_h}\|_X = \max\{\,\|w^{(r,m)}\|_X : \ \ (t^{(r)}, x^{(m)}) \in A_h\,\},$$

$$\|w|_{D_h}\|_X = \max\{\,\|w^{(r,m)}\|_X : \ \ (t^{(r)}, x^{(m)}) \in D_h\,\}.$$

For $z : E_{0.h} \cup E_h \to X$ we define

$$\|z\|_{h.r} = \sup\{\,\|z^{(i,m)}\|_X : \ \ -K_0 \leq i \leq r, \ \ m \in \mathbb{Z}^m\,\}, \ \ 0 \leq r \leq N_0.$$

Put $I_h = \{t^{(0)}, t^{(1)}, \ldots, t^{(N_0)}\}$. For $\beta : I_h \to \mathbb{R}$ we write $\beta^{(r)} = \beta(t^{(r)})$ on I_h. We formulate assumptions on comparison operators corresponding to (3), (4).

Assumption $H[\sigma]$**.** The function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

1) σ is continuous and it is nondecreasing with respect to the both variables,

2) $\sigma(t,0)=0$ for $t\in[0,a]$ and the function $\tilde{\omega}(t)=0$ for $t\in[0,a]$ is the maximal solution of the Cauchy problem

$$\omega'(t)=\sigma(t,\omega(t)),\ \ \omega(0)=0. \tag{5}$$

Now we formulate the main result of this section.

Theorem 1. *Suppose that* $F_h : E'_h\times\mathbb{F}(\Omega_h,X)\to X$, $\varphi_h : E_{0.h}\to\mathbb{R}$ *are given and*

1) $z_h : E_{0.h}\cup E_h\to X$ *is the solution of* (3), (4),

2) $v_h : E_{0.h}\cup E_h\to X$ *and*

(i) there are $\alpha_0,\ \gamma:\Delta\to\mathbb{R}_+$ *such that*

$$\|v_h^{(r+1,m)}-F_h[\,(v_h)_{\langle r,m\rangle}\,]^{(r,m)}\|_X\le h_0\gamma(h)\ on\ E'_h\ and\ \lim_{h\to0}\gamma(h)=0, \tag{6}$$

$$\|(v_h-z_h)^{(r,m)}\|_X\le\alpha_0(h)\ on\ E_{0.h}\ \ and\ \ \lim_{h\to0}\alpha_0(h)=0, \tag{7}$$

(ii) $(v_h)_{\langle r,m\rangle}\in Y_h$ *for* $(t^{(r)},x^{(m)})\in E_h$,

3) there exists $\sigma:[0,a]\times\mathbb{R}_+\to\mathbb{R}_+$ *such that Assumption* $H[\sigma]$ *is satisfied and*

$$\|F_h[\,w\,]^{(r,m)}-F_h[\,\bar{w}\,]^{(r,m)}\|_X\le\|(w-\bar{w})|_{A_h}\|_X+h_0\sigma(\,t^{(r)},\|(w-\bar{w})|_{D_h}\|_X\,) \tag{8}$$

where $(t^{(r)},x^{(m)},w)\in E'_h\times\mathbb{F}(\Omega_h,X)$ *and* $\bar{w}\in Y_h$. *Then there is* $\alpha:\Delta\to\mathbb{R}_+$ *such that*

$$\|(z_h-v_h)^{(r,m)}\|_X\le\alpha(h)\ on\ E_h\ and\ \lim_{h\to0}\alpha(h)=0. \tag{9}$$

Proof. Let us denote by $\beta_h : I_h\to\mathbb{R}_+$ the solution of the difference problem

$$\beta^{(r+1)}=\beta^{(r)}+h_0\sigma(\,t^{(r)},\beta^{(r)}\,)+h_0\gamma(h),\ 0\le r\le N_0-1, \tag{10}$$

$$\beta^{(0)}=\alpha_0(h). \tag{11}$$

We prove that

$$\|z_h-v_h\|_{h.r}\le\beta^{(r)}\ \ \text{for}\ \ 0\le r\le N_0. \tag{12}$$

It follows from (7) that estimate (12) holds for $r=0$. Assuming (12) to hold for r, $0\le r\le N_-1$, we will prove it for $r+1$. We conclude from (6) and (8) that for $0\le i\le r$ we have

$$\|(v_h-z_h)^{(i+1,m)}\|_X\le\|F_h[\,(z_h)_{\langle i,m\rangle}\,]^{(i,m)}-F_h[\,(v_h)_{\langle i,m\rangle}\,]^{(i,m)}\|_X+$$

$$+\|v_h^{(i+1,m)} - F_h[\,(v_h)_{\langle i,m\rangle}\,]^{(i,m)}\|_X \leq \beta^{(i)} + h_0\sigma(\,t^{(i)}, \|(z_h - v_h)_{\langle i,m\rangle}|_{D_h}\|_X\,) +$$

$$+h_0\gamma(h) \leq \beta_h^{(r)} + h_0\sigma(t^{(r)}, \beta_h^{(r)}) + h_0\gamma(h) = \beta_h^{(r+1)}$$

and consequently

$$\|v_h - z_h\|_{h.r+1} \leq \beta_h^{(r+1)}.$$

Hence the proof of (12) is completed by induction. Consider the Cauchy problem

$$\omega'(t) = \sigma(\,t, \omega(t)) + \gamma(h), \;\; \omega(0) = \alpha_0(h). \tag{13}$$

It follows from Assumption $H[\sigma]$ that the maximal solution $\omega(\,\cdot\,, h)$ of (13) is defined on $[0, a]$ and

$$\lim_{h\to 0} \omega(t, h) = 0 \;\text{ uniformly on }\; [0, a].$$

The function $\omega(\,\cdot\,, h)$ is convex on $[0, a]$. Then we have the difference inequality

$$\omega(t^{(r+1)}, h) \geq \omega(t^{(r)}, h) + h_0\sigma(t^{(r)}, \omega(t^{(r)}, h)) + h_0\gamma(h), \;\; r = 0, 1, \ldots, N_0 - 1.$$

Since β_h satisfies (10), (11), the above relations show that $\beta^{(r)} \leq \omega(t^{(r)}, h)$ for $0 \leq r \leq N_0$. Then condition (9) is satisfied with $\alpha(h) = \omega(a, h)$. This proves the theorem.

Remark 1. Suppose that $\sigma(t, \tau) = L_0\tau$ on $[0, a] \times \mathbb{R}_+$. Then assumption (8) has the form

$$\|F_h[\,w\,]^{(r,m)} - F_h[\,\bar{w}\,]^{(r,m)}\|_X \leq \|(w - \bar{w})|_{A_h}\|_X + h_0 L_0 \|(w - \bar{w})|_{D_h}\|_X$$

where $w \in \mathbb{F}(\Omega_h, X)$, $\bar{w} \in Y_h$. Then assertion (9) takes the form

$$\|(z_h - v_h)^{(r,m)}\|_X \leq \tilde{\alpha}(h) \;\text{ on }\; E_h$$

where

$$\tilde{\alpha}(h) = \alpha_0(h)\, \exp[L_0 a] + \gamma(h)\frac{\exp[L_0 a] - 1}{L_0} \;\text{ if }\; L_0 > 0, \tag{14}$$

$$\tilde{\alpha}(h) = \alpha_0(h) + a\gamma(h) \;\text{ if }\; L_0 = 0. \tag{15}$$

The above example is important in simple applications.

3. INITIAL PROBLEMS FOR HAMILTON–JACOBI FUNCTIONAL DIFFERENTIAL EQUATIONS

We will need a discrete version of the operator $(t,x) \to z_{(t,x)}$. If $z : E_{0.h} \cup E_h \to \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E_h$ then the function $z_{[r,m]} : D_h \to \mathbb{R}$ is defined by

$$z_{[r,m]}(\tau, y) = z(t^{(r)} + \tau, x^{(m)} + y), \quad (\tau, y) \in D_h.$$

For $w \in \mathbb{F}(D_h, \mathbb{R})$ we put

$$\|w\|_{D_h} = \max\{|w(\tau, y)| : \ (\tau, y) \in D_h\}.$$

We consider the following interpolating operator $T_h : \mathbb{F}(D_h, \mathbb{R}) \to \mathbb{F}(D, \mathbb{R})$. Set

$$S_+ = \{ \, s = (s_1, \ldots, s_n) : \ s_i \in \{0, 1\} \text{ for } 1 \le i \le n \, \}.$$

Let $w \in \mathbb{F}(D_h, \mathbb{R})$ and $(t,x) \in D$. There exists $(t^{(r)}, x^{(m)}) \in D_h$ such that $(t^{(r+1)}, x^{(m+1)}) \in D_h$ where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$. Write

$$T_h[w](t,x) = \Big(1 - \frac{t - t^{(t)}}{h_0}\Big) \sum_{s \in S_+} w^{(r,m+s)} \Big(\frac{x - x^{(m)}}{h'}\Big)^s \Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-s} +$$

$$+ \frac{t - t^{(t)}}{h_0} \sum_{s \in S_+} w^{(r+1,m+s)} \Big(\frac{x - x^{(m)}}{h'}\Big)^s \Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-s}$$

where

$$\Big(\frac{x - x^{(m)}}{h'}\Big)^s = \prod_{i=1}^{n} \Big(\frac{x_i - x_i^{(m_i)}}{h_i}\Big)^{s_i},$$

$$\Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-s} = \prod_{i=1}^{n} \Big(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\Big)^{1-s_i}$$

and we take $0^0 = 1$ in the above formulas It is easy to see that $T_h[w] \in C(D, \mathbb{R})$. The above interpolating operator has been first considered in [12].

The following Lemmas are important in our considerations.

Lemma 1. *If* $w \in \mathbb{F}(D_h, \mathbb{R})$ *then*

$$\|T_h[w]\|_D = \|w\|_{D_h}. \tag{16}$$

Proof. It is easy to prove by induction with respect to n that

$$\sum_{s\in S_+}\Big(\frac{x-x^{(m)}}{h'}\Big)^s\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-s}=1 \text{ for } x^{(m)}\le x\le x^{(m+1)}.$$

The equality (16) follows from the above relation.

Lemma 2. *Suppose that $w : D \to X$ is of class C^1 and denote by w_h the restriction of w to the set D_h. Let*

$$\tilde{C}=\max\{\,\|\partial_t w\|_D,\ \|\partial_{x_i} w\|_D,\ \ i=1,\ldots,n\,\}.$$

Then

$$\|T_h[w_h]-w\|_D\le \tilde{C}\|h\|,$$

where $\|h\|=h_0+h_1+\ldots+h_n$.

The above lemma can be proved by a method used in the proof of Theorem 5.27 in [12].

Lemma 3. *Suppose that $w : D \to X$ is of class C^2 and denote by w_h the restriction of w to the set D_h. Let*

$$\tilde{C}=\max\{\,\|\partial_{tt} w\|_D,\ \|\partial_{tx_i} w\|_D,\ \ \|\partial_{x_ix_j} w\|_D,\ \ i,j=1,\ldots,n\,\}.$$

Then

$$\|T_h[w_h]-w\|_D\le \tilde{C}\|h\|^2.$$

The Lemma 3 is a consequence of Theorem 5.27 in [12].

For $x=(x_1,\ldots,x_n)=x\in\mathbb{R}^n$ we put $\|x\|=|x_1|+\ldots+|x_n|$. Let D, E, E_0 and D_h, E_h, $E_{0.h}$, Ω_h, A_h be the sets defined in Sections 1 and 2. We formulate a difference method for initial problem (1), (2). For $1\le i\le n$ we define $e_i=(0,\ldots,0,1,0,\ldots,0)\in\mathbb{R}^n$ with 1 standing on the $i-$th place. Let δ_0 and $(\delta_1,\ldots,\delta_n)=\delta$ be the difference operators given by

$$\delta_0 z^{(r,m)}=\frac{1}{h_0}\big(z^{(r+1)}-\Lambda[z]^{(r,m)}\big) \tag{17}$$

where

$$\Lambda[z]^{(r,m)}=\frac{1}{2n}\sum_{i=1}^{n}\big[z^{(r,m+e_i)}+z^{(r,m-e_i)}\big]$$

and

$$\delta_j z^{(r,m)}=\frac{1}{2h_j}\big[z^{(r,m+e_j)}-z^{(r,m-e_j)}\big],\ \ j=1,\ldots,n. \tag{18}$$

In the same way we define the expressions $\Lambda[w]^{(0,\theta)}$ and

$$\delta w^{(0,\theta)} = (\delta_1 w^{(0,\theta)}, \ldots, \delta_n w^{(0,\theta)})$$

where $w \in \mathbb{F}(\Omega_h, \mathbb{R})$ and $\theta = (0, \ldots, 0) \in \mathbb{R}^n$. For a function $z : E_{0.h} \cup E_h \to \mathbb{R}$ and for a point $(t^{(r)}, x^{(m)}) \in E_h$ we write $T_h z_{[r,m]}$ instead of $T_h[z_{[r,m]}]$.

Given $\varphi_h : E_{0.h} \to \mathbb{R}$, we approximate classical solutions of (1), (2) with solutions of the difference functional problem

$$\delta_0 z^{(r,m)} = f(\, t^{(r)}, x^{(m)}, T_h z_{[r,m]}, \delta z^{(r,m)}\,), \tag{19}$$

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h}. \tag{20}$$

The above difference method is called the Lax scheme for (1), (2). We claim that difference problem (19), (20) is a particular case of (3), (4). Let $X = \mathbb{R}$ and $F_h : E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ be defined by

$$F_h[w]^{(r,m)} = \Lambda[w]^{(0,\theta)} + h_0 f(\, t^{(r)}, x^{(m)}, T_h[w|_{D_h}], \delta w^{(0,\theta)}\,). \tag{21}$$

It is easily seen that equation (19) is equivalent to (3) with F_h given by (21).

Assumption $H[f]$. The function $f : \Sigma \to \mathbb{R}$ of the variables (t, x, w, q), $q = (q_1, \ldots, q_n)$, is continuous and

1) there exists $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and

$$|f(t, x, w, q) - f(t, x, \bar{w}, q)| \le \sigma(\, t, \|w - \bar{w}\|_D) \ \text{ on } \ \Sigma, \tag{22}$$

2) the partial derivatives $\big(\partial_{q_1} f, \ldots, \partial_{q_n} f\,\big) = \partial_q f$ exist on Σ, $\partial_q f \in C(\Sigma, \mathbb{R}^n)$ and the function $\partial_q f$ is bounded on Σ.

Theorem 2. *Suppose that Assumption $H[f]$ is satisfied and*

1) the function $z_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution of (19), (20) *and there is $\alpha_0 : \Delta \to \mathbb{R}_+$ such that*

$$|(\varphi - \varphi_h)^{(r,m)}| \le \alpha_0(h) \text{ on } E_{0.h} \text{ and } \lim_{h \to 0} \alpha_0(h) = 0, \tag{23}$$

2) for $(t, x, w, q) \in \Sigma$ we have

$$\frac{1}{n} - \frac{h_0}{h_i} \,\big|\, \partial_{q_i} f(t, x, w, q) \,\big| \ge 0, \ \ i = 1, \ldots, n, \tag{24}$$

3) $v : E_0 \cup E \to \mathbb{R}$ is a classical solution of (1), (2) *and v is of class C^2 on $E_0 \cup E$,* *4) the functions*

$$\partial_{tt}v,\ \ \partial_{tx_i}v,\ \ \partial_{x_ix_j}v,\ \ i,j=1,\dots,n,$$

are bounded on $E_0\cup E$. *Then there is* $\alpha:\Delta\to\mathbb{R}_+$ *such that*

$$|(v_h-u_h)^{(r,m)}|\le\alpha(h)\ \ \text{on}\ \ E'_h\ \ \text{and}\ \ \lim_{h\to0}\alpha(h)=0, \tag{25}$$

where v_h *is the restriction of* v *to the set* $E_{0.h}\cup E_h$.

Proof. We apply Theorem 1 to prove (25). Put $Y_h=\mathbb{F}(\Omega_h,\mathbb{R})$. It follows that z_h satisfies (3) where F_h is defined by (21) and there is $\gamma:H\to\mathbb{R}_+$ such that condition (6) is satisfied. Now we estimate the difference $F_h[w]-F_h[\bar w]$, where $w,\ \bar w\in\mathbb{F}(\Omega_h,\mathbb{R})$. It follows from Lemmas 1, 3and from (22), (24) that

$$\big|F_h[w]^{(r,m)}-F_h[\bar w]^{(r,m)}\big|\le h_0\big|\,f(t^{(r)},x^{(m)},T_h[w|_{D_h}],\delta w^{(0,\theta)})-$$

$$-f(t^{(r)},x^{(m)},T_h[w|_{D_h}],\delta\bar w^{(0,\theta)})+\Lambda[w-\bar w]^{(0,\theta)}\big|+$$

$$+h_0\big|f(t^{(r)},x^{(m)},T_h[w|_{D_h}],\delta\bar w^{(0,\theta)})-f(t^{(r)},x^{(m)},T_h[\bar w|_{D_h}],\delta\bar w^{(0,\theta)})\big|\le$$

$$\le\frac12\sum_{j=1}^{n}\Big|\big(w-\bar w\big)^{(0,e_j)}\Big[\,\frac1n+\frac{h_0}{h_j}\,\partial_{q_j}f(\tilde P)\,\Big]\Big|+$$

$$+\frac12\sum_{j=1}^{n}\Big|\big(w-\bar w\big)^{(0,-e_j)}\Big[\,\frac1n-\frac{h_0}{h_j}\,\partial_{q_j}f(\tilde P)\,\Big]\Big|+h_0\sigma(t^{(r)},\|(w-\bar w)|_{D_h}\|)\le$$

$$\le\|(w-\bar w)|_{A_h}\|+h_0\sigma(t^{(r)},\|(w-\bar w)|_{D_h}\|),$$

where $\tilde P\in\Sigma$ is an intermediate point. Then the operator F_h satisfies (8). Thus we see that all the assumptions of Theorem 1 are satisfied and the assertion (25) follows.

Now we formulate a result o the error estimate for the Lax scheme.

Lemma 4. *Suppose that*

1) all the assumptions of Theorem 2 are satisfied with $\sigma(t,\tau)=L_0\tau$ *on* $[0,a]\times\mathbb{R}_+$ *where* $L_0\in\mathbb{R}_+$, *(then we have assumed that* f *satisfies the Lipschitz condition with respect to the functional variable),*

2) there are $B,\ M\in\mathbb{R}^n_+$ *such that*

$$\big(|\partial_{q_1}f(t,x,w,q)|,\dots,|\partial_{q_n}f(t,x,w,q)|\big)\le B\ \ \text{on}\ \ \Sigma$$

and $h'\le Mh_0$,

3) the constant $\bar{C}$ *is defined by the relations*

$$|\partial_{tt}v(t,x)|,\ |\partial_{tx_i}v(t,x)|,\ |\partial_{x_ix_j}v(t,x)| \le \bar{C} \text{ on } E_0 \cup E \text{ for } i,j = 1,\dots,n.$$

Then

$$|(v_h - z_h)^{(r,m)}| \le \tilde{\alpha}(h) \ \text{ on } \ E_h$$

where $\tilde{\alpha}$ *is given by* (14), (15) *with*

$$\gamma(h) = \tilde{A}h_0 + L_0\bar{C}\|h\|^2$$

and

$$\tilde{A} = \frac{1}{2}\bar{C}(1+\tilde{\Gamma}) + \frac{1}{2}\bar{C}\|M\|\,\|B\|, \quad \tilde{\Gamma} = \frac{1}{n}\sum_{i=1}^{n} M_i^2, \ \ (M_1,\dots,M_n) = M.$$

The above Lemma is a consequence of Theorem 2 and Lemmas 1 and 3.

Now we consider functional difference problem (19), (20) with δ_0 and $\delta = (\delta_1,\dots,\delta_n)$ defined in the following way:

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\big[z^{(r+1,m)} - z^{(r,m)}\big] \tag{26}$$

and

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\big[z^{(r,m+e_i)} - z^{(r,m)}\big] \ \text{ for } \ 1 \le i \le \kappa, \tag{27}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\big[z^{(r,m)} - z^{(r,m-e_i)}\big] \ \text{ for } \ \kappa+1 \le i \le n, \tag{28}$$

where $0 \le \kappa \le n$ is fixed. If $\kappa = 0$ then δz is given by (28), if $\kappa = n$ then δz is given by (27). In the same way we define the expressions

$$\delta w^{(0,\theta)} = \big(\delta_1 w^{(0,\theta)},\dots,\delta_n w^{(0,\theta)}\big)$$

where $w \in \mathbb{F}(\Omega_h,\mathbb{R})$. Difference scheme (19), (20) with δ_0 and δ defined by (26)–(28) is called the Euler difference method. Let $F_h : E'_h \times \mathbb{F}(\Omega_h,\mathbb{R}) \to \mathbb{R}$ be defined by

$$F_h[w]^{(r,m)} = w^{(0,\theta)} + h_0 f\big(t^{(r)}, x^{(m)}, T_h[w|_{D_h}], \delta w^{(0,\theta)}\big). \tag{29}$$

It is easily seen that the Euler difference method is equivalent to (3) for $X = \mathbb{R}$ and F_h defined by (29).

Theorem 3. *Suppose that Assumption* $H[f]$ *is satisfied and*

1) the function $z_h : E_{0.h} \cup E_h \to \mathbb{R}$ *is a solution of* (19), (20) *with* δ_0 *and* δ *given by* (26)–(28) *and there is* $\alpha : \Delta \to \mathbb{R}_+$ *such that condition* (23) *is satisfied,*

2) for $(t, x, w, q) \in \Sigma$ *we have*

$$\partial_{q_i} f(t, x, w, q) \geq 0 \ \ for \ \ 1 \leq i \leq \kappa, \tag{30}$$

$$\partial_{q_i} f(t, x, w, q) \leq 0 \ \ for \ \ \kappa + 1 \leq i \leq n \tag{31}$$

and

$$1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} |\partial_{q_i} f(t, x, w, q)| \geq 0, \tag{32}$$

3) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of (1), (2) and* v *is of class* C^2 *on* $E_0 \cup E$ *and the functions*

$$\partial_{tt} v, \ \ \partial_{tx_i} v, \ \ \partial_{x_i x_j} v, \ \ i, j = 1, \ldots, n,$$

are bounded on $E_0 \cup E$*. Then there is* $\alpha : \Delta \to \mathbb{R}_+$ *such that condition (25) is satisfied where* v_h *is the restriction of* v *to the set* $E_{0.h} \cup E_h$.

The proof of the above theorem is similar to the proof of Theorem 2. Details are omitted.

Now we formulate a result on the error estimate for the Euler difference method.

Lemma 5. *Suppose that*

1) all the assumptions of Theorem 3 are satisfied with $\sigma(t, \tau) = L_0 \tau$ *on* $[0, a] \times \mathbb{R}_+$ *where* $L_0 \in \mathbb{R}_+$,

2) the conditions 2), 3) of Lemma 4 are satisfied.

Then

$$|(v_h - z_h)^{(r,m)}| \leq \tilde{\alpha}(h) \ \ on \ \ E_h$$

where $\tilde{\alpha}$ *is given by* (14), (15) *with*

$$\gamma(h) = \frac{1}{2} \bar{C} (1 + \|M\| \, \|B\|) h_0 + L_0 \bar{C} \|h\|^2.$$

We use Lemmas 1, 3, 11 and Theorem 3 in a simple proof of the above relation.

4. GENERALIZED EULER METHOD FOR NONLINEAR DIFFERENTIAL FUNCTIONAL EQUATIONS

We have considered two difference method for (1), (2): the Lax scheme and the Euler difference method. Two types of assumptions are needed in theorems on the convergence of difference schemes generated by (1), (2). The first type conditions deal with the regularity of f. They are formulated in Assumption $H[f]$ and they the same for the both methods. The assumptions of the second type are called the Courant–Friedrichs–Levy (CFL) conditions. The (CFL) condition for (1), (2) and for the Lax difference scheme has the form (24) Assumptions (30)–(32) are the (CFL) conditions for the Euler difference method.

Note that assumptions (24) and (32) are quite similar. Some relations between h_0 an $h' = (h_1, \ldots, h_n)$ are required in (24) and (32). It follows from (30), (31) that we need more restrictive assumptions on f for the Euler difference method than for the Lax scheme.

There are initial problems (1), (2) such that both the above difference methods are convergent. It follows from the theory of bicharacteristics for nonlinear differential functional equations that in this case the numerical results obtained by the Euler difference method are better than corresponding results obtained by the Lax scheme. This property of difference methods can be easy illustrated by numerical experiments.

With the above motivation we are interested in proving of convergence results for the Euler method and for a possibly large class of nonlinear problems. More precisely, we will show that there are convergent difference methods of the Euler type for which the assumptions (30), (31) are omitted.

We denote by $CL(D, \mathbb{R})$ the class of all linear and continuous operators defined on $C(D, \mathbb{R})$ and taking values in $\mathbb{R}$. The norm in the space $CL(D, \mathbb{R})$ generated by the maximum norm in the space $C(D, \mathbb{R})$ will be denoted by $\| \cdot \|_\star$. Let $M_{n\times n}$ be a class of all $n \times n$ matrices with real elements. For $U \in M_{n\times n}$ we write

$$\|U\| = \max\left\{ \sum_{j=1}^{n} |u_{ij}| : \ 1 \le i \le n \right\} \ \text{ where } \ U = \left[u_{ij} \right]_{i,j=1,\ldots,n}.$$

If $U \in M_{n\times n}$ then U^T denotes the transpose matrix.

Assumption $H[f, \varphi]$**.** The functions $f : \Sigma \to \mathbb{R}$ and $\varphi : E_0 \to \mathbb{R}$ are continuous and

1) the partial derivatives $\partial_x f$, $\partial_q f$ exist on Σ and $\partial_x f$, $\partial_q f \in C(\Sigma, \mathbb{R}^n)$,

2) there exists the Fréchet derivative $\partial_w f(P)$ and $\partial_w f(P) \in CL(D, \mathbb{R})$ for $P \in \Sigma$,

3) $\varphi : E_0 \to \mathbb{R}$ is of class C^2.

Now we formulate a new class of difference methods corresponding to (1), (2). Let (z, u), $u = (u_1, \dots, u_n)$, be unknown functions of the variables $(t^{(r)}, x^{(m)}) \in E_{0.h} \cup E_h$. Write

$$P^{(r,m)}[z, u] = \big(t^{(r)}, x^{(m)}, T_h z_{[r,m]}, u^{(r,m)} \big).$$

We consider the system of difference functional equations

$$\delta_0 z^{(r,m)} = f(\, P^{(r,m)}[z, u]\,) + \sum_{i=1}^{n} \partial_{q_i} f(\, P^{(r,m)}[z, u]\,) \big(\delta_i z^{(r,m)} - u_i^{(r,m)} \big), \quad (33)$$

$$\delta_0 u^{(r,m)} = \partial_x f(\, P^{(r,m)}[z, u]\,) + \quad (34)$$

$$+ \partial_w f(\, P^{(r,m)}[z, u]\,)\, T_h u_{[r,m]} + \partial_q f(\, P^{(r,m)}[z, u]\,) \big[\, \delta u^{(r,m)} \,\big]^T$$

with the initial conditions

$$z^{(r,m)} = \varphi_h^{(r,m)}, \quad u^{(r,m)} = \psi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \quad (35)$$

where $\varphi_h : E_{0.h} \to \mathbb{R}$ and $\psi_h : E_{0.h} \to \mathbb{R}^n$ are given functions and

$$\partial_w f(\, P^{(r,m)}[z, u]\,)\, T_h u_{[r,m]} =$$

$$= \big(\partial_w f(\, P^{(r,m)}[z, u]\,) T_h (u_1)_{[r,m]}, \dots, \partial_w f(\, P^{(r,m)}[z, u]\,) T_h (u_n)_{[r,m]} \,\big),$$

and

$$\delta u = \big[\, \delta_j u_i \,\big]_{i,j=1,\dots n}.$$

The difference operator δ_0 is defined by

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} \big[z^{(r+1,m)} - z^{(r,m)} \big], \quad \delta_0 u^{(r,m)} = \frac{1}{h_0} \big[u^{(r+1,m)} - u^{(r,m)} \big]. \quad (36)$$

The difference operators $(\delta_1, \dots, \delta_n)$ are defined in the following way. Suppose that the functions (z, u) are known on the set $\big(E_{0.h} \cup E_h\big) \cap \big([-d_0, t^{(r)}] \times \mathbb{R}^n\big)$, $0 \le r < N_0$. We put

$$\text{if} \quad \partial_{q_j} f(\, P^{(r,m)}[z, u]\,) \ge 0 \quad (37)$$

then

$$\delta_j z^{(r,m)} = \frac{1}{h_j} \big[z^{(r,m+e_j)} - z^{(r,m)} \big], \quad \delta_j u^{(r,m)} = \frac{1}{h_j} \big[u^{(r,m+e_j)} - u^{(r,m)} \big]. \quad (38)$$

Moreover we put

$$\text{if } \partial_{q_j} f(\, P^{(r,m)}[z,u]\,) < 0 \tag{39}$$

then

$$\delta_j z^{(r,m)} = \frac{1}{h_j}\big[z^{(r,m)} - z^{(r,m-e_j)}\big], \;\; \delta_j u^{(r,m)} = \frac{1}{h_j}\big[u^{(r,m)} - u^{(r,m-e_j)}\big]. \tag{40}$$

We take $j = 1, \dots, n$ in (37)–(40). The above difference functional problem is called a generalized Euler method for (1), (2). It is clear that there exists exactly one solution $(z_h, u_h) : E_{0.h} \cup E_h \to \mathbb{R}^{1+n}$, $u_h = (u_{h.1}, \dots, u_{h.n})$, of (33)–(35) with δ_0 and δ defined by (36)–(40).

The generalized Euler method is obtained in the following way. Suppose that Assumption $H[f, \varphi]$ is satisfied. The method of quasilinearization for nonlinear equations consists in replacing problem (1), (2) with the following one. Let (z, u), $u = (u_1, \dots, u_n)$, be unknown functions of the variables $(t, x) \in E_0 \cup E$. First we introduce an additional unknown function $u = \partial_x z$ in (1). Then we consider the following linearization of (1) with respect to u:

$$\partial_t z(t,x) = f(t, x, z_{(t,x)}, u(t,x)) + \tag{41}$$

$$+ \sum_{i=1}^{n} \partial_{q_i} f(t, x, z_{(t,x)}, u(t,x)) \big(\partial_{x_i} z(t,x) - u_i(t,x)\big).$$

We get differential functional equations for u by differentiating equation (1), resulting is the following:

$$\partial_t u(t,x) = \partial_x f(t, x, z_{(t,x)}, u(t,x)) + \tag{42}$$

$$+ \partial_w f(t, x, z_{(t,x)}, u(t,x))\, u_{(t,x)} + \partial_q f(t, x, z_{(t,x)}, u(t,x)) \,\big[\, \partial_x u(t,x)\,\big]^T$$

where $u_{(t,x)} = ((u_1)_{(t,x)}, \dots, (u_n)_{(t,x)})$. We consider the following initial condition for (41), (42):

$$z(t,x) = \varphi(t,x), \;\; u(t,x) = \partial_x \varphi(t,x) \;\text{ on }\; E_0. \tag{43}$$

Under natural assumptions on given functions the above problem has the following properties:

(i) if $(\tilde z, \tilde u)$ is a classical solution of (41)–(43) then $\partial_x \tilde z = \tilde u$ and $\tilde z$ is a solution of (1), (2);

(ii) if $\tilde v$ is a solution of (1), (2) and $\tilde v$ is of class C^2 then $(\tilde v, \partial_x \tilde v)$ satisfies (41)–(43).

Difference problem (33)–(35) is a discretization of (41)–(43).

The above method of quasilinearization was first proposed in a nonfunctional setting by S. Cinquini and S. Cinquini Cibraio [7, 8]. It was extended in [4, 12] on nonlinear functional differential problems.

We claim that the generalized Euler method for (1), (2) is a particular case of (3), (4). Put $X = \mathbb{R}^{1+n}$. The norm $\| \cdot \|_X$ where $X = \mathbb{R}^{1+n}$ is denoted by $\| \cdot \|_{1+n}$. For $p \in \mathbb{R}^{1+n}$ where $p = (p_0, p')$, $p' \in \mathbb{R}^n$ we put $\|p\|_{1+n} = |p_0| + \|p'\|$. For $w \in \mathbb{F}(\Omega_h, \mathbb{R}^{1+n})$, $w = (\zeta, \eta)$, $\eta = (\eta_1, \ldots, \eta_n)$ we write

$$Q^{(r,m)}[w] = (t^{(r)}, x^{(m)}, T_h[\zeta|_{D_h}], \eta^{(0,\theta)})$$

and

$$\|\zeta|_{A_h}\| = \max\{|\zeta^{(r,m)}| : \ (t^{(r)}, x^{(m)}) \in A_h\},$$

$$\|\eta|_{A_h}\| = \max\{\|\eta^{(r,m)}\| : \ (t^{(r)}, x^{(m)}) \in A_h\}.$$

Consider the operator $F_h = (F_{h.0}, F_{h.I})$ defined by

$$F_{h.0}[w]^{(r,m)} = \zeta^{(0,\theta)} + h_0 f(Q^{(r,m)}[w]) + \tag{44}$$

$$+h_0 \sum_{i=1}^{n} \partial_{q_i} f(Q^{(r,m)}[w]) \big(\delta_i \zeta^{(0,\theta)} - \eta_i^{(0,\theta)}\big)$$

and

$$F_{h.I}[w]^{(r,m)} = \eta^{(0,\theta)} + h_0 \partial_x f(Q^{(r,m)}[w]) + \tag{45}$$

$$+h_0 \partial_w f(Q^{(r,m)}[w]) \, T_h[\eta|_{D_h}] + h_0 \partial_q f(Q^{(r,m)}[w]) \big[\, \delta \eta^{(0,\theta)} \,\big]^T,$$

where

$$\partial_w f(Q^{(r,m)}[w]) \, T_h[\eta|_{D_h}] =$$

$$= \big(\partial_w f(Q^{(r,m)}[w]) \, T_h[\eta_1|_{D_h}], \ldots, \partial_w f(Q^{(r,m)}[w]) \, T_h[\eta_n|_{D_h}]\big).$$

The difference expressions

$$(\delta_1 \zeta^{(0,\theta)}, \ldots, \delta_n \zeta^{(0,\theta)}) \ \text{ and } \ \delta \eta^{(0,\theta)} = \big[\delta_j \eta_i^{(0,\theta)}\big]_{i,j=1,\ldots,n}$$

are defined by (37)–(40) with $\zeta|_{D_h}$ and $\eta^{(0,\theta)}$ instead of $z_{[r,m]}$ and $u^{(r,m)}$.

Write $Z = (z, u)$, $\Psi_h = (\varphi_h, \psi_h)$, $Z_{\langle r,m \rangle} = (z_{\langle r,m \rangle}, u_{\langle r,m \rangle})$. Then problem (33)–(35) is equivalent to the functional difference equation

$$Z^{(r+1,m)} = F_h[Z_{\langle r,m \rangle}]^{(r,m)} \tag{46}$$

with the initial condition

$$Z^{(r,m)} = \Psi_h^{(r,m)} \ \text{ on } \ E_{0.h}. \tag{47}$$

Assumption $\tilde{H}[\sigma]$**.** The function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions

1) σ is continuous and it is nondecreasing with respect to the both variables,

2) $\sigma(t,0) = 0$ for $t \in [0,a]$ and for each $b \in \mathbb{R}_+$, and $c \geq 1$ the maximal solution of the Cauchy problem

$$\omega'(t) = b\,\omega(t) + c\,\sigma(t,\omega(t)), \;\; \omega(0) = 0,$$

is $\tilde{\omega}(t) = 0$ for $t \in [0,a]$.

Assumption $\tilde{H}[f,\varphi]$. The functions f and φ satisfy Assumption $H[f,\varphi]$ and

1) there is $A \in \mathbb{R}_+$ such that

$$\|\partial_x f(P)\|, \;\; \|\partial_q f(P)\|, \;\; \|\partial_w f(P)\|_\star \leq A$$

where $P \in \Sigma$,

2) there exists $\sigma : [0,a]\times\mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $\tilde{H}[\sigma]$ is satisfied and the terms

$$\|\partial_x f(t,x,w,q) - \partial_x f(t,x,\bar{w},\bar{q})\|, \;\; \|\partial_q f(t,x,w,q) - \partial_q f(t,x,\bar{w},\bar{q})\|,$$

$$\|\partial_w f(t,x,w,q) - \partial_w f(t,x,\bar{w},\bar{q})\|_\star,$$

are bounded from above by $\sigma(t, \|w-\bar{w}\|_D + \|q-\bar{q}\|)$ on Σ.

Theorem 4. *Suppose that Assumption* $\tilde{H}[f,\varphi]$ *is satisfied and*

1) $h \in H$ *and condition* (32) *holds,*

2) $(z_h,u_h) : E_{0.h} \cup E_h \to \mathbb{R}^{1+n}$ *is the solution of* (33)–(35) *with* δ_0 *and* δ *defined by* (36)–*40 and there is* $\alpha_0 : H \to \mathbb{R}_+$ *such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| + \|\partial_x\varphi^{(r,m)} - \psi_h^{(r,m)}\| \leq \alpha_0(h) \;\; on \;\; E_{0.h}$$

and $\lim_{h\to 0}\alpha_0(h) = 0$,

3) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of* (1), (2) *and* v *is of class* C^2 *and the functions*

$$\partial_t v, \; \partial_{x_i} v, \; \partial_{tt} v, \; \partial_{tx_i} v, \; \partial_{x_ix_j} v, \; i,j = 1,\ldots,n,$$

are bounded on $E_0 \cup E$*. Then there is* $\alpha : H \to \mathbb{R}_+$ *such that*

$$|z_h^{(r,m)} - v_h^{(r,m)}| + \|u_h^{(r,m)} - (\partial_x v)_h^{(r,m)}\| \leq \alpha(h) \;\; on \;\; E_h \;\; and \;\; \lim_{h\to 0}\alpha(h) = 0 \tag{48}$$

where v_h *and* $(\partial_x v)_h$ *are the restrictions of* v *and* $\partial_x v$ *to the set* $E_{0.h} \cup E_h$.

Proof. We apply Theorem 1 to prove (48). Suppose that $X = \mathbb{R}^{1+n}$ and $F_h = (F_{h.0}, F_{h.I})$ is given by (44), (45). Then $Z_h = (z_h, u_h)$ satisfies (46), (47). Write $V_h = (v_h, (\partial_x v)_h)$. Then the initial estimate

$$\|V_h^{(r,m)} - \Psi_h^{(r,m)}\|_{1+n} \leq \alpha_0(h) \text{ on } E_{0.h}$$

is satisfied. It follows from Lemma 2 that there is $\gamma : H \to \mathbb{R}_+$ such that

$$\| V_h^{(r+1,m)} = F_h[(V_h)_{\langle r,m\rangle}]^{(r,m)}\|_{1+n} \leq \gamma(h) \text{ on } E'_h \text{ and } \lim_{h\to 0}\gamma(h) = 0.$$

Let $\tilde{c} \in \mathbb{R}_+$ be defined by the relations

$$\|\partial_x v(t,x)\| \leq \tilde{c}, \ \ \|\partial_{xx} v(t,x)\| \leq \tilde{c} \text{ on } E_0 \cup E.$$

Write $Y_h = \{w = (\zeta, \eta) \in \mathbb{F}(\Omega_h, \mathbb{R}^{1+n}) : \|\delta\zeta^{(0,\theta)}\| \leq \tilde{c}, \|\delta\eta^{(0,\theta)}\| \leq \tilde{c}\}$. Then we have

$$(V_h)_{\langle r,m\rangle} = \big((v_h)_{\langle r,m\rangle}, (\,(\partial_x v)_h\,)_{\langle r,m\rangle} \in Y_h, \ \ 0 \leq r \leq N_0, \ m \in \mathbb{Z}^n.$$

Now we construct an estimate for the function

$$F_h[w] - F_h[\bar{w}] = \big(F_{h.0}[w] - F_{h.0}[\bar{w}], F_{h.I}[w] - F_{h.I}[\bar{w}]\big)$$

where $w \in \mathbb{F}(\Omega_h, \mathbb{R}^{1+n})$, $\bar{w} \in Y_h$ and $w = (\zeta, \eta)$, $\bar{w} = (\bar{\zeta}, \bar{\eta})$. It follows from (36)–(40) and from condition 2) of Assumption $\tilde{H}[f, \varphi]$ that

$$\big|F_{h.0}[w] - F_{h.0}[\bar{w}]\big| \leq$$

$$\leq \|(\zeta - \bar{\zeta})|_{A_h}\| + 2h_0 A\|(w - w)|_{D_h}\|_{1+n} + 2h_0\tilde{c}\sigma(t^{(r)}, \|(w - \bar{w})|_{D_h}\|_{1+n})$$

and

$$\big|F_{h.I}[w] - F_{h.I}[\bar{w}]\big| \leq$$

$$\leq \|(\eta - \bar{\eta})|_{A_h}\| + h_0 A\|(w - w)|_{D_h}\|_{1+n} + (1 + 2\tilde{c})h_0\tilde{c}\sigma(t^{(r)}, \|(w - \bar{w})|_{D_h}\|_{1+n}).$$

Adding the above inequalities we obtain the following final inequality

$$\|F_h[w]^{r,m)} - F_h[\bar{w}]^{(r,m)}\| \leq \|(w - \bar{w})|_{A_h}\|_{1+n} +$$

$$+ h_0(1 + 4\tilde{c})\sigma(t^{(r)}, \|(w - \bar{w})|_{D_h}\|_{1+n}) + 3Ah_0\|(w - \bar{w})|_{D_h}\|_{1+n}.$$

Thus we see that all the assumptions of Theorem 1 are satisfied and assertion (48) follows. This completes the proof.

Remark 2. Suppose that: 1) all the assumptions of Theorem 4 are satisfied with $\sigma(t,\tau) = L_0\tau$ on $[0,a] \times \mathbb{R}_+$ where $L_0 \in \mathbb{R}_+$; 2) the conditions 2), 3) of Lemma 4 are satisfied. Then there are C_0, $C_1 \in \mathbb{R}_+$ such that

$$|(v_h - z_h)^{(r,m)}| + \|u_h^{(r,m)} - (\partial_x v)_h^{(r,m)}\| \leq C_0\alpha_0(h) + C_0 h_0 \text{ on } E_h.$$

The proof of the above property of the generalized Euler method is similar to the proof of Lemma 4.

Remark 3. In the results on error estimates we need estimates for the derivatives of the solution of problem (1), (2). One may obtain them by the method of differential inequalities. The results given in [3, 12] for initial problems on the Haar pyramid can be easily extended to initial problems with solutions given on unbounded domains.

5. NUMERICAL EXAMPLES

Put $h = (h_0, h_1, h_2)$ and $t^{(r)} = rh_0$, $(x^{(m_1)}, y^{(m_2)}) = (m_1h_1, m_2h_2)$ where $r \in \mathbb{N}$, $(N_1, N_2) \in \mathbb{Z}^2$. Write

$$\Omega_h = \{(t^{(r)}, x^{(m_1)}, y^{(m_2)}) : \ 0 \leq r \leq N_0,$$
$$(-N_1 + r, -N_2 + r) \leq (m_1, m_2) \leq (N_1 - r, N_2 - r)\}$$

where $N_0h_0 = a$, $(N_1h_1, N_2h_2) = (b_1, b_2)$ and $N_1 > N_0$, $N_2 > N_0$.

Example 1. Put $n = 2$. Consider the differential equation with deviated variables

$$\partial_t z(t,x,y) = \cos\big[\partial_x z(t,x,y) + \partial_y z(t,x,y)\big] + \tag{49}$$
$$+(x-y)z(t,x,y) + z(0.5t,x,y) - z(t,0.5x,0.5y) - 1$$

and the initial condition

$$z(0,x,y) = 1 \ \text{ for } \ (x,y) \in \mathbb{R}^2. \tag{50}$$

The function $v(t,x,y) = \exp[t(x-y)]$ is the solution of (49), (50). Let $z_h : \Omega_h \to \mathbb{R}$ denote the function which is obtained by using the Lax difference scheme for (49), (50). Write

$$M(r) = \frac{1}{[2(N_1 - r) + 1][2(N_2 - r) + 1]}$$

and

$$\varepsilon_h^{(r)} = \frac{1}{M(r)} \sum_{\nu=-N_1+r}^{N_1-r} \sum_{\mu=-N_2+r}^{N_2-r} \big|(z_h - v_h)^{(r,\nu,\mu)}\big|, \ \ 0 \leq r \leq N_0, \tag{51}$$

where v_h is the restriction of v to the set Ω_h. The numbers $\varepsilon_h^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$.

In the table 1 we give experimental values of the function ε_h for the following parameters.

I. $a = 0.15$, $b_1 = b_2 = 3$, $h_0 = 10^{-4}$, $h_1 = h_2 = 10^{-3}$. The corresponding errors are denoted by $\tilde{\varepsilon}_h$.

II. $a = 0.4$, $b_1 = b_1 = 1$, $h_0 = 10^{-3}$, $h_1 = h_2 = 2.5 \cdot 10^{-3}$. The corresponding errors are denoted by $\bar{\varepsilon}_h$.

Table 1 of errors

$t^{(r)}$	$\tilde{\varepsilon}_k^{(r)}$	$t^{(r)}$	$\bar{\varepsilon}_h^{(r)}$
0.095	$3.09 \cdot 10^{-5}$	0.20	$1.54 \cdot 10^{-4}$
0.100	$3.24 \cdot 10^{-5}$	0.22	$1.70 \cdot 10^{-4}$
0.105	$3.41 \cdot 10^{-5}$	0.24	$2.11 \cdot 10^{-4}$
0.110	$3.59 \cdot 10^{-5}$	0.26	$2.87 \cdot 10^{-4}$
0.115	$3.78 \cdot 10^{-5}$	0.28	$4.00 \cdot 10^{-4}$
0.120	$3.99 \cdot 10^{-5}$	0.30	$5.35 \cdot 10^{-4}$
0.125	$4.19 \cdot 10^{-5}$	0.32	$6.85 \cdot 10^{-4}$
0.130	$4.40 \cdot 10^{-5}$	0.34	$8.51 \cdot 10^{-4}$
0.135	$4.60 \cdot 10^{-5}$	0.36	$1.03 \cdot 10^{-3}$
0.140	$4.80 \cdot 10^{-5}$	0.38	$1.23 \cdot 10^{-3}$

The results shown in the table are consistent with our mathematical analysis.

Example 2. Consider the differential integral equation

$$\partial_t z(t,x,y) = -\int_0^t z(\tau,x,y)d\tau + \cos x\, \cos y + \tag{52}$$

$$+\arctan\Big[2\partial_x z(t,x,y) + 2\partial_y z(t,x,y) - \int_{-x}^{x} z(t,s,y)ds - \int_{-y}^{y} z(t,x,s)ds\Big]$$

and the initial condition

$$z(0,x,y) = 0 \;\text{ for }\; (x,y) \in \mathbb{R}^2. \tag{53}$$

The solution of the above problem is known, it is $v(t,x,y) = \sin t\, \cos x\, \cos y$. Let $z_h : \Omega_h \to \mathbb{R}$ denote the function which is obtained by using the Lax difference scheme for (52), (53). We consider the errors ε_h defined by (51) for the above problem.

In the table 2 we give experimental values of the function ε_h for the following parameters.

I. $a = 0.5$, $b_1 = b_2 = 2$, $h_0 = 10^{-3}$, $h_1 = h_2 = 4\cdot 10^{-3}$. The corresponding errors are denoted by $\tilde{\varepsilon}_h$.

II. $a = 0.25$, $b_1 = b_1 = 1$, $h_0 = 5\cdot 10^{-4}$, $h_1 = h_2 = 2\cdot 10^{-3}$. The corresponding errors are denoted by $\bar{\varepsilon}_h$.

Table 2 of errors

$t^{(r)}$	$\tilde{\varepsilon}_h$	$t^{(r)}$	$\bar{\varepsilon}_h$
0.18	$1.43\cdot 10^{-4}$	0.11	$1.64\cdot 10^{-4}$
0.21	$2.00\cdot 10^{-4}$	0.12	$2.11\cdot 10^{-4}$
0.24	$2.67\cdot 10^{-4}$	0.13	$2.66\cdot 10^{-4}$
0.27	$3.47\cdot 10^{-4}$	0.14	$3.31\cdot 10^{-4}$
0.30	$4.38\cdot 10^{-4}$	0.15	$4.05\cdot 10^{-4}$
0.33	$5.41\cdot 10^{-4}$	0.16	$4.90\cdot 10^{-4}$
0.36	$6.58\cdot 10^{-4}$	0.17	$5.86\cdot 10^{-4}$
0.39	$7.88\cdot 10^{-4}$	0.18	$6.95\cdot 10^{-4}$
0.42	$9.33\cdot 10^{-4}$	0.19	$8.16\cdot 10^{-4}$
0.45	$1.09\cdot 10^{-3}$	0.20	$9.51\cdot 10^{-4}$

The results shown in the table are consistent with our mathematical analysis.

Example 3. Consider the differential integral equation

$$\partial_t z(t,x) = \sin\Big[\partial_x z(t,x) - \frac{1}{2}\int_{-x}^{x} z(t,s)ds\Big] + \cos x - \int_0^t z(\tau,x)d\tau, \qquad (54)$$

and the initial condition

$$z(0,x) = 0 \ \text{ for } \ x \in \mathbb{R}. \qquad (55)$$

The function $v(t,x) = \sin t\cos x$ is the solution of the above problem.

Put $h = (h_0, h_1)$ and $t^{(r)} = rh_0$, $x^{(m)} = mh_1$, where $r \in \mathbb{N}$, $m \in \mathbb{Z}$. Write

$$\Sigma_h = \{(r^{(r)}, x^{(m)}) : \ 0 \le r \le N_0, \ -N + r \le m \le N - r\}$$

where $N_0 h_0 = a$, $Nh_1 = b$ and $N > N_0$. Let $z_h : \Sigma_h \to \mathbb{R}$ denote the function which is obtained by using the Lax difference scheme for (54), (55). Write

$$\varepsilon_h^{(r)} = \frac{1}{2(N-r)+1}\sum_{j=-N+r}^{N-r} |(z_h - v_h)^{(r,j)}|, \ 0 \le r \le N_0.$$

where v_h is the restriction of v to the set Σ_h.

Consider the generalized Euler method to problem (54), (55) and its solution $(\tilde{z}_h, \tilde{u}_h) : \Sigma_h \to \mathbb{R}^2$. Write

$$\tilde{\varepsilon}_h^{(r)} = \frac{1}{2(N-r)+1} \sum_{j=-N+r}^{N-r} |(\tilde{z}_h - v_h)^{(r,j)}|, \; 0 \le r \le N_0.$$

The numbers $\varepsilon_h^{(r)}$ and $\tilde{\varepsilon}_h^{(r)}$ are the arithmetical means of errors with fixed $t^{(r)}$, $0 \le r \le N_0$.

We have solved problem (54), (55) for the following sets of parameters: $a = 0.5$, $b = 6$, $h_0 = 5{\cdot}10^{-4}$, $h_1 = 5{\cdot}10^{-3}$. In the Table 3 we give experimental values of the errors ε_h and $\tilde{\varepsilon}_h$.

Table 3 of errors

$t^{(r)}$	$\tilde{\varepsilon}_h$	ε_h
0.200	$3.47 \cdot 10^{-5}$	$1.32 \cdot 10^{-2}$
0.225	$4.29 \cdot 10^{-5}$	$1.61 \cdot 10^{-2}$
0.250	$5.17 \cdot 10^{-5}$	$1.89 \cdot 10^{-2}$
0.275	$6.04 \cdot 10^{-5}$	$2.14 \cdot 10^{-2}$
0.300	$6.85 \cdot 10^{-5}$	$2.37 \cdot 10^{-2}$
0.325	$7.60 \cdot 10^{-5}$	$2.57 \cdot 10^{-2}$
0.350	$8.39 \cdot 10^{-5}$	$2.75 \cdot 10^{-2}$
0.375	$9.29 \cdot 10^{-5}$	$2.93 \cdot 10^{-2}$
0.400	$1.05 \cdot 10^{-4}$	$3.02 \cdot 10^{-2}$
0.425	$1.26 \cdot 10^{-4}$	$3.09 \cdot 10^{-2}$

Note that $\tilde{\varepsilon}_h < \varepsilon_h$ for all the values r. Thus we see that the errors of the classical Lax difference scheme are larger than the errors of the generalized Euler method. This is due to the fact that approximation of the spatial derivaives of z in the generalized Euler method is better that the respective approximation of $\partial_x z$ in a classical case.

Remark 4. Results presented in the paper can be extended on weakly coupled differential functional systems.

Acknowledgements. The numerical computations reported in the paper were performed at the Academic Computer Center TASK in Gdańsk.

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СТІЙКІСТЬ НЕЛІНІЙНИХ РІЗНИЦЕВИХ ФУНКЦІОНАЛЬНИХ РІВНЯНЬ У НЕОБМЕЖЕНИХ ОБЛАСТЯХ

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Доведено теорему про оцінки похибок для наближених розв'язків різницево-функціональних рівнянь типу Вольтерра. Такі оцінки виражено за допомогою розв'язку початкової задачі для нелінійного диференціального рівняння.

Отриманий результат застосовано для дослідження стійкості різницевих схем, породжених початковими задачами для гіперболічних диференціально-функціональних рівнянь. Наведено приклади числових розрахунків.