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# ON CRITICAL CARDINALITIES RELATED TO $Q$-SETS

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In this note we collect some known information and prove new results about the small uncountable cardinal $\mathfrak{q}_0$. The cardinal $\mathfrak{q}_0$ is defined as the smallest cardinality of a subset $A \subset \mathbb{R}$ that is not a $Q$-set (a subspace $A \subset \mathbb{R}$ is a *$Q$-set* if each subset $B \subset A$ is $F_\sigma$ in $A$). We present a simple proof of a folklore fact that $\mathfrak{p} \le \mathfrak{q}_0 \le \min\{\mathfrak{b}, \mathrm{non}(\mathcal{N}), \log(\mathfrak{c}^+)\}$, and also establish the consistency of a number of strict inequalities between the cardinal $\mathfrak{q}_0$ and other standard small uncountable cardinals. In particular, we establish the consistency of $\mathfrak{p} < \mathfrak{lr} < \mathfrak{q}_0$, where $\mathfrak{lr}$ denotes the linear refinement number. We also prove that $\mathfrak{q}_0 \le \mathrm{non}(\mathcal{I})$ for any $\mathfrak{q}_0$-flexible cccc $\sigma$-ideal $\mathcal{I}$ on $\mathbb{R}$.

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Стаття є оглядом результатів, що стосуються малого незліченного кардинала $\mathfrak{q}_0$, рівного найменшій потужності підмножини $A \subset \mathbb{R}$, що не є $Q$-множиною (підпростір $A \subset \mathbb{R}$ називається *$Q$-множиною*, якщо кожна підмножина $B \subset A$ є $F_\sigma$-множиною в $A$).

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## 1. Introduction and results in ZFC

A subset $X$ of the real line is a *$Q$-set* if each subset $A \subset X$ is relative $F_\sigma$-set in $A$, see [32, §4]. The study of $Q$-sets was initiated by founders of Set-Theoretic Topology: Hausdorff [20], Sierpiński [39] and Rothberger [38]. $Q$-Sets are important as they appear naturally in problems related to (hereditary) normal or $\sigma$-discrete spaces; see [1], [10], [15] – [19], [21], [22], [25], [36], [37], [40], [41].

We shall be interested in two critical cardinals related to $Q$-sets:

- $\mathfrak{q}_0 = \min\{|X| : X \subset \mathbb{R},\ X \text{ is not a } Q\text{-set}\}$;
- $\mathfrak{q} = \min\{\kappa : \text{no subset } X \subset \mathbb{R} \text{ of cardinality } |X| \geq \kappa \text{ is a } Q\text{-set}\}$.

It is clear that $\mathfrak{q}_0 \leq \mathfrak{q}$. Since each countable subset of the real line is a $Q$-set and no subset $A \subset \mathbb{R}$ of cardinality continuum is a $Q$-set, the cardinals $\mathfrak{q}_0$ and $\mathfrak{q}$ are uncountable and lie in the interval $[\omega_1, \mathfrak{c}]$. So, these cardinals are another examples of small uncountable cardinals considered in [14] and [45]. It seems that for the first time the cardinals $\mathfrak{q}_0$ and $\mathfrak{q}$ appeared in the survey paper of J. Vaughan [45], who referred to the paper [19], which was not published yet at the moment of writing [45]. Unfortunately, the cardinal $\mathfrak{q}_0$ disappeared in the final version of the paper [19]. Our initial motivation was to collect known information on the cardinal $\mathfrak{q}_0$ in order to have a proper reference (in particular, in the paper [1] exploiting this cardinal). Studying the subject we have found a lot of interesting information on the cardinals $\mathfrak{q}_0$ and $\mathfrak{q}$ scattered in the literature. It seems that a unique paper devoted exclusively to the cardinal $\mathfrak{q}_0$ is [7] of Brendle (who denotes this cardinal by $\mathfrak{q}$). Among many other results, in [7] Brendle found a characterization of the cardinal $\mathfrak{q}_0$ in terms of weakly separated families.

Two families $\mathcal{A}$ and $\mathcal{B}$ of infinite subsets of a countable set $X$ are

- *orthogonal* if $A \cap B$ is finite for every sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$;
- *weakly separated* if there is a subset $D \subset X$ such that $D \cap A$ is infinite for every $A \in \mathcal{A}$ and $D \cap B$ is finite for every $B \in \mathcal{B}$.

Let us recall that a family $\mathcal{A}$ of infinite sets is *almost disjoint* if $A \cap B$ is finite for any distinct sets $A, B \in \mathcal{A}$.

**Theorem 1** (Brendle [7]). *The cardinal $\mathfrak{q}_0$ is equal to the smallest cardinality of a subset $A \subset 2^\omega$ such that the almost disjoint family $\mathcal{A} = \{B_x : x \in A\}$ of branches $B_x = \{x|n : n \in \omega\}$ of the binary tree $2^{<\omega}$ contains a subfamily $\mathcal{B} \subset \mathcal{A}$ that cannot be weakly separated from its complement $\mathcal{A} \setminus \mathcal{B}$.*

Having in mind this characterization, let us consider the following two cardinals ([7]):

- $\mathfrak{ap}$, equal to the smallest cardinality of an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ containing a subfamily $\mathcal{B} \subset \mathcal{A}$ that cannot be weakly separated from $\mathcal{A} \setminus \mathcal{B}$;
- $\mathfrak{dp}$, equal to the smallest cardinality of the union $A \cup B$ of two orthogonal families $\mathcal{A}, \mathcal{B} \subset [\omega]^\omega$ that cannot be weakly separated.

The notation $\mathfrak{dp}$ is an abbreviation of "Dow Principle" considered by Dow in [13].

It is clear that $\mathfrak{dp} \le \mathfrak{ap} \le \mathfrak{q}_0 \le \mathfrak{q}$. In [7] Brendle observed that the cardinals $\mathfrak{dp}$, $\mathfrak{ap}$, and $\mathfrak{q}_0$ are in the interval $[\mathfrak{p}, \mathfrak{b}]$. Let us recall that $\mathfrak{b}$ is the smallest cardinality of a subset $B$ of the Baire space $\omega^\omega$, that is not contained in a $\sigma$-compact subset of $\omega^\omega$.

The cardinal $\mathfrak{p}$ is the smallest cardinality of a family $\mathcal{F}$ of infinite subsets of $\omega$ such that

- $\mathcal{F}$ is *centered*, which means that for each finite subfamily $\mathcal{E} \subset \mathcal{F}$ the intersection $\cap\mathcal{E}$ is infinite, but
- $\mathcal{F}$ has no infinite *pseudo-intersection* $I \subset \omega$ (i.e., an infinite set $I \subset \omega$ such that $I \setminus F$ is finite for all $F \in \mathcal{F}$).

For a cardinal $\kappa$ its logarithm is defined as $\log(\kappa) = \min\{\lambda : 2^\lambda \ge \kappa\}$. It is clear that $\log(\mathfrak{c}) = \omega$ and $\log(\mathfrak{c}^+) \in [\omega_1, \mathfrak{c}]$, so $\log(\mathfrak{c}^+)$ is a small uncountable cardinal. König's Lemma implies that $\log(\mathfrak{c}^+) \le \mathrm{cf}(\mathfrak{c})$. We refer the reader to [14], [45] or [4] for the definitions and basic properties of small uncountable cardinals discussed in this note.

The following theorem collects some known lower and upper bounds on the cardinals $\mathfrak{dp}$, $\mathfrak{ap}$, $\mathfrak{q}_0$ and $\mathfrak{q}$. For the lower bound $\mathfrak{p} \le \mathfrak{dp}$ established in [7] (and implicitly in [13]) we give an elementary proof, which does not involve Bell's characterization [3] of $\mathfrak{p}$ (as the smallest cardinal for which Martin's Axiom for $\sigma$-centered posets fails). The inequality $\mathfrak{p} \le \mathfrak{q}_0$ is often attributed to Rothberger who actually proved in [38] that $\mathfrak{t} > \omega_1$ implies $\mathfrak{q}_0 > \omega_1$. According to a recent breakthrough result of Malliaris and Shelah [29], $\mathfrak{t} = \mathfrak{p}$.

**Theorem 2.** $\mathfrak{p} \le \mathfrak{dp} \le \mathfrak{ap} \le \mathfrak{q}_0 \le \min\{\mathfrak{b}, \mathfrak{q}\} \le \mathfrak{q} \le \log(\mathfrak{c}^+)$.

*Proof.* The equality $\mathfrak{q} \le \log(\mathfrak{c}^+)$ follows from the fact that each subset of a $Q$-set is Borel, and that a second countable space contains at most $\mathfrak{c}$ Borel subsets.

The inequality $\mathfrak{q}_0 \le \mathfrak{q}$ is trivial. To see that $\mathfrak{q}_0 \le \mathfrak{b}$, choose any countable dense subset $Q$ in the Cantor cube $2^\omega$ and consider its complement $2^\omega \setminus Q$, which is homeomorphic to the Baire space $\omega^\omega$ by the Aleksandrov-Urysohn Theorem [26, 7.7]. By the definition of the cardinal $\mathfrak{b}$, the space $2^\omega \setminus Q$ contains a subset $B$ of cardinality $|B| = \mathfrak{b}$ that is contained in no $\sigma$-compact subset of $2^\omega \setminus Q$. Then the union $A = B \cup Q$ is not a $Q$-set as the subset $B$ is not relative $F_\sigma$-set in $A$. Consequently, $\mathfrak{q}_0 \le |B \cup Q| = |B| = \mathfrak{b}$.

The inequality $\mathfrak{ad} \le \mathfrak{q}_0$ follows from Theorem 1 and $\mathfrak{dp} \le \mathfrak{ap}$ is trivial. Finally, we prove the inequality $\mathfrak{p} \le \mathfrak{dp}$. We need to check that any two orthogonal families $\mathcal{A}, \mathcal{B} \subset [\omega]^\omega$ with $|\mathcal{A} \cup \mathcal{B}| < \mathfrak{p}$ are weakly separated. By $[\omega]^{<\omega}$ we denote the family of all finite subsets of $\omega$.

For every $n \in \omega$ and $x \in \mathcal{A}$ and $y \in \mathcal{B}$ consider the families

$$\mathcal{A}_x = \{F \in [\omega]^{<\omega} : F \cap x = \varnothing\} \ \text{ and } \ \mathcal{B}_{y,n} = \{F \in [\omega]^{<\omega} : |F \cap y| \ge n\}.$$

It is easy to check that the family $\mathcal{F} = \{\mathcal{A}_x : x \in \mathcal{A}\} \cup \{\mathcal{B}_{y,n} : y \in \mathcal{B},\ n \in \omega\} \subset [[\omega]^{<\omega}]^\omega$ is centered. Since $|\mathcal{F}| < \mathfrak{p}$, this family has an infinite pseudointersection

$\mathcal{I} = \{F_k\}_{k\in\omega}$. It follows that the union $I = \bigcup_{k\in\omega} F_k$ has finite intersection with each set $x \in \mathcal{A}$ and infinite intersection with each set $y \in \mathcal{B}$. Hence $\mathcal{A}$ and $\mathcal{B}$ are weakly separated. □

According to [15], each $Q$-set $A \subset \mathbb{R}$ is meager and Lebesgue null and hence belongs to the intersection $\mathcal{M} \cap \mathcal{N}$ of the ideal $\mathcal{M}$ of meager subsets of $\mathbb{R}$ and the ideal $\mathcal{N}$ of Lebesgue null sets in $\mathbb{R}$. The ideal $\mathcal{M} \cap \mathcal{N}$ contains the $\sigma$-ideal $\mathcal{E}$ generated by closed null sets in $\mathbb{R}$. Cardinal characteristics of the $\sigma$-ideal $\mathcal{E}$ have been studied in [2, §2.6]. It turns out that each $Q$-set $A \subset \mathbb{R}$ belongs to the ideal $\mathcal{E}$, which implies that $\mathfrak{q}_0 \le \mathrm{non}(\mathcal{E})$. More generally, $\mathfrak{q}_0 \le \mathrm{non}(\mathcal{I})$ for any flexible cccc $\sigma$-ideal $\mathcal{I}$ on $\mathbb{R}$. Here $\mathrm{non}(\mathcal{I})$ stands for the smallest cardinality of a subset $A \subset X$ that does not belong to a $\sigma$-ideal $\mathcal{I}$. It is clear that $\omega_1 \le \mathrm{non}(\mathcal{I}) \le |X|$.

Let $\mathcal{I}$ be a $\sigma$-ideal on a set $X$. A bijective function $f : X \to X$ will be an *automorphism* of $\mathcal{I}$ if $\{f(A) : A \in \mathcal{I}\} = \mathcal{I}$. It is clear that automorphisms of $\mathcal{I}$ form a subgroup $\mathrm{Aut}(\mathcal{I})$ in the group of all bijections of $X$ endowed with the operation of composition. The group $\mathrm{Aut}(\mathcal{I})$ will be called *the automorphism group* of the ideal $\mathcal{I}$. A $\sigma$-ideal $\mathcal{I}$ will be *$\kappa$-flexible* for a cardinal number $\kappa$ if for any subsets $A, B \subset X$ with $|A \cup B| < \kappa$ there exists an automorphism $f \in \mathrm{Aut}(\mathcal{I})$ such that $f(A) \cap B = \varnothing$. A $\sigma$-ideal $\mathcal{I}$ on a set $X$ is *flexible* if it is $|X|$-flexible.

**Proposition 3.** *Each $\sigma$-ideal $\mathcal{I}$ on any set $X$ is $\mathrm{non}(\mathcal{I})$-flexible.*

*Proof.* Given any two subsets $A, B \subset X$ with $|A \cup B| < \mathrm{non}(\mathcal{I})$, we need to find an automorphism $f \in \mathrm{Aut}(\mathcal{I})$ such that $f(A) \cap B = \varnothing$. Since $|A \cup B| < \mathrm{non}(\mathcal{I}) \le |X|$, there is a subset $C \subset X \setminus (A \cup B)$ of cardinality $|C| = |A|$. Choose any bijective function $f : X \to X$ such that $f(A) = C$, $f(C) = A$ and $f$ is identity on the set $X \setminus (A \cup C)$. It is easy to see that $f$ is an automorphism of the $\sigma$-ideal $\mathcal{I}$ witnessing that $\mathcal{I}$ is $\mathrm{non}(\mathcal{I})$-flexible. □

**Example 4.** Each left-invariant $\sigma$-ideal $\mathcal{I}$ on a group $G$ is flexible.

*Proof.* First we observe that the group $G \notin \mathcal{I}$ is uncountable. Then for any subset $A, B \subset G$ with $|A \cup B| < |G|$, the set $BA^{-1} = \{ba^{-1} : b \in B,\ a \in A\}$ has cardinality $|BA^{-1}| < |G|$. So we can find a point $g \in G \setminus BA^{-1}$ and observe that $gA \cap B = \varnothing$. □

We shall say that a $\sigma$-ideal $\mathcal{I}$ on a topological space $X$ satisfies the *compact countable chain condition* (briefly, $\mathcal{I}$ is a *cccc ideal*) if for any uncountable disjoint family $\mathcal{C}$ of compact subsets of $X$ there is a set $C \in \mathcal{C}$ that belongs to the ideal $\mathcal{I}$. This is a bit weaker than the classical *countable chain condition* (briefly, *ccc*) saying that for any uncountable disjoint family $\mathcal{C}$ of Borel subsets of $X$ there is a set $C \in \mathcal{C}$ belonging to the ideal $\mathcal{I}$. A simple example of a cccc $\sigma$-ideal that fails to have ccc is the $\sigma$-ideal $\mathcal{K}_\sigma$ of subsets of $\sigma$-compact sets in the Baire space $\omega^\omega$.

A metrizable space $X$ is *analytic* if it is a continuous image of a Polish space (see [26]).

**Proposition 5.** *Each $\mathfrak{q}_0$-flexible cccc $\sigma$-ideal $\mathcal{I}$ on an analytic space $X$ has* $\mathrm{non}(\mathcal{I}) \ge \mathfrak{q}_0$.

*Proof.* We need to show that any subset $A \subset X$ of cardinality $|A| < \mathfrak{q}_0$ belongs to the ideal $\mathcal{I}$. This is trivial if $|A| < \omega_1$. So, we assume that $\omega_1 \le |A| < \mathfrak{q}_0$.

Using the $\mathfrak{q}_0$-flexibility of $\mathcal{I}$, by transfinite induction we can choose a transfinite sequence $(f_\alpha)_{\alpha\in\omega_1}$ of automorphisms of $\mathcal{I}$ such that for every $\alpha \in \omega_1$ the set $A_\alpha = f_\alpha(A)$ is disjoint with $\bigcup_{\beta<\alpha} f_\beta(A)$. The set $A_{\omega_1} = \bigcup_{\alpha\in\omega_1} A_\alpha$ has cardinality $|A_{\omega_1}| = \max\{\omega_1, |A|\} < \mathfrak{q}_0$ and hence is a $Q$-set (here we use the fact $Q$-sets are preserved by homeomorphisms and $A_{\omega_1}$ being zero-dimensional, is homeomorphic to a subspace of the real line). Consequently, for every $\alpha \in \omega_1$ the subset $A_\alpha$ is $F_\sigma$ in $A_{\omega_1}$ and we can find an $F_\sigma$-set $F_\alpha \subset X$ such that $F_\alpha \cap A_{\omega_1} = A_\alpha$. It follows that for every $\alpha \in \omega_1$ the set $B_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta$ is Borel in $X$, contains $A_\alpha$, and the family $(B_\alpha)_{\alpha\in\omega_1}$ is disjoint. Each space $B_\alpha$ is analytic, being a Borel subset of the analytic space $X$. Consequently, we can find a surjective map $g_\alpha : \omega^\omega \to B_\alpha$ and choose a subset $A'_\alpha \subset \omega^\omega$ of cardinality $|A'_\alpha| = |A_\alpha|$ such that $g_\alpha(A'_\alpha) = A_\alpha$. Since $|A'_\alpha| = |A_\alpha| < \mathfrak{q}_0 \le \mathfrak{b}$, the set $A'_\alpha$ is contained in a $\sigma$-compact set $K'_\alpha \subset \omega^\omega$ according to the definition of the cardinal $\mathfrak{b}$. Then $K_\alpha = g_\alpha(K'_\alpha)$ is a $\sigma$-compact subset of $B_\alpha$ containing the set $A_\alpha$. Since the family $(K_\alpha)_{\alpha\in\omega_1}$ is disjoint and the $\sigma$-ideal $\mathcal{I}$ satisfies cccc, the set $\{\alpha \in \omega_1 : K_\alpha \notin \mathcal{I}\}$ is at most countable. So, for some ordinal $\alpha \in \omega_1$ the set $K_\alpha$ belongs to $\mathcal{I}$ and so does its subset $A_\alpha$. Then $A = f_\alpha^{-1}(A_\alpha) \in \mathcal{I}$ as $f_\alpha \in \mathrm{Aut}(\mathcal{I})$. □

Let $\widetilde{\mathcal{I}}_{cccc}$ be the intersection of all flexible cccc $\sigma$-ideals on the real line. Proposition 5 implies that $\mathfrak{q}_0 \le \mathrm{non}(\widetilde{\mathcal{I}}_{cccc})$. So, any upper bound on the cardinal $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc})$ yields an upper bound on $\mathfrak{q}_0$.

In fact, in the definition of the cardinal $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc})$ we can replace the real line by any uncountable zero-dimensional Polish space. Given a topological space $X$ denote by $\widetilde{\mathcal{I}}_{cccc}(X)$ the intersection of all flexible cccc $\sigma$-ideals on $X$.

**Proposition 6.** *Any uncountable Polish space $X$ has* $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) \le \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(X))$. *If the space $X$ is zero-dimensional, then* $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) = \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(X))$.

*Proof.* Choose a subset $A \subset X$ of cardinality $|A| = \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(X))$ that does not belong to the ideal $\widetilde{\mathcal{I}}_{cccc}(X)$ and hence does not belong to some $\mathfrak{c}$-flexible cccc $\sigma$-ideal $\mathcal{I}$ on $X$. Let $X'$ be the (closed) subset of $X$ consisting of all points $x \in X$ that have no countable neighborhood in $X$. It follows that the space $X'$ is perfect (i.e., has no isolated points) and the complement $X \setminus X'$ is countable and hence belongs to the ideal $\mathcal{I}$. Fix any countable dense subset $D \subset X'$ and observe the space $Z = X' \setminus D$ is Polish and nowhere locally compact. By [26, 7.7, 7.8], the space $Z$ is the image of the space of irrationals $\mathbb{R}\setminus\mathbb{Q}$ under an injective continuous map $f : \mathbb{R} \setminus \mathbb{Q} \to Z$. It can be shown that $\mathcal{J} = \{A \subset \mathbb{R} : f(A \setminus \mathbb{Q}) \in \mathcal{I}\}$ is a $\mathfrak{c}$-flexible cccc $\sigma$-ideal on $\mathbb{R}$ such that $f^{-1}(A) \notin \mathcal{J}$. So, $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) \le \mathrm{non}(\mathcal{J}) \le |f^{-1}(A)| \le |A| = \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(X))$.

If the space $X$ is zero-dimensional, then by [26, 7.7] the space $Z$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ and we can assume that $f : \mathbb{R} \setminus \mathbb{Q} \to Z$ is a homeomorphism.

Since the complement $X \setminus Z$ is countable, for every $\mathfrak{c}$-flexible cccc $\sigma$-ideal $\mathcal{I}$ on $\mathbb{R}$ the family $f(\mathcal{I}) = \{A \subset X : f^{-1}(A) \in \mathcal{I}\}$ is a $\mathfrak{c}$-flexible cccc $\sigma$-ideal on $X$, which implies that $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(X)) \le \mathrm{non}(\widetilde{\mathcal{I}}_{cccc})$. □

For a Polish group $G$ let $\mathcal{I}_{ccc}(G)$ be the intersection of all invariant ccc $\sigma$-ideals with Borel base on $G$. It is clear that $\widetilde{\mathcal{I}}_{cccc}(G) \subset \mathcal{I}_{ccc}(G)$ and hence $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(G)) \le \mathrm{non}(\mathcal{I}_{ccc}(G))$. For the compact Polish group $\mathbb{Z}_2^\omega = \{0,1\}^\omega$ the ideal $\mathcal{I}_{ccc}(\mathbb{Z}_2^\omega)$, denoted by $\mathcal{I}_{ccc}$, was introduced and studied by Zakrzewski [46], [47] who proved that $\mathfrak{s}_\omega \le \mathrm{non}(\mathcal{I}_{ccc}) \le \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$. Here $\mathfrak{s}_\omega$ is the $\omega$-splitting number introduced in [30] and studied in [11], [27]. It is clear that *splitting number* $\mathfrak{s}$ is not greater than $\mathfrak{s}_\omega$. On the other hand, the consistency of $\mathfrak{s} < \mathfrak{s}_\omega$ is an open problem (attributed to Steprans). By Theorems 3.3 and 6.9 [4], the cardinal $\mathfrak{s}$ is localized in the interval $[\mathfrak{h}, \mathfrak{d}]$, where $\mathfrak{h}$ is the *distributivity number* and $\mathfrak{d}$ is the *dominating number* (it is equal to the smallest cardinality of a cover of $\omega^\omega$ by compact subsets). The proof of the inequality $\mathfrak{s} \le \mathfrak{d}$ in Theorem 3.3 of [4] can be easily modified to obtain $\mathfrak{s}_\omega \le \mathfrak{d}$. In the following theorem by $\mathcal{E}$ we denote the $\sigma$-ideal generated by closed Lebesgue null sets on the real line.

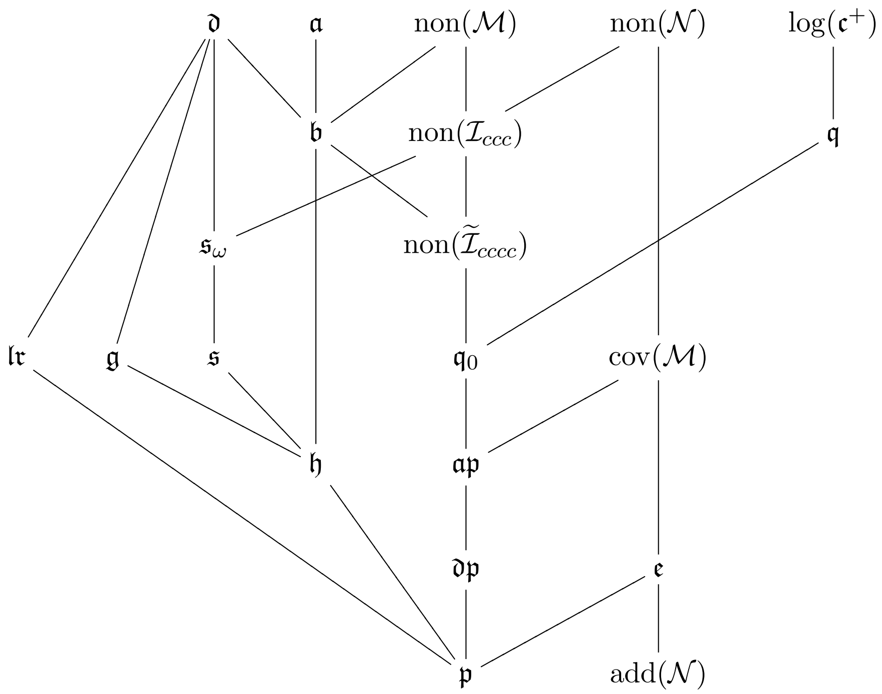
**Theorem 7.** *The following inequalities hold:*
$\mathfrak{q}_0 \le \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) \le \min\{\mathfrak{b}, \mathrm{non}(\mathcal{I}_{ccc})\} \le \min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{E})\}$.

*Proof.* The inequality $\mathfrak{q}_0 \le \mathrm{non}(\widetilde{\mathcal{I}}_{cccc})$ follows from Proposition 5. Since $\widetilde{\mathcal{I}}_{cccc}(\mathbb{Z}_2^\omega) \subset \mathcal{I}_{ccc}(\mathbb{Z}_2^\omega) = \mathcal{I}_{ccc}$, Proposition 6 guarantees that $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) = \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(\mathbb{Z}_2^\omega)) \le \mathrm{non}(\mathcal{I}_{ccc})$. Observe that the $\sigma$-ideal $\mathcal{K}_\sigma$ consisting of subsets of $\sigma$-compact sets in the topological group $\mathbb{Z}^\omega$ is a flexible cccc $\sigma$-ideal with $\mathrm{non}(\mathcal{K}_\sigma) = \mathfrak{b}$. Then Proposition 6 implies that $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) = \mathrm{non}(\widetilde{\mathcal{I}}_{cccc}(\mathbb{Z}^\omega)) \le \mathrm{non}(\mathcal{K}_\sigma) = \mathfrak{b}$. The inequality $\mathrm{non}(\mathcal{I}_{ccc}) \le \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$ follows from the fact that the ideals $\mathcal{M}$ and $\mathcal{N}$ are invariant ccc $\sigma$-ideals with Borel base. Taking into account that $\mathfrak{b} \le \mathrm{non}(\mathcal{M})$, we conclude that $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{I}_{ccc})\} \le \min\{\mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\} = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$. The equality $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{E})\}$ follows from Theorem 2.6.8 [2]. □

## 2. Consistency results

In this section we establish some consistent inequalities between the cardinals $\mathfrak{q}_0$, $\mathfrak{q}$ and some other known small uncountable cardinals. The definitions that are not included in this paper and provable relations between small cardinals one can be found in [4] and [45]. We consider also a relatively new cardinal $\mathfrak{lr}$, called the *linear refinement number*, that equals the minimal cardinality of a centered family $\mathcal{F} \subset [\omega]^\omega$ that has no linear refinement. A family $\mathcal{L} \subset [\omega]^\omega$ is a *linear refinement* of $\mathcal{F}$ if $\mathcal{L}$ is linearly ordered by the preorder $\subset^*$ and for every $F \in \mathcal{F}$ there is $L \in \mathcal{L}$ with $L \subset^* F$. Let us recall that $A \subset^* B$ whenever $A \setminus B$ is finite. The linear refinement number $\mathfrak{lr}$ was introduced by Tsaban in [44] (with the ad-hoc name $\mathfrak{p}^*$) and has been thoroughly studied in [28].

ZFC-inequalities between the cardinals $\mathfrak{dp}$, $\mathfrak{ap}$, $\mathfrak{q}_0$, $\mathfrak{q}$ and some other cardinal characteristics of the continuum are described in the following diagram (the inequality $\mathfrak{ap} \leq \mathrm{cov}(\mathcal{M})$ was proved by Brendle in [7]):




**Theorem 8.** *Each of the following inequalities is consistent with ZFC:*

*1)* $\omega_1 = \mathfrak{p} = \mathfrak{s} = \mathfrak{g} = \mathfrak{q}_0 = \mathfrak{q} = \log(\mathfrak{c}^+) < \mathrm{add}(\mathcal{N}) = \mathfrak{b} = \mathfrak{lr} = \mathfrak{c} = \omega_2$;

*2)* $\omega_1 = \mathfrak{q}_0 = \mathfrak{q} = \log(\mathfrak{c}^+) < \mathfrak{h} = \mathfrak{lr} = \mathfrak{c} = \omega_2$;

*3)* $\omega_1 = \mathfrak{p} = \mathfrak{s}_\omega < \mathfrak{dp} = \mathfrak{q} = \mathfrak{c} = \omega_2$;

*4)* $\omega_1 = \mathfrak{q}_0 = \mathfrak{q} = \mathfrak{b} < \mathfrak{g} = \omega_2$;

*5)* $\omega_1 = \mathfrak{q}_0 = \mathfrak{d} = \mathrm{non}(\mathcal{N}) < \mathfrak{q} = \mathfrak{c} = \omega_2$;

*6)* $\omega_1 = \mathfrak{q}_0 = \mathrm{non}(\mathcal{M}) = \mathfrak{a} < \mathfrak{q} = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$;

*7)* $\omega_1 = \mathfrak{dp} < \mathfrak{ap} = \mathfrak{c} = \omega_2$;

*8)* $\omega_1 = \mathfrak{ap} < \mathfrak{q}_0 = \mathfrak{c} = \omega_2$;

*9)* $\omega_1 = \mathfrak{p} < \mathfrak{lr} = \omega_2 < \mathfrak{q}_0 = \mathfrak{c} = \omega_3$.

*Proof.* 1. The consistency of $\omega_1 = \mathfrak{s} = \mathfrak{p} = \log(\mathfrak{c}^+) < \mathrm{add}(\mathcal{N}) = \mathfrak{b} = \mathfrak{c} = \omega_2$ is a direct consequence of [23, Theorems 3.2 and 3.3], see [23, Lemma 3.16] for some explanations. The equality $\omega_1 = \mathfrak{g}$ follows from the well-known fact that $\mathfrak{g}$ equals $\omega_1$ after iterations with finite supports of Suslin posets, see, e.g., [8]. The equality $\mathfrak{lr} = \omega_2$ follows from Theorem 2.2 [28] (saying that $\mathfrak{lr} = \omega_1$ implies $\mathfrak{d} = \omega_1$).

2. To obtain a model of $\omega_1 = \mathfrak{q}_0 = \log(\mathfrak{c}^+) < \mathfrak{h} = \omega_2$ let us consider an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \beta < \alpha \leq \omega_2\rangle$ with countable supports such that $\mathbb{Q}_0$ is the countably closed Cohen poset adding $\omega_3$-many subsets to $\omega_1$ with countable conditions. For every $0 < \alpha < \omega_2$ let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the Mathias forcing, see [31] or [4, p. 478]. It is standard to check that $2^{\omega_1} = \omega_3 > \omega_2 = 2^\omega$ holds in the final model, and hence $\log(\mathfrak{c}^+) = \mathfrak{q}_0 = \omega_1$ there. Also, $\mathfrak{h} = \omega_2 = 2^\omega$ in this model simply by the design of the Mathias poset, see the discussion on [4, p. 478]. The equality $\mathfrak{lr} = \omega_2$ follows from Theorem 2.2 [28].

3. A model with $\omega_1 = \mathfrak{p} < \mathfrak{dp} = \mathfrak{q} = \omega_2 = \mathfrak{c}$ was constructed by Alan Dow in [13], see Theorem 2 there. Below we shall also show that $\mathfrak{s}_\omega = \omega_1$ in that model. Following [9] we say that a forcing notion $\mathbb{P}$ *strongly preserves countable tallness* if for every sequence $\langle \tau_n : n \in \omega\rangle$ of $\mathbb{P}$-names for infinite subsets of $\omega$ there is a sequence $\langle B_n : n \in \omega\rangle$ of infinite subsets of $\omega$ such that for any $B \in [\omega]^\omega$, if $B \cap B_n$ is infinite for all $n$, then $\vDash_{\mathbb{P}}$ "$B \cap \tau_n$ is infinite for all $n$". In [13, Theorem 2] a poset $\mathbb{P}$ has been constructed such that $\mathfrak{q}_0 = \mathfrak{b} = \mathfrak{c} > \omega_1$ holds in $V^{\mathbb{P}}$. By the definition, $\mathbb{P}$ is an iteration with finite supports of posets of the form $\mathbb{Q}_{\mathcal{A}}$, see [13, Def. 2]. Observe that the notion of posets strongly preserving countable tallness remains the same if we demand the existence of the sequence $\langle B_n : n \in \omega\rangle$ with the property stated there just for a single $\mathbb{P}$-name $\tau$ for an infinite subset of $\omega$. Therefore it follows from Lemmata 2,3 in [13] that the posets $\mathbb{Q}_{\mathcal{A}}$ strongly preserve countable tallness. Applying [9, Lemma 5] we conclude that $\mathbb{P}$ strongly preserves countable tallness as well. The latter easily implies that the ground model reals are splitting, and hence $\mathfrak{s}_\omega = \omega_1$. Indeed, given a sequence of $\mathbb{P}$-names $\langle \tau_n : n \in \omega\rangle$ for an infinite subsets of $\omega$ find an appropriate sequence $\langle B_n : n \in \omega\rangle$ of ground model infinite subsets of $\omega$. Now let $X \in [\omega]^\omega \cap V$ be such that $X$ splits all the $B_n$'s. Then $\Vdash$ "$X$ splits every $\tau_n$".

4. The condition $\omega_1 = \mathfrak{b} = \mathfrak{q}_0 < \mathfrak{g} = \omega_2 = \mathfrak{c}$ holds, e.g., in the model of Blass and Shelah constructed in [6], and in the Miller's model constructed in [33], see [5] for the proof. If, as in item 2, these forcings are preceded by the countably closed Cohen poset adding $\omega_3$-many subsets to $\omega_1$ with countable conditions, then we get in addition $2^{\omega_1} = \omega_3 > \omega_2 = 2^\omega$ in the extension, and hence $\mathfrak{q}$ equals $\omega_1$ as well.

5. The consistency of $\omega_1 = \mathfrak{q}_0 = \mathfrak{d} = \mathrm{non}(\mathcal{N}) < \mathfrak{q} = \mathfrak{c} = \omega_2$ was proved by Judah and Shelah [24] (see also [34]).

6. A model with $\omega_1 = \mathfrak{q}_0 = \mathrm{non}(\mathcal{M}) = \mathfrak{a} < \mathfrak{q} = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \omega_2$ was constructed by Miller [34].

7 and 8. For every regular cardinal $\kappa > \omega_1$ the consistency of the strict inequalities $\omega_1 = \mathfrak{dp} < \kappa = \mathfrak{ap} = \mathfrak{c}$ and $\omega_1 = \mathfrak{ap} < \kappa = \mathfrak{q}_0 = \mathfrak{c}$ was proved by

Brendle [7].

9. The consistency of $\omega_1 = \mathfrak{p} < \mathfrak{lr} = \omega_2 < \mathfrak{q}_0 = \mathfrak{c} = \omega_3$ follows from Theorem 9 below. □

**Theorem 9.** *Assume the Generalized Continuum Hypothesis and let $\kappa$ and $\lambda$ be uncountable regular cardinal numbers such that $\kappa < \lambda = \lambda^{<\kappa}$. There is a forcing notion $\mathbb{P}$ such that in a generic extension $V[G]$: $\mathfrak{p} = \kappa$, $\mathfrak{lr} = \kappa^+$, and $\mathfrak{q}_0 = \lambda = \mathfrak{c}$.*

*Proof.* A forcing notion we use is very similar to one in Theorem 3.9 from [28]. The difference is that we use Dow's forcings $\mathbb{Q}_{\mathcal{A}}$ instead of Hechler forcing and a length of iteration is the ordinal $\lambda \cdot \lambda$.

More precisely the forcing $\mathbb{P}$ is given by an iteration:

1. $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda \cdot \lambda, \beta < \lambda \cdot \lambda \rangle$ is a finite support iteration;
2. $\mathbb{P} = \mathbb{P}_{\lambda \cdot \lambda}$;
3. $\mathbb{P}_0$ is the trivial forcing;
4. if $\alpha = \lambda \cdot \xi$ where $\xi > 0$, then:
   (a) $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is the Dow forcing $\mathbb{Q}_{\mathcal{A}_\xi}$ defined for a family $\dot{\mathcal{A}}_\xi$;
   (b) $\dot{\mathcal{A}}_\xi$ is a $\mathbb{P}_\alpha$-name for an ideal on $\omega$ generated by an almost disjoint family of cardinality $< \lambda$;
   (c) for each $\beta$ if $\Vdash_{\mathbb{P}_\beta} \dot{\mathcal{A}}$ is an ideal on $\omega$ generated by an almost disjoint family of cardinality $< \lambda$, then exists $\alpha > \beta$ such that $\alpha = \lambda \cdot \xi$ and $\Vdash_{\mathbb{P}_\alpha} \dot{\mathcal{A}} = \dot{\mathcal{A}}_\xi$.
5. if $\alpha \notin \{\lambda \cdot \xi : \xi > 0\}$, then
   (a) $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is the $\dot{\mathcal{F}}_\alpha$-Mathias forcing;
   (b) $\dot{\mathcal{F}}_\alpha$ is a name for a filter generated by a centered family $\{\dot{A}_{\alpha,\iota} : \iota < \iota_\alpha\}$ which contains cofinite sets, where $\iota_\alpha$ is an ordinal $< \kappa$;
   (c) $\iota_\alpha = 0$ for $\alpha < \lambda$ (thus $\mathbb{Q}_\alpha$ is isomorphic to Cohen's forcing for $\alpha < \lambda$);
   (d) $\dot{A}_{\alpha,\iota}$ is a $\mathbb{P}_\alpha$-name for a subset of $\omega$;
   (e) $b_{\alpha,\iota} \colon (2^\omega)^\omega \to [\omega]^\omega$ is a Borel function coded in the ground model;
   (f) $\Vdash_{\mathbb{P}_\alpha} \dot{A}_{\alpha,\iota} = b_{\alpha,\iota}(\langle \dot{B}_{\gamma(\alpha,\iota,n)} : n < \omega \rangle)$, where $B_\alpha \subset [\omega]^\omega$ denotes the $\alpha$-th generic real;
   (g) If $\alpha = \lambda \cdot \xi + \nu$, then $\gamma(\alpha, \iota, n) < \lambda \cdot \xi$.
   (h) For each $\zeta < \lambda$ and each sequence $\langle b_\iota : \iota < \iota_* \rangle$ of Borel functions $b_\iota \colon (2^\omega)^\omega \to [\omega]^\omega$ of length $\iota_* < \kappa$, and all ordinal numbers $\delta(\iota, n) < \lambda \cdot \zeta$ such that $\mathbb{P}$ forces that the filter generated by the cofinite sets together with the family $\{b_\iota(\langle B_{\delta(\iota,n)} : n < \omega \rangle) : \iota < \iota_*\}$, is proper, there are arbitrarily large $\alpha < \lambda \cdot (\zeta + 1)$ such that:

i. $\iota_\alpha = \iota_*$;
ii. $b_{\alpha,\iota} = b_\iota$ for all $\iota < \iota_*$;
iii. $\gamma(\alpha,\iota,n) = \delta(\iota,n)$ for all $\iota < \iota_*$ and all $n$.

A proof of equalities $\mathfrak{p} = \kappa$, $\mathfrak{lr} = \kappa^+$ is essentially the same as in Lemmata 3.11 – 3.15 in [28]. The only difference is in the iteration Lemma 3.10. By Dow's in Lemma 2 in [13], Dow's forcing notions cannot add a pseudointersection to eventually narrow families and, in particular, to the family of the Cohen reals. The usage of the Dow forcings instead of Hechler forcing give us an inequality $\mathfrak{q}_0 \geq \lambda$ instead of $\mathfrak{b} \geq \lambda$.

□

The argument in the remark below is usually attributed to Devlin and Shelah [12]. We have learned it from David Chodounsky.

**Remark 10.** We did not have to start with the countably closed Cohen poset adding $\omega_3$-many subsets to $\omega_1$ in items 2 and 4 of Theorem 8 in order to guarantee that $\mathfrak{q} = \omega_1$. However, the argument presented in the proof of Theorem 8 seems to be easier and more direct, and hence we presented it for those readers who are interested just in the consistency of corresponding constellations.

Following [35] we denote by $\diamondsuit(2,=)$ the following statement: *For every Borel* $F : \omega^{<\omega_1} \to 2$ *there exists* $g : \omega_1 \to 2$ *such that for every* $f : \omega_1 \to 2$ *the set* $\{\alpha : F(g \restriction \alpha) = f(\alpha)\}$ *is stationary*. Here $F : \omega^{<\omega_1} \to 2$ is Borel iff $f \restriction \omega^\alpha \to 2$ is Borel for all $\alpha \in \omega_1$. $\diamondsuit(2,=)$ implies that $\mathfrak{q} = \omega_1$, which means that no uncountable $Q$-set of reals exists. Indeed, suppose $X = \{x_\alpha : \omega < \alpha < \omega_1\}$ is a $Q$-set of reals. Choose some nice coding for $G_\delta$ sets of reals by elements of $2^\omega$. For each $\alpha \in (\omega,\omega_1)$ define $F_\alpha : 2^\alpha \to 2$ as follows: For $x$ in $2^\alpha$ put $F_\alpha(x) = 1$ iff $x_\alpha$ is in the $G_\delta$ set coded by $x \restriction \omega$. $F_\alpha$ is Borel and thus $F = \bigcup_{\alpha\in\omega_1} F_\alpha$ is also Borel. Therefore there exists a guessing function $g : \omega_1 \to 2$ for $F$. Put $Y = \{x_\alpha : g(\alpha) = 0\}$. Then $Y$ is not a $G_\delta$ subset of $X$. In order to show this choose a $G_\delta$ set $G$ and any $f : \omega_1 \to 2$ such that $f \restriction \omega$ codes $G$. Then there is $\beta$ such that $F(f \restriction \beta) = g(\beta)$, and hence $x_\beta$ is in $G \Delta Y$ which means $G \cap X \neq Y$. Finally, it suffices to note that $\diamondsuit(2,=)$ holds in any model considered in items 2,4 of Theorem 8, see [35, Theorem 6.6].

It would be nice to know more about the relation of the cardinals $\mathfrak{q}_0$ and $\mathfrak{q}$ to the cardinals $\mathfrak{g}$, $\mathfrak{e}$, $\mathrm{cov}(\mathcal{M})$, and $\mathrm{cov}(\mathcal{N})$. Here $\mathfrak{e}$ is the *evasion number* considered by A. Blass in [4, §10]. It follows from [4, 10.4] that $\mathfrak{q}_0 = \mathfrak{b} < \mathfrak{e}$ is consistent.

**Problem 1.** Is any of the inequalities $\mathfrak{q}_0 > \mathrm{cov}(\mathcal{M})$, $\mathfrak{q}_0 > \mathfrak{e}$, $\mathfrak{q}_0 > \mathfrak{g}$, $\mathrm{non}(\widetilde{\mathcal{I}}_{cccc}) > \mathfrak{q}_0$ consistent? In particular, what are the values of $\mathfrak{e}$ and $\mathfrak{g}$ in the model of Dow (or its modifications)?

The question whether $\mathfrak{q}_0 > \mathrm{cov}(\mathcal{M})$ is consistent seems the most intriguing among those mentioned above. In [7] this question is attributed to A. Miller.

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