HYPERSPACE OF COMPACT BODIES OF CONSTANT WIDTH ON SPHERE

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We investigate the hyperspace of convex bodies of constant width in two-dimensional spheres. The main result asserts that the mentioned hyperspace is a manifold modeled on the Hilbert cube (Q-manifold).

1. INTRODUCTION

The hyperspace of compact convex subsets $\mathrm{cc}(\mathbb{R}^n)$ in the euclidean space $\mathbb{R}^n$ is endowed with the Hausdorff metric. It is well-known (and is often referred as the Blaschke completeness theorem) that $\mathrm{cc}(\mathbb{R}^n)$ is a complete metric space. The investigation of this space from the point of view of infinite-dimensional topology is initiated in [7]. One of the main results of [7] is that the space $\mathrm{cc}(\mathbb{R}^n)$, $n \leq 2$, is homeomorphic to the punctured Hilbert cube $Q \setminus \{*\}$. Similar results are obtained by the author for the hyperspace $\mathrm{cw}(\mathbb{R}^n)$ of convex bodies of constant width [2] (see also [3]).

The notion of convex set as well as a convex body of constant width can be naturally defined for every riemannian manifold. The convex bodies of constant width in the hyperbolic plane were considered in [1]; a close to the notion of body of constant width that of spherical rotor in [4] and [5].

In this paper we consider the hyperspace of bodies of constant width in two-dimensional sphere; the main result is a counterpart of the mentioned result of [7].

Theorem 1. *The hyperspace* $\mathrm{cw}(\mathbb{R}^n)$ *is a* Q*-manifold.*

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2. PRELIMINARIES

Let either $S = S^2 \subset \mathbb{R}^3$, $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 + (x_3)^2 = 1\}$, or $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 + (x_3 - 1)^2 = 1\}$, like in Theorem 4. We assume that S is endowed with the topology induced from $\mathbb{R}^3$.

Every two points $a, b \in S$ can be connected with a geodesic. For every $a \in S$, we denote by a^- the antipodal point to a.

By segments we mean the geodesic segments. The segment connection a and b is denoted by $[a, b]$. If $b = a^-$, then the notation $[a, b]$ is not determined.

Let U be an open hemisphere and $a, b \in U$. Then $[a, b] \subset U$. We have $[a, b] = \bigcap\{U \mid a, b \in U, U \text{ is a hemisphere}\}$.

We denote by d the geodesic metric on the sphere.

Let $K(a, d)$ denote the circumference of radius d centered at $a \in S$, i.e. the set $K(a, d) = \{b \in S \mid d(a, b) = d\}$. Every circumference $K(a, d)$ is the intersection of S with some plane and in $\mathbb{R}^3$ this is a circumference of radius $\sin d$. Obviously, $d \in (0, \frac{\pi}{2}]$. If $d = \frac{\pi}{2}$, then we obtain a great circumference. Similarly, by $B(a, d)$ we denote the circle of radius d centered at a: $B(a, d) = \{b \in S \mid d(a, b) \leq d\}$.

The **angle** $\angle bac$ between arbitrary segments $[a, b]$ and $[a, c]$ is evaluated counterclockwise. The notion of angle is not symmetric: $\angle cab = \pi - \angle bac$.

Definition 1. Suppose that a closed subset A is contained in some open hemisphere U. The *diameter* of the set A is the number

$$\operatorname{diam} A = \max\{d(a, b) \mid a, b \in A\}.$$

Definition 2. Suppose that a subset $A \subset$ does not contain pairs of antipodal points. A set A is called **convex**, if, for every two points $a, b \in A$, we have $[a, b] \subset A$.

Definition 3. A compact convex subset with nonempty interior is called a **convex body**.

It follows from the definition of convex body that $\operatorname{diam}(A) < \pi$, for every convex $A \in \operatorname{cc}(U)$.

The boundary $\operatorname{Bd}A$ of an arbitrary convex body A on S is homeomorphic to S^1 and the body A itself to the disc.

For $r \leq \pi$, the disc $B(a, r)$ is convex; for $r > \pi$, this is not the case. The following simple statements have their counterparts in the space $\mathbb{R}^n$:

Theorem 2. *The intersection of an arbitrary family of open sets is open.*

Theorem 3. *Let $\{A_\alpha\}, \alpha \in \Lambda$, be a family of sets linearly ordered by inclusion, i.e. $A_\alpha \subset A_\beta$ if and only if $\alpha \leq \beta$. Then $A = \bigcup_\alpha A_\alpha$ is a convex set.*

The following lemma can be proved by elementary arguments.

Lemma 1. *Let A be an arbitrary convex body and $a \in A$. Let P_1, P_2 and P_3 be great halfcircumferences connecting a and a^- such that the set $S \setminus (P_1 \cup P_2 \cup P_3)$ does not contain a hemisphere. If the set A contains on all semi-circumferences P_i points distinct of a, then a is an interior point of A.*

Corollary 1. *For any convex body A and any $a \in \mathrm{Bd}A$ there exists a closed hemisphere $\mathrm{Cl}\,U$ such that $a \in \mathrm{Bd}U$ and $A \subset \mathrm{Cl}\,U$.*

Definition 4. The boundary $K = \mathrm{Bd}U$ of the hemisphere U from Corollary 1 will be called the **supporting circumference of the convex body A at** a.

Definition 5. A convex body A is called **smooth**, if, at any point $a \in \mathrm{Bd}A$ of its boundary, there exists a unique support circumference.

Remark 1. Note that every convex body A is contained in an open hemisphere.

We keep the following notation till the end of this section. Let U be a fixed hemisphere formed by $K^* = K(O^*, \pi/2)$ and $O^* \in U$. Choose an initial point p^* on the circumference K^*. Let q be a point of the circumference K^* such that its length from p^* to q (counterclockwise) is equal to φ. By K_φ we denote the great circumference through O^* and q. Then $S = \bigcup\{K_\varphi \mid \varphi \in [0, \pi)\}$. The space of convex bodies K_φ, $\varphi \in [0, \pi)$ passing through O^* is homeomorphic to S^1. The value of parameter φ depends on the choice of initial point p^*. Denote $p(\varphi) = K^* \cap K_\varphi$ and consider the set $\mathcal{S}$ of the pairs $(p(\varphi)), p(\varphi)^-)$. This set is homeomorphic to the projective space $\mathbb{RP}^1$.

By $\mathrm{cc}(U)$ we denote the hyperspace of compact convex subsets in U.

Theorem 4. *The hyperspace $\mathrm{cc}(U)$ is homeomorphic to the hyperspace $\mathrm{cc}(\mathbb{R}^2)$.*

Proof. Without loss of generality, one may assume that

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1)^2 + (x_2)^2 + (x_3 - 1)^2 = 1,\ x_3 < 1\}.$$

Identify $\mathbb{R}^2$ with the plane $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$. Let l be any ray emanating from $(0, 0, 1)$ not parallel to Π. Then l intersects U and Π at the points a and b respectively. Let $F\colon U \to \Pi$ denote the map $F(a) = b$. Clearly, F is a homeomorphism.

The geodesics on U are exactly great circumferences. It is easy to see that their images under F are precisely lines in the plane. Conversely, the lines in the plane Π are mapped onto the great circles on the hemisphere.

Therefore, this map sends the convex sets into convex ones and vice versa. Therefore, $\mathrm{cc}(U) \cong \mathrm{cc}(\mathbb{R}^2)$.

Definition 6. The center of a convex body $A \in \text{cw}(U)$ is a point $c(A) \in S$ such that $d_H(A, \{c(A)\}) = \min\{d > 0 \mid B(a,d) \supset A, a \in S\}$.

One can easily prove that every convex body $A \in \text{cw}(U)$ possesses a unique center $c(A) \in A$ (it is easy to construct an example when $c(A) \in \text{Bd}A$). The map sending $A \in \text{cw}(U)$ to its center $c(A)$,

$$\text{cw}(U) \mapsto U, \tag{1}$$

is continuous.

3. BODIES OF CONSTANT WIDTH

Definition 7. A convex body A is called a **body of constant width** d, $d < \frac{\pi}{2}$, if for every $a \in \text{Bd}A$ we have $d(a,b) \leq d$ for all $b \in A$ and there is $c \in \text{Bd}A$ such that $d(a,c) = d$. The segment $[a,c]$ is then called a diameter of A.

By $\text{cw}(U)$ we denote the hyperspace of all convex bodies of constant width lying in U.

It is easy to see that every body of constant width lies in the intersection of discs of radius d.

Remark 2. For every $A, B \subset U$ and $t \in [0,1]$, we define the set $(1-t)A + tB$ as follows. For every $a \in A$ and $b \in B$, define $c = (1-t)a + tb$ as the unique point of the segment $[a,b]$ which divides it in the ratio $(1-t) : t$. Then $(1-t)A + tB = \{(1-t)a + tb \mid a \in A,\ b \in B\}$. However, this operation, in general, does not preserve the class of convex bodies as well as the class of convex bodies of constant width. Indeed, let $A, B \in \text{cw}(U)$ be convex bodies of constant width d_1 and d_2 respectively. In order to demonstrate that $(1-t)A + tB$ is not a body of constant width, it suffices to consider small balls K_1 and K_2 of radius $r \ll \pi$ around the poles q and q^- (see Fig. 1). Then the intersection of the body $\frac{K_1+K_2}{2}$ with the circumference

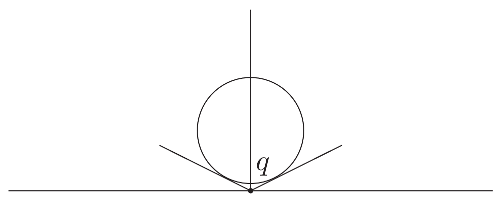

Figure 1:

that passes through the points q and q^- equals r and the intersection with the circumference $K(q, \pi/2)$ is larger.

There are counterparts of the Reuleaux triangles on the sphere. Consider a triangle ABC with $d(A, B) = d(B, C) = d(C, A) = d \leq \pi$. Consider circumferences of radius d centered at the points A, B, C. The smaller arcs of these circumferences connect the vertices and form the Reuleaux triangle. The following statement is obvious.

Proposition 1. *For every body A of constant width, any sequence of diameters tends to a diameter.*

Let $T(a, b) = B(a, d) \cap B(b, d)$. Obviously, if A is a body of constant width d and $D = [a, b]$ is some of its diameters, then $A \subset T(a, b)$.

Lemma 2. *Let $[a, b]$ be a segment of length $d < \pi/2$ and let $[c, e]$ be another segment of length d such that $[c, e] \subset T(a, b)$. Then also $[a, b] \subset T(c, e)$.*

The diameter $[a, b]$ decomposes the set $T(a, b)$ into two parts, T_1 and T_2: $T(a, b) = [a, b] \cup T_1 \cup T_2$. The points c and d either lie in different parts T_i, or one of them coincides with one of the points a and b.

Proposition 2. *For every body A of constant width d, every two its diameters $[a, b]$ and $[c, e]$ intersect each other. If $a = c$, then the boundary* BdA *of A between the points b and e coincides with the arc of the circumference $K(a, d)$.*

Proposition 3. *Every point a of a body A of constant width d belongs to some diameter. In other words, every body of constant width is the union of its diameters.*

Proof. Assume the contrary, i.e. that there exists a point $a \in \operatorname{Int} A$ which does not belong to any diameter. Since the union of diameters is a closed set, there exists $t > 0$ such that every disc $B(a, t)$ does not meet any diameter. Consider a great circle through a and let c and e be the points at which it intersects the boundary of the body A. Evidently, $d(c, e) < d$ and the segment $[c, e]$ decomposes the body A into two parts. Consider the diameters $[c, c']$ and $[e, e']$. By Proposition 2, they necessarily intersect and therefore are located in the same side with respect to the segment $[c, e]$ outside the circle $K(a, t)$ (see Fig. 2).

Denote by γ the arc from BdA that connects the points e' and c' and does not contain the points e and c. All the diameters with the endpoints on the arc γ have their another endpoints on the arc β that connects the points e and c and does not contain the points e' and c', because they meet the diameters $[c, c']$ and $[e, e']$. They intersect either the segment $[a, c]$ or

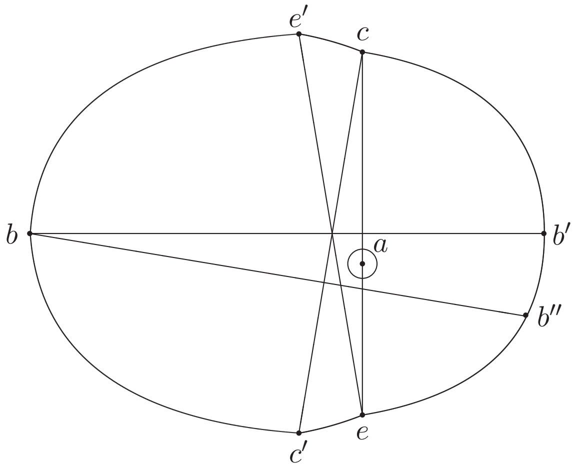

Figure 2:

the segment $[a, e]$ (outside $K(a, t)$). Using Proposition 1 we conclude that there exists a point b with a diameter $[b, b']$ intersecting the segment $[a, c]$ and the diameter $[b, b'']$ intersecting the segment $[a, e]$. By Proposition 2, this means that every arc $\smile b'b$" consists of the endpoints of some diameters and the whole sector $bb'b''$ is the union of diameters. This contradicts to our assumption that there is no point of any diameter inside the circumference $K(a, t)$.

Proposition 4. *Let A be a body of constant width d. Then*

$$\begin{aligned} A = \bigcap\{T(c, e) \mid [c, e] \text{ is a diameter of } A\} = \\ = \bigcup\{[c, e] \mid [c, e] \text{ is a diameter of } A\}. \end{aligned} \tag{2}$$

Proposition 5. *Let A be a body of constant width d and $[c, e]$ some of its diameters from c to e. Then for every $\varphi \in (0, \pi)$ there exists a diameter $[f, g]$ of A that forms with the diameter $[c, e]$ the angle φ.*

Proof. Assume the contrary. Let $\varphi \in (0, \pi)$ be such that no diameter of A forms the angle φ with the diameter $[c, e]$. Then, by Proposition 1, the number φ satisfies this property together with its neighborhood on the interval $(0, \pi)$. Let $(\alpha_1, \alpha_2) \in (0, \pi)$, $\varphi \in (\alpha_1, \alpha_2)$ be such a maximal interval, i.e. no diameter of the body A forms an angle from this interval with the diameter $[c, e]$ and there exist diameters $[f, g]$ and $[h, i]$, that form with it the angles α_1 and α_2 respectively (see Fig. 3).

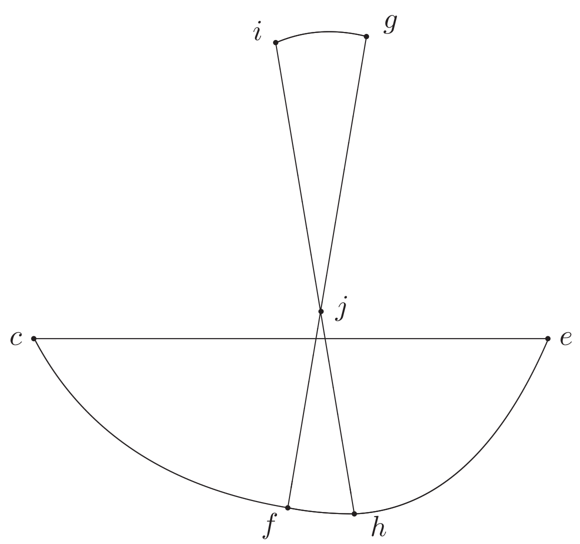

Figure 3:

By Proposition 2, the given diameters intersect at the point j and no arc $\smile ig$ and $\smile fh$ of the boundary BdA degenerate (otherwise this is an arc of a circumference of radius d and the diameters with the endpoints on this arc intersect with the diameter $[c, e]$ and the angles at the points of intersection fill the whole segment (α_1, α_2)). Let $k \in\smile ig$ be an interior point of this arc and $[k, l]$ is the corresponding diameter. We are going to show that $l \in\smile fh$. Indeed, since the diameter $[k, l]$ meets the diameter $[f, g]$, we see that $l \in\smile fe$, and since the diameter $[k, l]$ meets the diameter $[h, i]$, we see that $l \in\smile ch$. But then the angle between this diameter and the diameter $[c, e]$ belongs to the segment (α_1, α_2). The obtained contradiction finishes the proof.

Corollary 2. *Let $A \subset U$ be a body of constant width d and $[c, e]$ be its fixed diameter. By Proposition 5, for arbitrary angle $\varphi \in (0, \pi)$ there exists another diameter $[f, g]$ of A intersecting $[c, e]$ at h under the angle φ (we assume that the endpoints of the diameter $[f, g]$ are denoted so that $\varphi = \angle ehg$; if $h = e$ then in order to define the angle we extend the diameter $[f, g]$ beyond e). Let $p(\varphi) = h$. For a fixed body of constant width $A \subset U$ and for his fixed diameter $[c, e]$, we therefore defined a continuous function*

$$p\colon (0, \pi) \to [c, e],\ \ p(\varphi) = h. \qquad (3)$$

Fix a counterclockwise direction on the boundary $K^* = \mathrm{Bd}U$ of the hemisphere U. Let $p \in K^*$. Consider the family of great circumferences through p:

$$K_\varphi(p), \quad \varphi \in [0, \pi], \tag{4}$$

such that $K_0(p) = K^*$ and the angle between the circumferences $K_\varphi(p)$ and K^* equals $\varphi \in [0, \pi]$.

Evidently, for every convex body A and every pair $(p, p^-) \in \mathcal{S}$ we have $\mathrm{diam}(A, (p, p^-)) \leq \mathrm{diam}\, A$.

Proposition 6. *For every point $p \in K^*$ and every body of constant width $A \in \mathrm{cw}(U)$ there exists a unique diameter $[n, m] = [n(p), m(p)] \subset A$ which lies on the circumference $K_\varphi(p)$. Then the map*

$$\Phi\colon\ \mathrm{cw}(U) \times \mathcal{S} \to \exp(\mathbb{R}), \tag{5}$$

that sends $A \in \mathrm{cw}(U)$ and any pair $(p, p^-) \in \mathcal{S}$ to the diameter $[n, m] = [n(p), m(p)] \subset A$ is continuous.

Proof. Fix an arbitrary point $p \in K^*$. Let $[c, e]$ be an arbitrary diameter of A. If it belongs to some circumference $K_\varphi(p)$, then no other diameter possesses such a property because the circumferences $K_\varphi(p)$, $\varphi \in (0, \pi)$, in the hemisphere S^+ do not intersect and the proposition is proved. Indeed, assume the contrary. Then it intersects the family of circumferences $K_\varphi(p)$, $\varphi \in [\alpha_1, \alpha_2] \subset (0, \pi)$. Let $q(x)$, where $x \in [c, e]$, be equal to the angle between the circumference K_φ and the diameter $[c, e]$ at the point x. Thus we have defined a function $q\colon [c, e] \to [\alpha_1, \alpha_2]$. Being monotone, this function admits the inverse one. In addition, earlier we have introduced the function (3) $p\colon (0, \pi) \to [c, e]$. It is easy to see that there exists $j \in (c, e)$ such that $p(q(j)) = j$. Therefore, there exists a diameter $[n, m]$ that passes through the point j and lies on some circumference $K_\varphi(p)$.

Now we are able to provide another, equivalent definition of the body of constant width.

Definition 8. A convex set A is said to be a body of constant width d if, for every pair $(p, p^-) \in \mathcal{S}$ and every circumference $K_\varphi(p)$ from (4), the intersection $K_\varphi(p) \cap A$ for all $\varphi \in [0, \pi)$ is either empty or is a segment of length not exceeding d, and the equality is attained for precisely one of the values φ.

Definition 9. A convex body A is a body of width at least d if for every pair $(p, p^-) \in \mathcal{S}$ there exists a circumference $K_{\varphi_0}(p)$ from formula (4) such that $K_{\varphi_0}(p) \cap A$ is a segment of length at least d.

Let

$$\operatorname{diam}(A,(p,p^-)) = \max\{\operatorname{diam}(K_\varphi(p)\cap A) \mid \varphi\in[0,\pi)\}.$$

It is easy to see that a convex body A is of width at least d if for every pair $(p,p^-)\in\mathcal{S}$ we have $\operatorname{diam}(A,(p,p^-))\geq d$.

Corollary 3. *Let $[a,b]\in K_\varphi(p)$, $d(a,b)=d$ and $T(a,b)\subset U$. Then $\operatorname{diam}(A,(p,p^-))=d$ and for arbitrary another pair $(q,q^-)\in\mathcal{S}$ we have $\operatorname{diam}(A,(q,q^-))>d$.*

Let $A\,\mathrm{cw}(U)$ be an arbitrary convex body. Denote by $C(A)$ the set of the points $a\in A$ that belong to more than one diameter. The following statement is obvious and we leave the proof for the reader.

Lemma 3. *Let $A\in\mathrm{cw}(U)$ be an arbitrary convex body and $\varepsilon>0$ be such that $\overline{O}_\varepsilon(A)=\{b\in S\mid d(A,\{b\})\leq\varepsilon\}\subset U$. Then $d_H(\mathrm{Bd}A,C(A))\geq\varepsilon$.*

Conversely, if $d_H(\mathrm{Bd}A,C(A))=\varepsilon>0$, then there exists a convex body $B\in\mathrm{cw}(U)$, $B=\{b\in A\mid d_H(C(A),\{b\})\geq\varepsilon\}$ such that $A=\overline{O}_\varepsilon$.

We provide a universal method of construction of the convex bodies of constant width $d\in(0,\pi/2]$ in the hemisphere. The method is a modification of the method of construction of the bodies of constant width in $\mathbb{R}^n$ proposed in [2].

Fix an arbitrary dense sequence (p_k,p_k^-) of pairs $(p_k,p_k^-)\in\mathcal{S}$. By induction in k, construct a convex body A of constant width $d\in(0,\pi/2]$.

Let $k=1$. By $[a_1,b_1]$ we denote a segment of length d which lies on some circumference $K_\varphi(p_1)$ (see (4)) and the set $T(a_1,b_1)\cap U$ is a convex body of width at least d. Let $A_1=T(a_1,b_1)\cap U$. Clearly, $\operatorname{diam}(A_1,(p_1,p_1^-))=d$.

Let $k=2$. We choose a segment $[a_2,b_2]\subset K_\varphi(p_2)$, $\varphi\in(0,\pi)$, of length d so that $[a_2,b_2]\subset A_1$. Clearly, then $[a_1,b_1]\subset T(a_2,b_2)$. Let $A_2=A_1\cap T(a_2,b_2)$. The set A_2 is a convex body of length at least d and

$$\operatorname{diam}(A_2,(p_i,p_i^-))=d$$

for $i=1,2$. Let us assume that the construction is already performed for $k=1,2,\dots,n-1$ and perform it for $k=n$. We choose a segment $[a_n,b_n]\subset K_\varphi(p_n)$ of length d from the condition $[a_n,b_n]\subset A_{n-1}$. This is possible, because the convex body A_{n-1} is of width at least d. Let $A_n=A_{n-1}\cap T(a_n,b_n)$. The set A_n is a convex body of width at least d and $\operatorname{diam}(A_2,(p_i,p_i^-))=d$ for $i=1,\dots,n$.

Let

$$A=\bigcap_{i=1}^{\infty}A_i=\bigcap_{i=0}^{\infty}T(a_i,b_i),\quad\text{where}\quad A_0=U.$$

The set A is a convex body of constant width d and

$$A = \mathrm{Cl}\left(\bigcup_{i=1}^{\infty}[a_i, b_i]\right).$$

It is easy to see that this construction gives all the convex bodies of constant width. The following are some properties of the construction.

Proposition 7. *The construction is uniformly continuous in the following sense: for every $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that, for any convex bodies*

$$A' = \mathrm{Cl}\left(\bigcup_{i=1}^{\infty}[a'_i, b'_i]\right), \qquad A'' = \mathrm{Cl}\left(\bigcup_{i=1}^{\infty}[a''_i, b''_i]\right)$$

of constant width d, if $[a'_i, b'_i] = [a''_i, b''_i]$ for $i = 1, 2, \ldots, n_\varepsilon$, then

$$d_H(A', A'') < \varepsilon d.$$

Proposition 8. *Let $A = \bigcap_{i=1}^{\infty} T(a_i, b_i)$ be a convex body of constant width d such that $d_H(C(A), \mathrm{Bd}A) \geq \varepsilon > 0$. Then, for every n, there exists $\theta(\varepsilon, n)$, $\theta(\varepsilon, n) \to 0$ as $n \to \infty$, such that arbitrary segments of the form $[a_j, c]$, $[b_j, c]$ of length d and direction differing from that of the segment $[a_j, b_j]$ by angle not exceeding $\theta(\varepsilon, n)$, belong to the set $A_n = \bigcap_{i=1}^{n} T(a_i, b_i)$.*

Proposition 9. *Let B be a convex body of constant width at least d. For every segment $[a, b]$ of length $d' \leq d$, there exists a unique segment $[a', b'] \subset B$ of the same length and direction as $[a, b]$ and which is the closest to $[a, b]$ with respect to the Hausdorff metric. The assignment $[a, b] \mapsto [a', b']$ continuously depends on B.*

In the sequel, the endpoints of the diameters of the same direction are denoted according to the orientation of the direction: for any diameters $[a', b']$ and $[a'', b'']$ of convex bodies A' and A'' respectively, we have $[a', a''] \cap [b', b''] = \emptyset$.

4. PROOF OF THE MAIN RESULT

The following statements can be proved by using elementary geometric arguments.

Lemma 4. *Let $A \in \mathrm{cc}(U)$ be an arbitrary convex body lying in the hemisphere U. Denote $\beta(A) = \min\{d(a, k) \mid a \in A, k \in K^*\}$ and let $\delta \in (0, \beta(A))$.*

By $\overline{O}_\delta(A) = \{a \in S \mid d_H(a, A) \leq \delta\}$ *we denote the closed* δ*-neighborhood of the body* A*. Then* $\overline{O}_\delta(A)$ *is a smooth convex body.*

If $A \in \mathrm{cw}(U)$ *is a body of constant width* d*, then* $\overline{O}_\delta(A)$ *is a body of constant width* $d + \delta$*.*

If $\delta\colon \mathrm{cc}(U) \to (0, \pi/2)$*,* $\delta(A) < \beta(A)$ *is a continuous function, then the map* $A \mapsto \overline{O}_\delta(A)$ *is continuous as well.*

Lemma 5. *Let* $[a, b]$ *and* $[c, e]$ *be two diameters of a convex body* A *of constant width* d *that intersect at an interior point* q *(see Fig. 4). Let* $\delta =$

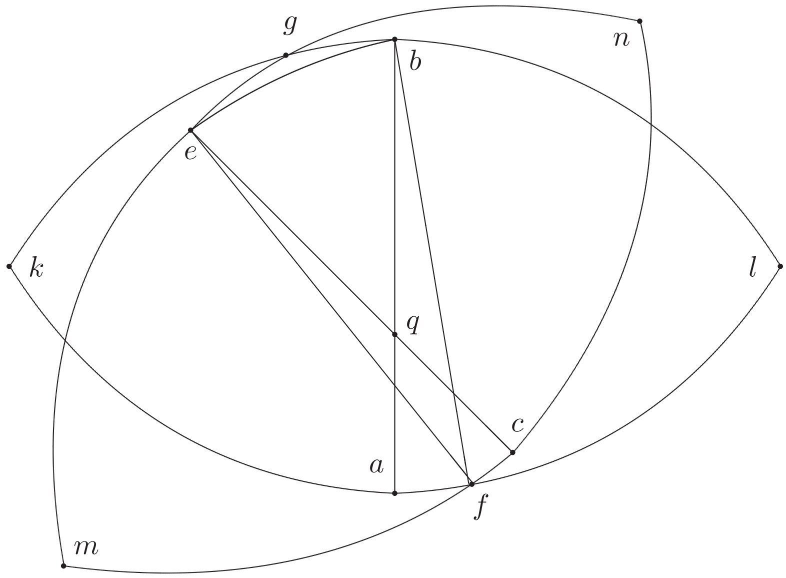

Figure 4:

$d(q, \{a, b, c, e\})$ *and let* α *be the angle between the segments* $[q, a]$ *and* $[q, c]$*. Then, if the boundary* $\mathrm{Bd}A$ *of the body* A *between the points* b *and* e *is replaced by the arc* $\smile be$ *of the circumference* $K(f, d)$*, where* f *is the point of intersection of the circumferences* $\mathrm{Bd}K(e, d)$ *and* $\mathrm{Bd}K(b, d)$ *between the points* a *and* c*, and the part of the boundary* $\mathrm{Bd}A$ *between the points* a *and* c *is replaced by two arcs,* $\smile af$ *and* $\smile fc$*, of the circumferences* $\mathrm{Bd}K(b, d)$ *and* $\mathrm{Bd}K(e, d)$ *respectively, then we obtain a new body* A^* *of constant width* d *with the boundary* $\mathrm{Bd}A^*$*, with at least two support circumferences at the point* f *(i.e.* A^* *is not smooth). The Hausdorff distance* $d_H(A, A^*)$ *between the bodies* A *and* A^* *does not exceed the number* $\sigma(\delta, \alpha) < \delta$ *which tends to*

0 as $\alpha \to 0$.

By ANR we denote the class of absolute neighborhood retracts for the class of metric spaces.

We say that a metric space X satisfies the *disjoint approximation property* (DAP) if for every continuous function $\varepsilon\colon X \to (0,\infty)$ there exist continuous maps $f_1, f_2\colon X \to X$ such that $d(f_i(x),(x)) < \varepsilon(x)$, for every $x \in X$, $i = 1,2$, and $f_1(X) \cap f_2(X) = \emptyset$.

The following is a characterization theorem for Q-manifolds.

Theorem 5. (Toruńczyk [8]). *A locally compact ANR X is a Q-manifold if and only if X satisfies the DAP.*

Theorem 6. *The space* $\mathrm{cw}(U)$ *satisfies the disjoint approximation property.*

Proof. Let ε^* be an arbitrary number. Define an arbitrary continuous function $\varepsilon\colon \mathrm{cw}(U) \to (0,\pi/2)$, $\varepsilon(A) = \min\{\varepsilon^*, \beta(A)/2\}$, where the function $\beta(A)$ is introduced in Lemma 4. We use this statement in order to construct a map $f_\varepsilon\colon \mathrm{cw}(S^+) \to \mathrm{cw}(S^+)$ and put $f_\varepsilon = \overline{O}_{\varepsilon(A)}(A)$. This map is continuous, its images are the smooth bodies of constant width and

$$d_H(A, f_\varepsilon(A)) < \varepsilon(A).$$

Let us construct another continuous map $g_\varepsilon\colon \mathrm{cw}(U) \to \mathrm{cw}(U)$ such that $d_H(A, g_\varepsilon(A)) < \varepsilon(A)$ and $f_\varepsilon(\mathrm{cw}(U)) \cap g_\varepsilon(\mathrm{cw}(U)) = \emptyset$.

On the circumference $K^* = \mathrm{Bd}U$, choose a fixed point p and in every body of constant width $g_{\varepsilon(A)/2}(A) \in \mathrm{cw}(U)$ fix a diameter $[n(p), m(p)]$ lying on a great circumference that passes through p (Proposition 6). From Lemma 5, determine an angle $\alpha(A)$ such that $\sigma(\varepsilon(A)/2, \alpha(A)) < \varepsilon(A)/2$. Let $[a,b]$ and $[c,e]$ be the two other diameters of $g_{\varepsilon/2}(A)$ whose angle of intersection is $\varepsilon(A)/2$ and that form equal angles (from different sides) with the diameter $[n(p), m(p)]$. Apply Lemma 5 and replace $g_{\varepsilon/2}(A)$ by a non-smooth body of constant width A^* such that $d_H(g_{\varepsilon/2}(A), A^*) < \varepsilon(A)/2$. We put $g_\varepsilon(A) = A^*$ and thus obtain a required map g_ε.

Proposition 10. *The hyperspace* $\mathrm{cw}(U)$ *is a retract of the space* $\mathrm{cc}(U)$.

Proof. For every $A \in \mathrm{cc}(U)$, by $[\varphi_1(A), \varphi_2(A)] \subset [0,\pi]$ we denote the set of all $\varphi \in [\varphi_1(A), \varphi_2(A)]$ such that $K_\varphi(p^*) \cap A \neq \emptyset$. Evidently, $\varphi_1(A) \neq \varphi_2(A)$. We make the following convention: in the segment

$$[a(A), b(A)] = A \cap K_{(\varphi_1(A)+\varphi_2(A))/2},$$

we have $d(p^*, a(A)) < d(p^*, b(A))$.

By $\mathcal{V}$ we denote the set of all convex bodies $B \in \mathrm{cw}(U)$ that lie in the convex set A and one of their diameters lies on the segment $[a(A), b(A)]$. Further, let $d^* = \max\{\operatorname{diam} B \mid B \in \mathcal{V}\}$ and $\mathcal{V}^* = \{B \in \mathcal{V} \mid \operatorname{diam} B = d^*\}$.

From the set $\mathcal{V}^*$ we are going to choose a unique element $B(A)$ which continuously depends on A. Denote by $\mathcal{V}_1 \subset \mathcal{V}^*$ the set of all the bodies whose diameter $[a_1, b_1]$ which lies on the segment $[a(A), b(A)]$, is closest to the point p^* (and therefore to the point $a(A)$). In other words, the segment $[a_1, b_1]$ is a diameter of all the bodies $B \in \mathcal{V}_1$ of constant width d^*.

By $\{\varphi_i\}$, we denote a dense sequence of angles $\varphi_i \in [0, \pi)$, $\varphi_1 = 0$, (e.g. $\{0, \pi/2, \pi/4, 3\pi/4, \pi/8, 3\pi/8, \dots\}$).

For any angle φ_2 and any body $B \in \mathcal{V}_1$, denote by $h_2(B)$ the intersection point under the angle φ_2 of a diameter of this body (see Proposition 5 and Corollary 2) with the diameter $[a_1, b_1]$. Let h_2 be one of these points which is the nearest to p^*. By $\mathcal{V}_2 \subset \mathcal{V}_1$ we denote the set of the bodies whose diameters that form the angle φ_2 with the segment $[a_1, b_1]$ pass through the point h_2 and are closest to it with respect to the Hausdorff metric. In other words, all the bodies $B \in \mathcal{V}_2$ have at least two common diameters: $[a_1, b_1]$ and the diameter $[a_2, b_2]$ passing under the angle φ_2 to $[a_1, b_1]$ through the point h_2.

We then proceed by induction. We look for a sequence of embedded $\mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots \supset \mathcal{V}_2 \supset \dots$ whose intersection is a singleton (note that a singleton can be obtained even at a finite stage of the construction; then the induction is finished).

Suppose that a set $\mathcal{V}_{n-1}$ is obtained. It consists of bodies of constant width d^* that have common diameters $[a_i, b_i]$, $i = 1, \dots, n-1$, forming the angles φ_i with the diameter $[a_1, b_1]$.

Now we construct the set $\mathcal{V}_n$. For the angle φ_n and any body $B \in \mathcal{V}_{n-1}$, denote by $h_n(B)$ the point of intersection with angle φ_n of a diameter of this body and the diameter $[a_1, b_1]$. Let h_n be the closest of these points to the point p^*. By $\mathcal{V}_n \subset \mathcal{V}_{n-1}$ we denote the set of bodies we denote the set of the bodies whose diameters that form the angle φ_n with the segment $[a_1, b_1]$ pass through the point h_n and are closest to it with respect to the Hausdorff metric. This means that all the bodies $B \in \mathcal{V}_n$ have at least n common diameters: $[a_1, b_1]$ and the diameters $[a_i, b_i]$passing under the angle φ_i to $[a_1, b_1]$ at h_i, $i = 2, \dots, n$. From the construction it follows that the set $\mathcal{V} = \bigcap_{n=1}^{\infty} \mathcal{V}_n$ is a singleton: $\mathcal{V} = \{B(A)\}$ and the body of constant width $B(A) \in \mathrm{cw}(U)$ continuously depends on the convex body A.

Corollary 4. *The hyperspace* $\mathrm{cw}(U)$ *is an absolute retract.*

Using Toruńczyk's Characterization Theorem we conclude that the hyperspace $\mathrm{cw}\,\mathbb{R}^n$ is a Q-manifold. This finishes the proof of Theorem 1.

5. REMARKS AND OPEN QUESTIONS

Note that our methods work only in dimension 2. It is a natural to ask whether a counterpart of the main result is valid for the spheres of higher dimension.

Montejano [6] proved that the hyperspace $\mathrm{cc}(U)$, where U is an open subset of $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to $U \times Q \times [0, 1)$.

Question. Is the hyperspace $\mathrm{cw}(V)$, where V is an open subset of a hemisphere, homeomorphic to $U \times Q \times [0, 1)$?

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ГІПЕРПРОСТІР КОМПАКТНИХ ТІЛ СТАЛОЇ ШИРИНИ НА СФЕРІ

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