ON MINIMUM MODULUS OF BOUNDED ANALYTIC FUNCTIONS

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We describe asymptotic behavior of nonvanishing bounded analytic functions in the unit disk in terms of the moduli of continuity of the Stieltjes measures from their representation. Under Frostman's type conditions new lower estimates for Blaschke products is found.

1. INTRODUCTION

Let $U = \{z \in \mathbb{C} : |z| < 1\}$. For an analytic function g on U we define the minimum modulus $\mu(r, g) = \min\{|g(z)| : |z| = r\}$, and the maximum modulus $M(r, g) = \max\{|g(z)| : |z| = r\}$, $0 < r < 1$.

The following result due to M. Heins [10] is known

Theorem A. *If f is analytic in U, $f \not\equiv const$, f is bounded in U then there exist a constant $K > 0$ and a sequence (r_n), $r_n \uparrow 1$ such that*

$$\ln \mu(r_n, f) \geq -\frac{K}{1 - r_n}, \quad n \to +\infty. \tag{1}$$

For the function $f(z) = \exp\left\{\frac{1}{z-1}\right\}$ we have $\log M(r, f) = O(1)$, $\ln \mu(r, f) = -\frac{1}{1-r}$, $r \uparrow 1$. Thus, inequality (1) is sharp in the class of bounded analytic functions in the unit disk.

Brothers Riesz' theorem [9, Theorem 6.13] gives a very useful representation for bounded analytic functions in U.

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Theorem B. *An analytic function f in U is bounded if and only if f has the form*

$$f(z) = z^p \prod_{k=1}^{\infty} \frac{\overline{a_k}(a_k - z)}{|a_k|(1 - \overline{a_k} z)} \exp\left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\psi(t) + iC \right\} \equiv B(z)g(z) \tag{2}$$

where ψ is a non-decreasing function on $[-\pi, \pi]$, $C \in \mathbb{R}$, $p \in \mathbb{Z}_+$, $B(z)$ is a Blaschke product constructed by the zeros a_k of f, $0 < |a_k| < 1$, $\sum_k (1 - |a_k|) < \infty$.

In this paper we consider separately behavior of factors from (2), namely, a nonvanishing analytic function in U and a Blaschke product.

In the case when g has no zeros in U, the following theorem is known [12, Theorem 3.2.3]

Theorem C. *Let $g(z)$ be analytic and nonvanishing in U, $|g(z)| < 1$, $z \in U$. Then*

$$\lim_{r \to 1} (1 - r) \ln \mu(r, g) = -2\beta_0,$$

where $\beta_0 \equiv \max_k \left\{ \frac{h_k}{2\pi} \right\}$, $\{h_k\}$ are jumps of the function ψ, ψ is of bounded variation on $[-\pi, \pi]$.

For a function $\psi \colon [-\pi, \pi] \to \mathbb{R}$ of bounded variation on $[-\pi, \pi]$ we define the modulus of continuity $\omega(\tau, \varphi; \psi) = \sup\{|\psi(x_1) - \psi(x_2)| : |x_i - \varphi| < \tau, x_i \in [-\pi; \pi], i = 1, 2\}$, $\omega(\tau; \psi) = \sup_\varphi \omega(\tau, \varphi; \psi)$.

It seems reasonable to ask in which classes inequality (1) can be improved. The assertion of Theorem C corresponds to the case when $\omega(\tau, \psi) \asymp 1$, $\tau \downarrow 0$. Results of [3, 4, 6, 11] allow us to assume that it is the classes defined by restrictions on the modulus of continuity of ψ.

On the other hand, for a Blaschke product the following result due to O.Frostman [2, 8] is known.

Theorem D. *In order that*

$$\lim_{r \uparrow 1} f(re^{i\theta_0}) = L \tag{3}$$

and $|L| = 1$ for $f = B$, and every subproduct of $B(z)$, it is necessary and sufficient that

$$\sum_{k=1}^{\infty} \frac{1 - |a_k|}{|e^{i\theta_0} - a_k|} < \infty. \tag{4}$$

Strengthening (4) one can obtain additional information on asymptotic behavior of f (see e.g. [5, 7]). If condition (4) is not satisfied counterparts of Frostman's results are not known. The main results of the paper are necessary and sufficient conditions on minimum modulus of a nonvanishing bounded analytic function on U, that generalizes Theorem C, and new lower estimates for the logarithm of modulus of a Blaschke product complementing Theorem D. These results complements those obtained in [13].

2. MINIMUM MODULUS OF A NONVANISHING ANALYTIC FUNCTION

Theorem 1. *Let $\nu : [0;1] \to \mathbb{R}_+$, $\nu \nearrow$, $g(z)$ be a nonvanishing analytic function in U of the form (2). Then*

1) If $\exists K_1 > 0$, $\exists E \subset (0;1]$: 0 is a limit point of E, and $\omega(\tau;\psi) \geq K_1\nu(\tau)$, $\tau \in E$, then

$$\exists K_2 > 0 : \quad \ln \mu(1-\tau, g) \leq -K_2 \cdot \frac{\nu(\tau)}{\tau}, \quad \tau \to 0, \ \tau \in E.$$

2) If $\exists K_2 > 0$, $\exists F \subset [0;1)$: 1 is a limit point of F, $\ln|\mu(r,g)| \leq -K_2 \cdot \frac{\nu(1-r)}{1-r}$, $r \in F$, then $\exists K_1 > 0$:

$$\int\limits_{1-r}^{\pi} \frac{\omega(\tau;\psi)}{\tau^2} d\tau \geq K_1 \frac{\nu(1-r)}{1-r}, \quad r \in F, \quad r \to 1.$$

It immediately follows from Theorem 1 that

Corollary 1. *Let $\nu : [0;1] \to \mathbb{R}_+$, $\nu \nearrow$, $g(z)$ be a nonvanishing analytic function on U of the form (2). Suppose that*

$$\int\limits_{\delta}^{\pi} \frac{\omega(\tau;\psi)}{\tau^2} d\tau = O\Big(\frac{\omega(\delta;\psi)}{\delta}\Big), \quad \delta \downarrow 0.$$

If $E \subset (0;1]$ such that 0 is a limit point of E, then $\exists K_1 > 0$: $\omega(\tau;\psi) \geq K_1\nu(\tau)$, $\tau \in E$, $\tau \to 0$ if and only if $\exists K_2 > 0$:

$$\ln \mu(1-\tau, g) \leq -K_2 \cdot \frac{\nu(\tau)}{\tau}, \quad \tau \to 0, \quad \tau \in E.$$

For a function ψ of bounded variation on $[-\pi, \pi]$ we define $\tau[\psi] = \sup\{\gamma \geq 0 : \psi \in \Lambda_\gamma\}$, where Zygmund's class Λ_γ consists of ψ satisfying $\omega(t;\psi) = O(t^\gamma)$, $t \to 0+$.

Corollary 2. *Let $g \not\equiv const$ be defined by* (2)*, ψ be nondecreasing, $\tau[\psi] = \gamma$. Then*

$$\rho_\mu[g] \stackrel{\text{def}}{=} \varlimsup_{r\uparrow 1} \frac{\ln(-\ln \mu(r,g))}{-\ln(1-r)} = 1-\gamma. \tag{5}$$

Proof of Theorem 1. 1) Let $\omega(\tau;\psi) \geq K_1\nu(\tau)$. Then for an arbitrary $\varepsilon > 0$ there exists $\varphi = \varphi(\varepsilon, \tau)$ such that $\omega(\tau, \varphi; \psi) > (K_1-\varepsilon)\nu(\tau)$. Therefore, for $z = re^{i\varphi}$, $r = 1-\tau$, we have (see [4, p. 143])

$$-\ln\left|g(re^{i\varphi})\right| \geq \frac{1}{2\pi}(1-r^2)\int\limits_{\varphi-\tau}^{\varphi+\tau} \frac{d\psi(t)}{|e^{it}-re^{i\varphi}|^2}.$$

Since $\left|e^{it}-re^{i\varphi}\right| \leq \sqrt{2}\tau$, $|\varphi - t| < \tau$,

$$-\ln\left|g(re^{i\varphi})\right| \geq \frac{1+r}{4\pi\tau}\int\limits_{\varphi-\tau}^{\varphi+\tau} d\psi(t).$$

Note that $\int\limits_{\varphi-\tau}^{\varphi+\tau} d\psi(t) = \psi(\varphi+\tau) - \psi(\varphi-\tau) = \omega(\tau, \varphi; \psi)$, hence

$$\ln\left|g(re^{i\varphi})\right| \leq -\frac{1+r}{4\pi\tau}\omega(\tau,\varphi;\psi).$$

Since $\omega(\tau, \varphi; \psi) \geq (K_1-\varepsilon)\nu(\tau)$ for $\tau = 1-r \in E$, we obtain

$$\ln\mu(r,g) \leq \ln\left|g(re^{i\varphi})\right| \leq -\frac{1+r}{4\pi(1-r)}\nu(1-r)(K_1-\varepsilon) \leq -K_2\frac{\nu(1-r)}{1-r},$$

where $K_2 < \frac{K_1}{2\pi}$ as $r \to 1$, $1-r \in E$.

2) Let $\mu(r,g) = |g(re^{i\varphi})|$. Then, by the assumptions of the theorem

$$\exists K_2 > 0 : \ln\left|g(re^{i\varphi})\right| \leq -K_2 \cdot \frac{\nu(1-r)}{1-r}.$$

It is well-known [1] that

$$\left|\frac{\partial}{\partial t}P(r,t)\right| \leq \frac{\pi^2}{t^2}, \quad \left|\frac{\partial}{\partial t}P(r,t)\right| \leq \frac{2}{(1-r)^2} \quad \text{if } r \geq \frac{1}{2},\ t \in [-\pi;\pi]. \tag{6}$$

We expand ψ on $\mathbb{R}$ by the formula $\psi(t+2\pi) - \psi(t) = \psi(2\pi) - \psi(0)$. The standard arguments yield (see [6, p. 38])

$$-\ln\left|g(re^{i\varphi})\right| = \int\limits_{-\pi+\varphi}^{\pi+\varphi} P(r,\theta)d(\psi(\theta)-\psi(\varphi)) =$$
$$= (\psi(\theta)-\psi(\varphi))P(r,\theta-\varphi)\Big|_{-\pi+\varphi}^{\pi+\varphi} -$$
$$-\int\limits_{-\pi+\varphi}^{\pi+\varphi} \frac{\partial}{\partial\theta}(P(r,\theta-\varphi))\cdot(\psi(\theta)-\psi(\varphi))d\theta =$$
$$= (\psi(2\pi)-\psi(0))\cdot P(r,\pi) - \int\limits_{-\pi}^{\pi} \frac{\partial}{\partial t}P(r,\tau)\cdot(\psi(\tau+\varphi)-\psi(\varphi))d\tau.$$

Using (6), we obtain $(r\to 1)$

$$-\ln\left|g(re^{i\varphi})\right| \le \frac{C_1(\psi)(1-r)}{1+r} +$$
$$+\left(\int\limits_{|\tau|\le 1-r} + \int\limits_{1-r\le|\tau\le\pi|}\right)\left|\frac{\partial}{\partial t}P(r,\tau)\right|\omega(|\tau|,\psi;\varphi)d\tau \le$$
$$\le o(1) + 2\int\limits_{|\tau|\le 1-r} \frac{\omega(|\tau|;\psi)}{(1-r)^2}d\tau + \int\limits_{1-r\le|\tau|\le\pi} \frac{\pi^2}{\tau^2}\omega(|\tau|;\psi)d\tau \le$$
$$\le o(1) + 4\frac{\omega(1-r;\psi)}{1-r} + 2\pi^2\int\limits_{1-r\le|\tau|\le\pi} \frac{\omega(\tau;\psi)}{\tau^2}d\tau \le K_3\int\limits_{1-r\le|\tau|\le\pi} \frac{\omega(\tau;\psi)}{\tau^2}d\tau,$$

where $K_3 > 4+2\pi^2$. By our assumption $\ln\left|g(re^{i\varphi})\right| \le -K_2\frac{\nu(1-r)}{1-r}$, $r\in F$. Then

$$K_3\int\limits_{1-r\le|\tau|\le\pi} \frac{\omega(\tau;\psi)}{\tau^2}d\tau \ge K_2\cdot\frac{\nu(1-r)}{1-r}, \quad r\in F, r\to 1.$$

Hence, 2) holds with $K_1 < K_2/(4+2\pi^2)$. Theorem 1 is proved.

Proof of Corollary 2. It is clear that $0\le\gamma\le 1$. First, we suppose that $0<\gamma<1$. We take an arbitrary ε, $0<\varepsilon<1-\gamma$. By the definition of $\tau[\psi]$, there exists a sequence (τ_n), $\tau_n\downarrow 0$ such that $\omega(\tau_n,\psi)\ge\tau_n^{\gamma+\varepsilon}$ $(n\to\infty)$. According to item 1 of the theorem

$$\ln\mu(1-\tau_n,g) \le -K_2\tau_n^{\gamma+\varepsilon-1}, \quad n\to+\infty.$$

Hence

$$\rho_\mu[g] \geq 1 - \gamma - \varepsilon. \tag{7}$$

Suppose that $\rho_\mu[g] > 1 - \gamma + \varepsilon$, $\gamma > \varepsilon > 0$. Then

$$\ln \mu(r_n, g) \leq -\frac{1}{(1-r_n)^{1-\gamma+\varepsilon}}, \quad n \to +\infty. \tag{8}$$

But $\psi \in \Lambda_{\gamma-\varepsilon/3}$. Therefore, for some $r_0 \in [0,1)$ we have

$$\int\limits_{1-r}^{\pi} \frac{\omega(\tau,\psi)}{\tau^2} d\tau \leq \int\limits_{1-r}^{r_0} \frac{\tau^{\gamma-\varepsilon/2}}{\tau^2} d\tau + \int\limits_{r_0}^{\pi} \frac{\omega(\tau,\psi)}{\tau^2} d\tau = O\Big(\frac{1}{(1-r)^{1-\gamma+\frac{\varepsilon}{2}}}\Big), \ \ r \uparrow 1.$$

Choosing $\nu(t) = t^{\gamma-\varepsilon}$ we see that it contradicts (8) in view of item 2 of the theorem. Consequently, $\rho_\mu[g] = 1 - \gamma$ for $0 < \gamma < 1$.

If $\gamma = 0$ the proof (7) is the same, and (8) follows from Theorem A.

If $\gamma = 1$ the assertion of the corollary immediately follows from (8).

3. LOWER ESTIMATES OF A BLASCHKE PRODUCT

Let

$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{a}_k(a_k - z)}{1 - \overline{a}_k z}, \tag{9}$$

$0 < |a_k| < 1$, $\sum_k (1 - |a_k|) < \infty$. It is suitable for us to omit a factor $|a_k|$ in the denominator. In view of Blaschke's condition $B(z)$ differs from a classical definition (see (2)) on a constant factor.

In this section under Frostman's type conditions we obtain new lower estimates for (9).

Theorem 2. *Let $B(z)$ be a Blaschke product* (9), *$a_n \neq 0$, $0 < q < 1$, $\theta_0 \in [0; 2\pi]$, $\delta > 0$, and*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta_0} - a_n|^{1-q}} < K_0, \tag{10}$$

where K_0 is a constant. Then there exist a set $F \subset [0;1)$ of values $r = |z|$ and a constant $K_2(q)$ such that $\operatorname{mes}([r;1) \cap F) < \delta(1-r)$ *as $r \to 1$, and*

$$\ln \left| \frac{1}{B(z)} \right| \leq \frac{K_2(q)}{(1-r)^q} \cdot \ln \frac{1}{1-r}, \quad r \in [0;1) \setminus F, \quad z \in S_\sigma(\theta_0), \tag{11}$$

where $S_\sigma(\theta_0)=\{z\in U: |1-ze^{-i\theta_0}|\le\sigma(1-|z|)\}$ *is a Stolz angle with the vertex* $e^{i\theta_0}$.

From the last theorem it is easy to obtain the following minimum type result

Corollary 3. *Let* $B(z)$ *be defined by* (9), $a_n\ne 0$, $0<q<1$, $\delta>0$, *and*

$$\sup_\theta\sum_{n=1}^\infty \frac{1-|a_n|}{|e^{i\theta}-a_n|^{1-q}}<K_0. \tag{12}$$

Then there exists a set $F\subset[0;1)$ *of values* $r=|z|$ *such that* $\mathrm{mes}([r;1)\cap F)<\delta(1-r)$ *as* $r\to 1$, *and*

$$\ln\mu(r,B)\ge -\frac{K_2}{(1-r)^q}\ln\frac{1}{1-r},\quad r\in[0;1)\setminus F. \tag{13}$$

Proof of Theorem 2. Let condition (10) be satisfied with $\theta_0=0$. We write $A_n(z)=\frac{1-|a_n|^2}{1-\bar a_n z}$. We have

$$\ln\left|\frac{1}{B(z)}\right|\le\sum_{|A_n(z)|\ge\frac12}\ln\left|\frac{1-\overline{a_n}z}{\overline{a_n}(a_n-z)}\right|+\sum_{|A_n(z)|<\frac12}\ln\left|\frac{1-\overline{a_n}z}{\overline{a_n}(a_n-z)}\right|. \tag{14}$$

Following M.Tsuji [14,15] we consider the case when $|A_n(z)|<\frac12$. We obtain

$$\left|\ln\frac{1-\overline{a_n}z}{\overline{a_n}(a_n-z)}\right|=|\ln(1-A_n(z))|\le\sum_{k=1}^\infty|A_n(z)|^k=\frac{|A_n(z)|}{1-|A_n(z)|}\le 2\,|A_n(z)|\,.$$

Thus,

$$\sum_{|A_n(z)|<\frac12}\left|\ln\frac{1-\overline{a_n}z}{\overline{a_n}(a_n-z)}\right|\le 2\sum_{|A_n(z)|<\frac12}|A_n(z)|. \tag{15}$$

If z lays outside the union of the disks $D=\bigcup_{k=1}^\infty D_k$, where $D_k=\{z: |z-a_k|\le\frac{(1-|a_k|^2)^{1+q}}{K_1}\}$, K_1 being a positive constant which will be specified late, then if $\left|\frac{1-|a_n|^2}{1-\overline{a_n}z}\right|\ge\frac12$ we deduce

$$\left|\frac{1-\overline{a_n}z}{\overline{a_n}(a_n-z)}\right|\le\frac{K_1\cdot|1-\overline{a_n}z|}{|\overline{a_n}|\,(1-|a_n|^2)^{1+q}}\le\frac{K_1\,|1-\overline{a_n}z|}{|\overline{a_n}|\,(\frac{|1-\overline{a_n}z|}{2})^{1+q}}=$$

$$=\frac{K_1 2^{q+1}}{|\overline{a_n}|\,|1-\overline{a_n}z|^q}\le\frac{K(q)}{(1-r)^q},$$

where $K(q) = \frac{2^{q+1} \cdot K_1}{\min\limits_n |a_n|}$ is a constant. Then

$$\ln \left| \frac{1 - \overline{a_n} z}{\overline{a_n}(a_n - z)} \right| \le 2K_2(q)|A_n(z)| \ln \frac{1}{1-r},$$

because $|A_n(z)| \ge \frac{1}{2}$. Therefore,

$$\sum_{|A_n(z)| \ge \frac{1}{2}} \ln \left| \frac{1 - \overline{a_n} z}{\overline{a_n}(a_n - z)} \right| \le 2 \sum_{|A_n(z)| \ge \frac{1}{2}} |A_n(z)| \cdot K_2(q) \ln \frac{1}{1-r}. \tag{16}$$

Summarizing (14)–(16), we obtain

$$\ln \left| \frac{1}{B(z)} \right| \le 2K_2(q) \ln \frac{1}{1-r} \sum_{|A_n(z)| \ge \frac{1}{2}} |A_n(z)| + 2 \sum_{|A_n(z)| < \frac{1}{2}} \left| \frac{1 - |a_n|^2}{1 - \overline{a_n} z} \right| =$$

$$= \left(2K_2(q) \ln \frac{1}{1-r} + 2 \right) \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|}, \quad z \notin D. \tag{17}$$

Let us multiply (17) by $(1-r)^q$. We have

$$(1-r)^q \ln \left| \frac{1}{B(z)} \right| \le K_3(q) \ln \frac{1}{1-r} \sum_{n=1}^{\infty} \frac{\left(1 - |a_n|^2\right)(1-r)^q}{|1 - \overline{a_n} z|^{q+1-q}} \le$$

$$\le K_3(q) \ln \frac{1}{1-r} \sum_{n=1}^{\infty} \frac{(1 - |a_n|^2)}{|1 - \overline{a_n} z|^{1-q}} \le K_4(q) \ln \frac{1}{1-r}.$$

Hence,

$$\ln \left| \frac{1}{B(z)} \right| \le \frac{1}{(1-r)^q} K_4(q) K_0 \ln \frac{1}{1-r}, \quad z \notin D,$$

as required. It remains to estimate an exceptional set. We consider those zeros a_n which are located in the Stloz angle $S_{\sigma_1}(0)$, $\sigma_1 > \sigma$, i.e.

$$\sigma_1 (1 - |a_n|) \ge |1 - a_n|. \tag{18}$$

We renumber them by b_n. Then

$$+\infty > K_0 > \sum_n \frac{1 - |b_n|}{|1 - b_n|^{1-q}} \ge \sum_n \frac{1 - |b_n|}{\sigma_1^{1-q} (1 - |b_n|)^{1-q}} = \sigma_1^{q-1} \sum_n (1 - |b_n|)^q. \tag{19}$$

Let us prove that the points $z \in S_\sigma(0)$ do not belong to $D(a_k, \frac{(1-|a_k|^2)^{1+q}}{K_1})$ for $a_k \notin S_{\sigma_1}(0)$ and sufficiently large k.

By the definition of the Stolz angle, $z \in S_\sigma$ is equivalent to

$$|\arg(1-z)| \le b(r,\sigma), \quad b(r,\sigma) = \arccos\frac{1}{\sigma} + o(1) = \beta(\sigma) + o(1),\ r \uparrow 1. \tag{20}$$

Let $a_k \ne b_n$, $n \in \mathbb{N}$. We consider two cases. If $|1-z| > |1-a_n|$, we have

$$|z-a_n| \ge \Re a_n - \Re z = \Re(1-z) - \Re(1-a_n) \ge$$
$$\ge |1-z|\cos(\beta(\sigma)+o(1)) - |1-a_n|\cos(\beta(\sigma_1)+o(1)) \ge$$
$$\ge |1-a_n|(\cos(\beta(\sigma)+o(1)) - \cos(\beta(\sigma_1)+o(1))) \ge$$
$$\ge \frac{(1-|a_n|^2)^{1+q}}{K_1}, \quad n \ge n_0(\sigma,\sigma_1,q,K_1).$$

Similarly, if $|1-z| \le |1-a_n|$ we obtain for $n \ge n_1(\sigma,\sigma_1,q,K_1)$

$$|z-a_n| \ge \Im a_n - \Im z \ge |1-a_n|\sin(\beta(\sigma_1)+o(1)) - |1-z|\sin(\beta(\sigma)+o(1)) \ge$$
$$\ge |1-a_n|(\sin(\beta(\sigma_1)+o(1)) - \sin(\beta(\sigma)+o(1))) \ge \frac{(1-|a_n|^2)^{1+q}}{K_1}.$$

Hence, $z \notin D\Big(a_n, \frac{(1-|a_n|^2)^{q+1}}{K_1}\Big)$, $n \ge \max\{n_0, n_1\}$.

Estimate the counting function $n(r)$ of the sequence (b_n). Inequality (19) implies

$$\frac{K_0}{\sigma_1 2^{q-1}} > \sum_{n=1}^{\infty}(1-|b_n|)^q \ge \sum_{|b_n|\le r}(1-|b_n|)^q \ge n(r)(1-r)^q.$$

Consequently,

$$n(r) \le \frac{K_4}{\sigma_1^{q-1}(1-r)^q}. \tag{21}$$

Let $G_n = D\Big(b_n, \frac{(1-|b_n|^2)^{q+1}}{K_1}\Big)$, $G = \bigcup_{n=1}^{\infty} G_n$, F_n be the circular projection of G_n,

$$F_n \stackrel{\text{def}}{=} \left[|b_n| - \frac{(1-|b_n|^2)^{1+q}}{K_1}; |b_n| + \frac{(1-|b_n|^2)^{1+q}}{K_1}\right).$$

We estimate the measure of $\mathrm{mes}([r;1)\cap F)$. It is sufficient to take into account only those F_n, for which $r \le |b_n| + (1-|b_n|^2)^{1+q}/K_1$, i.e.

$$1-r \ge 1-|b_n| - \frac{(1-|b_n|^2)^{1+q}}{K_1} = (1-|b_n|)\,(1-o(1)).$$

Then $2(1-r) \geq (1-|b_n|)$, $r \uparrow 1$, and

$$\mathrm{mes}([r,1)\cap F) \leq \sum_{|b_n|\geq 1-2(1-r)} \frac{2(1-|b_n|^2)^{1+q}}{K_1} \leq$$

$$\leq \frac{2^{2+q}}{K_1} \int\limits_{1-2(1-r)}^{1} (1-t)^{1+q} dn(t).$$

We denote the integral by I. Integrating by parts and using (21), we obtain

$$I = n(t)(1-t)^{1+q}\Big|_{1-2(1-r)}^{1} + \int\limits_{1-2(1-r)}^{1} n(t)(1+q)(1-t)^{q} dt =$$

$$= \lim_{t\to 1} n(t)(1-t)^{1+q} - n(1-2(1-r))(2(1-r))^{1+q} +$$

$$+(1+q)\int\limits_{1-2(1-r)}^{1} n(t)(1-t)^{q} dt \leq (1+q)\frac{K_4}{2^{q-1}}(2(1-r)).$$

Therefore,

$$\mathrm{mes}([r;1)\cap F) \leq \frac{2^4}{K_1}(1+q)(1-r)K_4 \leq \delta(1-r),$$

for $K_1 = \frac{\delta K_3}{16(1+q)}$. Theorem 2 is proved.

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ПРО МІНІМУМ МОДУЛЯ ОБМЕЖЕНИХ АНАЛІТИЧНИХ ФУНКЦІЙ

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Описано асимтотичну поведінку обмеженої аналітичної функції без нулів в одиничному крузі в термінах модуля неперервності міри Стільтьєса її зображення. Знайдено нові оцінки знизу для добутків Бляшке за умов типу Фростмана.