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EXTENDED STOCHASTIC INTEGRALS WITH RESPECT TO A LÉVY PROCESS ON SPACES OF GENERALIZED FUNCTIONS

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One of the known generalizations of the chaotic representation property (CRP) for a Lévy process is based on orthogonalization of continuous monomials in the space (L^2) of square integrable random variables. Using this generalization of the CRP, we introduce riggings of (L^2) by spaces of test and (regular and nonregular) generalized functions, construct extended Skorohod stochastic integrals with respect to a Lévy process as linear continuous operators on the mentioned spaces of generalized functions, and establish some properties of these integrals.

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Одне з відомих узагальнень властивості хаотичного розкладу (ВХР) для процесу Леві базується на ортогоналізації неперервних мономів у просторі (L^2) квадратично інтегровних випадкових величин. Використовуючи це узагальнення ВХР, ми вводимо оснащення (L^2) просторами основних та (регулярних і нерегулярних) узагальнених функцій, будуємо розширені стохастичні інтеграли Скорохода за процесом Леві як лінійні неперервні оператори на згаданих просторах узагальнених функцій та встановлюємо деякі властивості цих інтегралів.

Introduction

Let $L = (L_t)_{t\in[0,+\infty)}$ be a Lévy process, i.e., a random process on $[0, +\infty)$ with stationary independent increments and such that $L_0 = 0$ (see, e.g., [5, 25, 26] for detailed information on Lévy processes). In particular cases, when L is a Wiener or Poisson process, any square integrable random variable can be decomposed into a series of repeated stochastic integrals

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of nonrandom functions with respect to L. This property of L is called the *chaotic representation property* (CRP), see, e.g., [22] for more information. The CRP plays a very important role in the stochastic analysis (in particular, it can be used to construct extended stochastic integrals, see, e.g., [14, 31, 13]), but, unfortunately, for a general Lévy process this property does not hold (e.g., [29]).

There are different generalizations of the CRP for Lévy processes. In particular, under Itô's approach [12] one decomposes a Lévy process L into the sum of a Gaussian process and a stochastic integral with respect to a Poisson random measure, and then uses the CRP for both terms in order to obtain a generalized CRP for L. Nualart-Schoutens' approach [23] (see also [27]) consists in decomposition of a square integrable random variable into a series of repeated stochastic integrals of nonrandom functions with respect to the so-called *orthogonalized centered power jump processes*; these processes are constructed using a cádlág version of L (i.e., a random process which is stochastically equivalent to L and has right continuous trajectories with finite left limits). Lytvynov's approach [21] is based on orthogonalization of continuous monomials in the space of square integrable random variables.

The interconnection between the above-mentioned generalizations of the CRP is described in, e.g., [21, 2, 28, 16]; one more example of a generalized CRP is given in [8, 7].

Let from now on L be a Lévy process without Gaussian part and drift (it is comparatively easy to study such processes from technical point of view). In order to construct an extended stochastic integral with respect to L, one can take any generalization of the CRP mentioned above. Namely, in the case of "Itô's CRP" the construction of this integral is analogous to the corresponding construction in the Poisson case, cf., e.g., [8] and [13]. In the case of "Nualart–Schoutens' CRP" one can use term by term integration of a Nualart–Schoutens decomposition of an integrand with respect to a random measure corresponding to L ([16]). In the case of "Lytvynov's CRP" one can construct the extended stochastic integral using a "special symmetrization" for kernels from the Lytvynov decomposition of an integrand [16] (see also [15, 18]), or as the operator adjoint to the Hida stochastic derivative. The reader can find more details on extended stochastic integrals with respect to Lévy processes in, e.g., [3, 20, 8, 6, 10, 24, 7, 16]; for a general background on stochastic integration on infinite-dimensional spaces see, e.g., [1, 9].

In the paper [16] the extended Skorohod stochastic integral with respect to a Lévy process, and the Hida stochastic derivative, in terms of the Lytvynov's generalization of the CRP, on the space of square integrable random variables were constructed; some properties of these operators were established; and it was shown that the extended stochastic integrals constructed with the use of three above-mentioned generalizations of the CRP coincide. But when we consider the stochastic integral as an operator on the space of square integrable random variables, then this operator is unbounded and, moreover, its domain depends on the interval of integration. This drawback essentially restricts an area of possible applications. Therefore, an important problem is to modify the definition of the extended stochastic integral in order to get a linear *bounded* (i.e., *continuous*) operator. A possible solution of this problem—to define the stochastic integrals as linear *continuous* operators acting on spaces of generalized functions (in the simplest case one can define the integral as an operator acting from the space of square integrable random variables to a space of regular generalized functions, see [17]). So, the aims of the present paper are to introduce riggings of the space of square integrable random variables by spaces of test and (regular and nonregular) generalized functions; to define the extended Skorohod stochastic integrals with respect to a Lévy process in terms of Lytvynov's generalization of the CRP as linear *continuous* operators on spaces of these riggings; and to describe some properties of these operators.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a convenient for our considerations probability triplet connected with L; then, following [16], we describe in detail Lytvynov's generalization of the CRP, the extended stochastic integral with respect to L, and the Hida stochastic derivative, on the space of square integrable random variables. In the second section we introduce riggings of the space of square integrable random variables by spaces of test and (regular and nonregular) generalized functions, and construct natural orthogonal bases in these spaces (we need these bases in order to define stochastic integrals). In the third section we introduce and study extended stochastic integrals on spaces of generalized functions.

1. Preliminaries

1.1. Lévy processes

Denote $\mathbb{R}_+ := [0, +\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t\in\mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. By the Lévy–Khintchine formula such a process can be represented in the form (e.g., [8]) $L_t = \int_0^t \int_{\mathbb{R}} x\widetilde{N}(du, dx)$, where $\widetilde{N}(du, dx, \cdot)$ is the compensated Poisson random measure of L, and the characteristic function of L is

$$\mathbb{E}[e^{iuL_t}] = \exp\Big[t\int_{\mathbb{R}}(e^{iux} - 1 - iux)\nu(dx)\Big], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$; here and below $\mathcal{B}$ denotes the Borel σ-algebra and $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$, *there exists* $\varepsilon > 0$ *such that*

$$\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty,$$

and

$$\int_{\mathbb{R}} x^2\nu(dx) = 1. \tag{2}$$

Let us define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [4]; see also Subsection 2.2). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle\omega, \varphi\rangle$; note that one can understand $\langle\cdot, \cdot\rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$. The notation $\langle\cdot, \cdot\rangle$ will be preserved for dual pairings in tensor powers of spaces.

Definition 1.1. A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x)du\nu(dx)\Big], \quad \varphi \in \mathcal{D}, \tag{3}$$

is called the *Lévy white noise measure.*

The existence of μ follows from the Bochner–Minlos theorem (e.g., [11]). Below we will assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is complete with respect to μ*, i.e., $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O) = 0$.

Denote $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) real-valued functions on $\mathcal{D}'$ that are square integrable with respect to μ; let also $\mathcal{H} := L^2(\mathbb{R}_+)$. Substituting in (3) $\varphi = t\psi$, $t \in \mathbb{R}$, $\psi \in \mathcal{D}$, and using the Taylor decomposition in t and (2), one can show that

$$\int_{\mathcal{D}'} \langle\omega,\psi\rangle^2\mu(d\omega) = \int_{\mathbb{R}_+} \big(\psi(u)\big)^2 du \tag{4}$$

(this statement follows also from the results of [21] and [8]). Let $f \in \mathcal{H}$ and $\mathcal{D} \ni \varphi_k \to f$ in $\mathcal{H}$ as $k \to \infty$. It follows from (4) that $\{\langle\circ,\varphi_k\rangle\}_{k\geq 1}$ is a Cauchy sequence in (L^2), therefore one can define $\langle\circ, f\rangle := \lim_{k\to\infty}\langle\circ,\varphi_k\rangle \in (L^2)$ (the limit in the topology of (L^2)). It is easy to show (by the method of “mixed sequences”) that $\langle\circ, f\rangle$ does not depend on the choice of an approximating sequence for f and therefore is well defined in (L^2).

Let us consider $\langle\circ, 1_{[0,t)}\rangle \in (L^2)$, $t \in \mathbb{R}_+$ (here and below 1_A denotes the indicator of a set A). It follows from (1) and (3) that $\big(\langle\circ, 1_{[0,t)}\rangle\big)_{t\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$, i.e., one can write $L_t = \langle\circ, 1_{[0,t)}\rangle \in (L^2)$.

1.2. Lytvynov's generalization of the CRP

Denote by $\widehat{\otimes}$ a symmetric tensor product. Let $\mathcal{P} \equiv \mathcal{P}(\mathcal{D}')$ be the set of continuous polynomials on $\mathcal{D}'$, i.e., elements of $\mathcal{P}$ have the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle\omega^{\otimes n}, f^{(n)}\rangle, \quad \omega \in \mathcal{D}',\ N_f \in \mathbb{Z}_+,\ f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n},\ f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle\omega^{\otimes 0}, f^{(0)}\rangle := f^{(0)} \in \mathcal{D}^{\widehat{\otimes} 0} := \mathbb{R}$. Since the Laplace transform of the Lévy white noise measure μ is holomorphic at zero (this follows from (3) and properties of the measure ν, see also [21]), $\mathcal{P}$ is a dense set in (L^2) ([30]). Denote by $\mathcal{P}_n$ the set of continuous polynomials of power $\leq n$, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal complement in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \bigoplus_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\!\langle\circ^{\otimes n}, f^{(n)}\rangle\!:$ the orthogonal projection of a monomial $\langle\circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$. Let us define the scalar products $\langle\cdot,\cdot\rangle_{ext}$ on $\mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$

$$\langle f^{(n)}, g^{(n)}\rangle_{ext} := \frac{1}{n!}\int_{\mathcal{D}'} :\!\langle\omega^{\otimes n}, f^{(n)}\rangle\!::\!\langle\omega^{\otimes n}, g^{(n)}\rangle\!:\mu(d\omega),$$

and let $|\cdot|_{ext}$ be the corresponding norms, i.e., $|f^{(n)}|_{ext} = \sqrt{\langle f^{(n)}, f^{(n)}\rangle_{ext}}$. Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the closures of $\mathcal{D}^{\widehat{\otimes} n}$ with respect to the norms $|\cdot|_{ext}$. For $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, f^{(n)}\rangle: \stackrel{\text{def}}{=} (L^2) - \lim_{k\to\infty} :\langle \circ^{\otimes n}, f_k^{(n)}\rangle:$, where $\mathcal{D}^{\widehat{\otimes} n} \ni f_k^{(n)} \to f^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$ (well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)}\rangle:, f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}\}$ is dense in $\mathbf{P}_n$, the following statement is fulfilled.

Theorem 1.2. *Let $F \in (L^2)$. Then there exists a unique sequence of kernels $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle: \tag{5}$$

and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{ext}^2. \tag{6}$$

Vice versa, any series (5) *with finite norm* (6) *is an element of* (L^2).

Note that for $F, G \in (L^2)$ the scalar product has the form

$$(F, G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega) G(\omega) \mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n! \langle f^{(n)}, g^{(n)}\rangle_{ext},$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G respectively. In particular, for $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $g^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned}\big(:\langle \circ^{\otimes n}, f^{(n)}\rangle:, :\langle \circ^{\otimes m}, g^{(m)}\rangle:\big)_{(L^2)} &= \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, f^{(n)}\rangle::\langle \omega^{\otimes m}, g^{(m)}\rangle: \mu(d\omega) \\ &= \mathbb{E}\big[:\langle \circ^{\otimes n}, f^{(n)}\rangle::\langle \circ^{\otimes m}, g^{(m)}\rangle:\big] = \delta_{n,m} n! \langle f^{(n)}, g^{(n)}\rangle_{ext}.\end{aligned}$$

Remark 1.3. It was shown in [21] that in the space (L^2) $:\langle \circ^{\otimes 0}, f^{(0)}\rangle: = \langle \circ^{\otimes 0}, f^{(0)}\rangle = f^{(0)}$ and $:\langle \circ, f^{(1)}\rangle: = \langle \circ, f^{(1)}\rangle$. But for $n > 1$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ is not a continuous polynomial, generally speaking. Moreover, in this case the elements $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ are continuous polynomials (and even generalized Appell polynomials, or Schefer polynomials in another terminology) if and only if our Lévy process L belongs to the so-called *Meixner class of random processes*, see [21] for details.

In order to work with spaces $\mathcal{H}_{ext}^{(n)}$, it is necessary to know the explicit formulas for the scalar products $\langle \cdot, \cdot\rangle_{ext}$. Let us write out these formulas. Denote by $\|\cdot\|_\nu$ the norm in the space $L^2(\mathbb{R}, \nu)$ of (classes of) real-valued functions on $\mathbb{R}$ that are square integrable with respect to ν. Let

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R},\ j \in \{1, \ldots, n-1\},\ n \in \mathbb{N}, \tag{7}$$

be polynomials orthogonal in $L^2(\mathbb{R}, \nu)$, i.e., for natural numbers n, m such that $n \neq m$, $\int_{\mathbb{R}} p_n(x) p_m(x) \nu(dx) = 0$.

Proposition 1.4. ([21]) *For* $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, *we have*

$$\langle f^{(n)}, g^{(n)}\rangle_{ext} = \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}$$

$$\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} f^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k}) \tag{8}$$

$$\times g^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})du_1\cdots du_{s_1+\dots+s_k}.$$

In particular, for $n=1$ $\langle f^{(1)}, g^{(1)}\rangle_{ext} = \langle f^{(1)}, g^{(1)}\rangle$; *if* $n=2$ *then* $\langle f^{(2)}, g^{(2)}\rangle_{ext} = \langle f^{(2)}, g^{(2)}\rangle + \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+} f^{(2)}(u,u)g^{(2)}(u,u)du$, *etc.*

As is easy to see, formulas (8) hold true for $f^{(n)}, g^{(n)} \in \mathcal{H}^{(n)}_{ext}$.

It follows from (8) that $\mathcal{H}^{(1)}_{ext} = \mathcal{H} \equiv L^2(\mathbb{R}_+)$: by (7) $p_1(x) = x$ and therefore by (2) $\|p_1\|_\nu = 1$; and for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}^{(n)}_{ext}$ that consists of "vanishing on diagonals" elements (i.e., $f^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k, j \in \{1,\dots,n\}$ such that $k \neq j$ but $u_k = u_j$). In this sense the *space* $\mathcal{H}^{(n)}_{ext}$ *is an extension of* $\mathcal{H}^{\widehat{\otimes} n}$ (this explains why we use the subscript *ext* in the designations $\mathcal{H}^{(n)}_{ext}$, $\langle\cdot,\cdot\rangle_{ext}$ and $|\cdot|_{ext}$).

1.3. An extended stochastic integral on the space of square integrable random variables

Let $F \in (L^2)\otimes\mathcal{H}$. It follows from representation (5) for elements of (L^2) that F can be presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}_{\cdot}\rangle:, \quad f^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}. \tag{9}$$

Let us describe the construction of an extended stochastic integral that is based on this decomposition and is correlated with the structure of the spaces $\mathcal{H}^{(n)}_{ext}$. Note that in the case when L is a process of Meixner type (e.g., [21]), such an integral is constructed and studied in [15].

Let $f^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}^{(n)}_{\cdot} \in f^{(n)}_{\cdot}$ such that

$$\dot{f}^{(n)}_u(u_1,\dots,u_n) = 0 \text{ if for some k} \in \{1,\dots,\text{n}\},\ u = u_k. \tag{10}$$

Accept by default that $t_1, t_2 \in [0,+\infty]$, $t_1 < t_2$. Let $\widetilde{f}^{(n)}_{[t_1,t_2)}$ be the symmetrization of $\dot{f}^{(n)}_{\cdot} 1_{[t_1,t_2)}(\cdot)$ by $n+1$ variables. Define $\widehat{f}^{(n)}_{[t_1,t_2)} \in \mathcal{H}^{(n+1)}_{ext}$ as the equivalence class in $\mathcal{H}^{(n+1)}_{ext}$ generated by $\widetilde{f}^{(n)}_{[t_1,t_2)}$. The next statement is a trivial modification of the corresponding result from [16].

Lemma 1.5. *For each* $f^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}$, $n \in \mathbb{N}$, *the element* $\widehat{f}^{(n)}_{[t_1,t_2)} \in \mathcal{H}^{(n+1)}_{ext}$ *is well defined (in particular,* $\widehat{f}^{(n)}_{[t_1,t_2)}$ *does not depend on the choice of a representative* $\dot{f}^{(n)}_{\cdot} \in f^{(n)}_{\cdot}$ *satisfying* (10)*) and*

$$|\widehat{f}^{(n)}_{[t_1,t_2)}|_{ext} \le |f^{(n)}_{\cdot} 1_{[t_1,t_2)}(\cdot)|_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}} \le |f^{(n)}_{\cdot}|_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}}. \tag{11}$$

Definition 1.6. For $F \in (L^2)\otimes\mathcal{H}$, $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, we define an extended stochastic integral $\int_{t_1}^{t_2} F(u)\widehat{d}L_u \in (L^2)$ by setting

$$\int_{t_1}^{t_2} F(u)\widehat{d}L_u := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{f}^{(n)}_{[t_1,t_2)}\rangle :, \tag{12}$$

where $\widehat{f}^{(0)}_{[t_1,t_2)} := f^{(0)}_{\cdot} 1_{[t_1,t_2)}(\cdot) \in \mathcal{H} = \mathcal{H}^{(1)}_{ext}$, and $\widehat{f}^{(n)}_{[t_1,t_2)} \in \mathcal{H}^{(n+1)}_{ext}$, $n \in \mathbb{N}$, are constructed by the kernels $f^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}$ from decomposition (9) for F, if the series in the right hand side of (12) converges in (L^2).

The domain of this integral, i.e., of the operator

$$\int_{t_1}^{t_2} \circ(u)\widehat{d}L_u : (L^2)\otimes\mathcal{H} \to (L^2), \tag{13}$$

consists of $F \in (L^2)\otimes\mathcal{H}$ such that (see (6))

$$\Big\| \int_{t_1}^{t_2} F(u)\widehat{d}L_u \Big\|^2_{(L^2)} = \sum_{n=0}^{\infty} (n+1)! |\widehat{f}^{(n)}_{[t_1,t_2)}|^2_{ext} < \infty. \tag{14}$$

Theorem 1.7. ([16]) *Let $F \in (L^2)\otimes\mathcal{H}$ be integrable by Itô (i.e., F is adapted with respect to the flow of σ-algebras generated by the Lévy process L). Then for any $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, F is integrable in the extended sense and*

$$\int_{t_1}^{t_2} F(u)\widehat{d}L_u = \int_{t_1}^{t_2} F(u)dL_u,$$

where $\int_{t_1}^{t_2} F(u)dL_u$ is the Itô stochastic integral.

1.4. A Hida stochastic derivative and its interconnection with the extended stochastic integral

In order to define a stochastic derivative on (L^2) we need some preparation. Let $g^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in g^{(n)}$ be a representative of $g^{(n)}$. We consider $\dot{g}^{(n)}(\cdot)$, i.e., separate one argument of $g^{(n)}$, and define $g^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ as the equivalence class in $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ generated by $\dot{g}^{(n)}(\cdot)$.

Lemma 1.8. ([16]) *For each $g^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, the element $g^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ is well defined (in particular, $g^{(n)}(\cdot)$ does not depend on the choice of a representative $\dot{g}^{(n)} \in g^{(n)}$) and*

$$|g^{(n)}(\cdot)|_{\mathcal{H}^{(n-1)}_{ext}\otimes\mathcal{H}} \le |g^{(n)}|_{ext}. \tag{15}$$

Remark 1.9. Note that, in spite of estimate (15), the space $\mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}\backslash\{1\}$, can not be considered as a subspace of $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ because different elements of $\mathcal{H}^{(n)}_{ext}$ can coincide as elements of $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$.

Definition 1.10. Let $t_1, t_2 \in [0,+\infty]$, $t_1 < t_2$. For $G \in (L^2)$ we define a Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial_\cdot G \in (L^2) \otimes \mathcal{H}$ by setting

$$1_{[t_1,t_2)}(\cdot)\partial_\cdot G := \sum_{n=0}^{\infty}(n+1){:}\langle \circ^{\otimes n}, g^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot)\rangle{:}, \tag{16}$$

where $g^{(n+1)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (5) for G, considered as elements of $\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, if the series in the right hand side of (16) converges in $(L^2) \otimes \mathcal{H}$.

The domain of this derivative, i.e., of the operator

$$1_{[t_1,t_2)}(\cdot)\partial_\cdot : (L^2) \to (L^2) \otimes \mathcal{H}, \tag{17}$$

consists of $G \in (L^2)$ such that

$$\|1_{[t_1,t_2)}(\cdot)\partial_\cdot G\|^2_{(L^2)\otimes\mathcal{H}} = \sum_{n=0}^{\infty}(n+1)!(n+1)|g^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot)|^2_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}} < \infty.$$

Theorem 1.11. ([16]) *For arbitrary $t_1, t_2 \in [0,+\infty]$, $t_1 < t_2$, extended stochastic integral* (13) *and Hida stochastic derivative* (17) *are mutually adjoint:*

$$\int_{t_1}^{t_2} \circ(u)\widehat{d}L_u = (1_{[t_1,t_2)}(\cdot)\partial_\cdot)^*\circ, \qquad 1_{[t_1,t_2)}(\cdot)\partial_\cdot = \Big(\int_{t_1}^{t_2} \circ\widehat{d}L\Big)^*_\cdot. \tag{18}$$

In particular, integral (13) *and derivative* (17) *are closed operators.*

Note that equalities (18) can be used as alternative definitions of the extended stochastic integral and the Hida stochastic derivative.

2. Spaces of test and generalized functions

2.1. A regular rigging of (L^2)

Denote $\mathcal{P}_W := \big\{f = \sum_{n=0}^{N_f} {:}\langle \circ^{\otimes n}, f^{(n)}\rangle{:}, f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\big\} \subset (L^2)$. Accept by default that $q \in \mathbb{Z}_+$, $\beta \in [0,1]$, and define scalar products $(\cdot,\cdot)_{q,\beta}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} {:}\langle \circ^{\otimes n}, f^{(n)}\rangle{:},\ g = \sum_{n=0}^{N_g} {:}\langle \circ^{\otimes n}, g^{(n)}\rangle{:} \in \mathcal{P}_W \tag{19}$$

$$(f,g)_{q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta}2^{qn}\langle f^{(n)}, g^{(n)}\rangle_{ext}.$$

Let $\|\cdot\|_{q,\beta}$ be the corresponding norms, i.e., $\|f\|_{q,\beta} = \sqrt{(f,f)_{q,\beta}}$.

Definition 2.1. We define parametrized Kondratiev-type spaces of test functions $(L^2)_q^\beta$ as closures of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{q,\beta}$; and set $(L^2)^\beta := \mathrm{pr\,lim}_{q\in\mathbb{Z}_+}(L^2)_q^\beta$ (the projective limit of spaces).

As is easy to see, $f \in (L^2)_q^\beta$ if and only if f can be presented in form (5) with $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and

$$\|f\|_{q,\beta}^2 := \|f\|_{(L^2)_q^\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |f^{(n)}|_{ext}^2 < \infty; \tag{20}$$

and for $f, g \in (L^2)_q^\beta$

$$(f, g)_{(L^2)_q^\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} \langle f^{(n)}, g^{(n)} \rangle_{ext},$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for f and g correspondingly. Further, $f \in (L^2)^\beta$ if and only if f can be presented in form (5) and norm (20) is finite for each $q \in \mathbb{Z}_+$.

Proposition 2.2. *For any $q \in \mathbb{Z}_+$ and $\beta \in [0, 1]$ the space $(L^2)_q^\beta$ is densely and continuously embedded into (L^2).*

Proof. Let $q \in \mathbb{Z}_+$, $\beta \in [0, 1]$ and $f \in (L^2)_q^\beta$. It is obvious that $\|f\|_{q,\beta} \geq \|f\|_{(L^2)}$. Further, if $\|f\|_{q,\beta} \neq 0$ then there exists at least one non-zero $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ from decomposition (5) for f, therefore $\|f\|_{(L^2)} \neq 0$. Hence $(L^2)_q^\beta$ is continuously embedded into (L^2). The density of this embedding follows from the fact that $\mathcal{P}_W$ is a dense set in (L^2). □

In view of this proposition, one can consider a chain

$$(L^2)^{-\beta} \supset (L^2)_{-q}^{-\beta} \supset (L^2) \supset (L^2)_q^\beta \supset (L^2)^\beta, \tag{21}$$

where $(L^2)_{-q}^{-\beta}$, $(L^2)^{-\beta} = \text{ind}\lim_{q \in \mathbb{Z}_+} (L^2)_{-q}^{-\beta}$ (the inductive limit of spaces) are the spaces dual of $(L^2)_q^\beta$, $(L^2)^\beta$ correspondingly with respect to (L^2).

Definition 2.3. The spaces $(L^2)_{-q}^{-\beta}$, $(L^2)^{-\beta}$ are called parametrized Kondratiev-type spaces of regular generalized functions.

The next statement follows from the definition of the spaces $(L^2)_{-q}^{-\beta}$ and the general duality theory.

Proposition 2.4. *1) Any regular generalized function $F \in (L^2)_{-q}^{-\beta}$ can be presented as a formal series*

$$F = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F^{(m)} \rangle:, \;\; F^{(m)} \in \mathcal{H}_{ext}^{(m)}, \tag{22}$$

that converges in $(L^2)_{-q}^{-\beta}$, i.e.,

$$\|F\|_{-q,-\beta}^2 := \|F\|_{(L^2)_{-q}^{-\beta}}^2 = \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} |F^{(m)}|_{ext}^2 < \infty; \tag{23}$$

and, vice versa, any formal series (22) with finite norm (23) is a regular generalized function from $(L^2)_{-q}^{-\beta}$;

2) for $F, G \in (L^2)^{-\beta}_{-q}$ *the scalar product has the form*

$$(F,G)_{(L^2)^{-\beta}_{-q}} = \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} \langle F^{(m)}, G^{(m)} \rangle_{ext},$$

where $F^{(m)}, G^{(m)} \in \mathcal{H}^{(m)}_{ext}$ *are the kernels from decompositions* (22) *for* F *and* G *respectively;*

3) the dual pairing between $F \in (L^2)^{-\beta}_{-q}$ *and* $f \in (L^2)^{\beta}_{q}$ *that is generated by the scalar product in* (L^2)*, has a form*

$$\langle\!\langle F, f \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, f^{(n)} \rangle_{ext},$$

where $F^{(n)}, f^{(n)} \in \mathcal{H}^{(n)}_{ext}$ *are the kernels from decompositions* (22) *and* (5) *for* F *and* f *respectively.*

Corollary 2.5. $F \in (L^2)^{-\beta}$ *if and only if* F *can be presented in form* (22) *and norm* (23) *is finite for some* $q \in \mathbb{Z}_+$.

Remark 2.6. We use the term "*regular* generalized functions" for elements of $(L^2)^{-\beta}_{-q}$ and $(L^2)^{-\beta}$ because the kernels from decompositions (22) of these elements and the kernels from decompositions (5) of test functions belong to the same spaces.

2.2. A nonregular rigging of (L^2)

Denote by T the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let $\mathcal{H}_\tau$ be the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., the scalar product in $\mathcal{H}_\tau$ is given by the formula

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{(k)}(u)\psi^{(k)}(u) \Big) \tau_2(u) du,$$

here $\varphi^{(k)}(\cdot)$ and $\psi^{(k)}(\cdot)$ are derivatives of order k of functions φ and ψ correspondingly. Denote the norms in $\mathcal{H}_\tau$ and its tensor powers by $|\cdot|_\tau$, i.e., for $\varphi_n \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, $|\varphi_n|_\tau = \sqrt{(\varphi_n, \varphi_n)_{\mathcal{H}_\tau^{\widehat{\otimes} n}}}$. Note that $\mathcal{D} = \mathrm{pr}\lim_{\tau \in T} \mathcal{H}_\tau$ and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$ (cf. [4]), therefore one can consider the rigging

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Denote the norms in $\mathcal{H}_{-\tau}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, by $|\cdot|_{-\tau}$.

Accept by default that $q \in \mathbb{Z}_+$, $\tau \in T$, and define scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ (see Subsection 2.1) by setting for $f, g \in \mathcal{P}_W$ of form (19)

$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f, N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes} n}}.$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f,f)_{\tau,q}}$.

Definition 2.7. We define Kondratiev spaces of test functions $(\mathcal{H}_\tau)_q$ as closures of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{\tau,q}$; and set $(\mathcal{H}_\tau) := \mathrm{pr}\lim_{q\in\mathbb{Z}_+}(\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \mathrm{pr}\lim_{q\in\mathbb{Z}_+,\tau\in T}(\mathcal{H}_\tau)_q$.

As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \;\; f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}, \tag{24}$$

with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_\tau)_q}^2 = \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f^{(n)}|_\tau^2 < \infty; \tag{25}$$

and for $f, g \in (\mathcal{H}_\tau)_q$

$$(f, g)_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty}(n!)^2 2^{qn}(f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes} n}},$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ are the kernels from decompositions (24) for f and g correspondingly (here for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ is the projection of a monomial $\langle \circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be presented in form (24) and norm (25) is finite for each $q \in \mathbb{Z}_+$ (for each $q \in \mathbb{Z}_+$ and each $\tau \in T$).

In order to construct an analog of chain (21) with Kondratiev spaces of test functions, we need some preparation. By analogy with [15] one can easily show that the Lévy white noise measure μ is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Removing from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each* $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.

Lemma 2.8. *There exists $\tau' \in T$ such that for each $n \in \mathbb{N}$ the space $\mathcal{H}_{\tau'}^{\widehat{\otimes} n}$ is densely and continuously embedded into $\mathcal{H}_{ext}^{(n)}$. Moreover, for all $f^{(n)} \in \mathcal{H}_{\tau'}^{\widehat{\otimes} n}$*

$$|f^{(n)}|_{ext}^2 \le n! c^n |f^{(n)}|_{\tau'}^2, \tag{26}$$

where $c > 0$ is some constant.

Proof. At first we will show that there exists $\tau' \in T$ such that for all $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$ estimate (26) is valid. By analogy with, e.g., [19] one can prove that, since the Laplace transform of the Lévy white noise measure μ is holomorphic at zero, there exist $\tau' \in T$ and $\varepsilon > 0$ such that $K(\tau', \varepsilon) := \int_{\mathcal{H}_{-\tau'}} e^{2\varepsilon|\omega|_{-\tau'}}\mu(d\omega) < \infty$. Further, it follows from the Taylor decomposition of the exponential function that for arbitrary $\omega \in \mathcal{H}_{-\tau'}$ and $n \in \mathbb{Z}_+$

$$|\omega^{\otimes n}|_{-\tau'} = |\omega|_{-\tau'}^n \le n! \frac{e^{\varepsilon|\omega|_{-\tau'}}}{\varepsilon^n},$$

therefore

$$\||\circ^{\otimes n}|_{-\tau'}\|_{(L^2)}^2 = \int_{\mathcal{H}_{-\tau'}} |\omega^{\otimes n}|_{-\tau'}^2 \mu(d\omega) \le \frac{(n!)^2}{\varepsilon^{2n}} K(\tau', \varepsilon). \tag{27}$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}$. We have

$$n!|f^{(n)}|_{ext}^2 = \int_{\mathcal{D}'} |:\langle \omega^{\otimes n}, f^{(n)}\rangle :|^2 \mu(d\omega) \le \int_{\mathcal{H}_{-\tau'}} |\langle \omega^{\otimes n}, f^{(n)}\rangle|^2 \mu(d\omega)$$
$$\le |f^{(n)}|_{\tau'}^2 \int_{\mathcal{H}_{-\tau'}} |\omega^{\otimes n}|_{-\tau'}^2 \mu(d\omega) \le \frac{(n!)^2}{\varepsilon^{2n}} K(\tau', \varepsilon) |f^{(n)}|_{\tau'}^2,$$

whence estimate (26) for $f^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}$ follows.

In order to prove that $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ is embedded into $\mathcal{H}_{ext}^{(n)}$ it remains to show that if a sequence $\big(f_k^{(n)}\big)_{k=0}^\infty \subset \mathcal{D}^{\widehat{\otimes}n}$ is a Cauchy one in $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ and simultaneously $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{ext}^{(n)}$ then $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ (see, e.g., [4]). In fact, if $\big(f_k^{(n)}\big)_{k=0}^\infty$ is such a sequence, then by the inequality $|\cdot|_{\mathcal{H}^{\widehat{\otimes}n}} \le |\cdot|_{\mathcal{H}_{ext}^{(n)}}$ (see (8)) $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}^{\widehat{\otimes}n}$. But $\mathcal{D}^{\widehat{\otimes}n} \subset \mathcal{H}_{\tau'}^{\widehat{\otimes}n} \subset \mathcal{H}^{\widehat{\otimes}n}$, therefore $\lim_{k\to\infty} f_k^{(n)} = 0$ in $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$. Estimate (26) for a general $f^{(n)} \in \mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ can be obtained by the corresponding passage to the limit. Finally, the embedding of $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ into $\mathcal{H}_{ext}^{(n)}$ is dense because $\mathcal{D}^{\widehat{\otimes}n} \subset \mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ and $\mathcal{D}^{\widehat{\otimes}n}$ is a dense set in $\mathcal{H}_{ext}^{(n)}$; and the continuity of this embedding follows from (26). □

Remark 2.9. It is not difficult to see that if for some $\tau \in T$ $\mathcal{H}_\tau$ is continuously embedded into $\mathcal{H}_{\tau'}$ then for each $n \in \mathbb{N}$ $\mathcal{H}_\tau^{\widehat{\otimes}n}$ is densely and continuously embedded into $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$

$$|f^{(n)}|_{ext}^2 \le n! c(\tau)^n |f^{(n)}|_\tau^2.$$

In what follows, it will be convenient to assume that the *indexes τ such that $\mathcal{H}_\tau$ is not continuously embedded into $\mathcal{H}_{\tau'}$, are removed from T.*

Proposition 2.10. *For each $\tau \in T$ there exists $q_0 = q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0} := \{q_0, q_0+1, \cdots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2).*

Proof. Let $\tau, \tau' \in T$, where τ' is the parameter from Lemma 2.8. Since for any $n \in \mathbb{N}$ $|\circ|_{\mathcal{H}_{-\tau}^{\widehat{\otimes}n}} \le c(\tau)^n |\circ|_{\mathcal{H}_{-\tau'}^{\widehat{\otimes}n}}$ with some $c(\tau) > 0$ (now $\mathcal{H}_{-\tau'} \subseteq \mathcal{H}_{-\tau}$ due to our modification of the set T), it follows from estimate (27) that there exist $\varepsilon(\tau) > 0$ and $K \in (0, +\infty)$ such that

$$\| |\circ^{\otimes n}|_{-\tau} \|_{(L^2)} \le \frac{n!}{\varepsilon(\tau)^n} K. \tag{28}$$

Let $f \in (\mathcal{H}_\tau)_q$. Using decomposition (24), estimate (28), and (25), we obtain

$$\|f\|_{(L^2)} \le \sum_{n=0}^\infty \|:\langle \circ^{\otimes n}, f^{(n)}\rangle :\|_{(L^2)} \le \sum_{n=0}^\infty \|\langle \circ^{\otimes n}, f^{(n)}\rangle\|_{(L^2)}$$
$$\le \sum_{n=0}^\infty \| |\circ^{\otimes n}|_{-\tau} \|_{(L^2)} |f^{(n)}|_\tau \le \sum_{n=0}^\infty \frac{n!}{\varepsilon(\tau)^n} K |f^{(n)}|_\tau$$
$$\le \sqrt{\sum_{n=0}^\infty (n!)^2 2^{qn} |f^{(n)}|_\tau^2} \sqrt{\sum_{n=0}^\infty \frac{K^2}{(2^q \varepsilon(\tau)^2)^n}} = \|f\|_{\tau,q} \sqrt{\sum_{n=0}^\infty \frac{K^2}{(2^q \varepsilon(\tau)^2)^n}} < \infty,$$

if $q \in \mathbb{N}_{q_0}$, where $q_0 = q_0(\tau)$ is such that $2^{q_0}\varepsilon(\tau)^2 > 1$. Further, let $\|f\|_{(L^2)} = 0$. Then for each $n \in \mathbb{Z}_+$ $|f^{(n)}|_{ext} = 0$, where $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \subset \mathcal{H}_{ext}^{(n)}$ are the kernels from decomposition (24) for f (see Lemma 2.8; by definition $\mathcal{H}_\tau^{\widehat{\otimes} 0} = \mathbb{R}$), therefore $|f^{(n)}|_\tau = 0$. But this means that $\|f\|_{\tau,q} = 0$, so, the continuous embedding of $(\mathcal{H}_\tau)_q$ into (L^2) is proved. Finally, this embedding is dense because $\mathcal{P}_W$ is a dense set in $(\mathcal{H}_\tau)_q$ and in (L^2). □

In view of this proposition for $\tau \in T$ and $q \geq q_0(\tau)$ one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}),$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \text{ind}\lim_{q\in\mathbb{N}_{q_0}}(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{D}') = \text{ind}\lim_{\tau\in T}(\mathcal{H}_{-\tau})$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$, $(\mathcal{D})$ correspondingly with respect to (L^2).

Definition 2.11. The spaces $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$, $(\mathcal{D}')$ are called Kondratiev spaces of generalized functions.

Remark 2.12. Let $q \in \mathbb{Z}_+$, $\tau \in T$ and $\beta \in [0,1]$. One can introduce on $\mathcal{P}_W$ scalar products $(\cdot,\cdot)_{\tau,q,\beta}$ by setting for $f, g \in \mathcal{P}_W$ of form (19)

$$(f,g)_{\tau,q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta} 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes} n}},$$

and define "parametrized Kondratiev spaces of test functions" $(\mathcal{H}_\tau)_q^\beta$ as closures of $\mathcal{P}_W$ with respect to the norms generated by these scalar products. But $(\mathcal{H}_\tau)_q^\beta \not\subset (L^2)$ if $\beta < 1$, generally speaking, so, we can not consider $(\mathcal{H}_\tau)_q^\beta$ with $\beta < 1$ as spaces of test functions.

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider the chains

$$\mathcal{D}'^{(m)} \supset \mathcal{H}_{-\tau}^{(m)} \supset \mathcal{H}_{ext}^{(m)} \supset \mathcal{H}_\tau^{\widehat{\otimes} m} \supset \mathcal{D}^{\widehat{\otimes} m}, \tag{29}$$

$m \in \mathbb{Z}_+$ (for $m = 0$, $\mathcal{D}^{\widehat{\otimes} 0} = \mathcal{H}_\tau^{\widehat{\otimes} 0} = \mathcal{H}_{ext}^{(0)} = \mathcal{H}_{-\tau}^{(0)} = \mathcal{D}'^{(0)} = \mathbb{R}$), where $\mathcal{H}_{-\tau}^{(m)}$, $\mathcal{D}'^{(m)} = \text{ind}\lim_{\tau\in T}\mathcal{H}_{-\tau}^{(m)}$ are the spaces dual of $\mathcal{H}_\tau^{\widehat{\otimes} m}$, $\mathcal{D}^{\widehat{\otimes} m}$ correspondingly with respect to $\mathcal{H}_{ext}^{(m)}$. The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [15]).

Proposition 2.13. *There exists a system of generalized functions*

$$\left\{ :\langle \circ^{\otimes m}, F_{ext}^{(m)} \rangle: \in (\mathcal{H}_{-\tau})_{-q} \mid F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)},\ m \in \mathbb{Z}_+ \right\}$$

such that

1) for $F_{ext}^{(m)} \in \mathcal{H}_{ext}^{(m)} \subset \mathcal{H}_{-\tau}^{(m)}$ $:\langle \circ^{\otimes m}, F_{ext}^{(m)} \rangle:$ *is a Wick monomial defined in Subsection 1.2;*

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ *can be presented as a formal series*

$$F = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{ext}^{(m)} \rangle:,\ F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}, \tag{30}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, *i.e.,*

$$\|F\|^2_{-\tau,-q} := \|F\|^2_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} |F^{(m)}_{ext}|^2_{\mathcal{H}^{(m)}_{-\tau}} < \infty; \tag{31}$$

and, vice versa, any formal series (30) *with finite norm* (31) *is a generalized function from* $(\mathcal{H}_{-\tau})_{-q}$;

3) for $F, G \in (\mathcal{H}_{-\tau})_{-q}$ *the scalar product has a form*

$$(F, G)_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} (F^{(m)}_{ext}, G^{(m)}_{ext})_{\mathcal{H}^{(m)}_{-\tau}},$$

where $F^{(m)}_{ext}, G^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau}$ *are the kernels from decompositions* (30) *for* F *and* G *respectively;*

4) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ *and* $f \in (\mathcal{H}_{\tau})_q$ *that is generated by the scalar product in* (L^2), *has the form*

$$\langle\!\langle F, f \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}_{ext}, f^{(n)} \rangle_{ext}, \tag{32}$$

where $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau}$ *and* $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}_{\tau}$ *are the kernels from decompositions* (30) *and* (24) *for* F *and* f *respectively,* $\langle \cdot, \cdot \rangle_{ext}$ *is the dual pairing between elements of negative and positive spaces from chain* (29).

Corollary 2.14. $F \in (\mathcal{H}_{-\tau})$ *(resp.,* $F \in (\mathcal{D}')$*) if and only if* F *can be presented in form* (30) *and norm* (31) *is finite for some* $q \in \mathbb{N}_{q_0(\tau)}$ *(resp., for some* $\tau \in T$ *and some* $q \in \mathbb{N}_{q_0(\tau)}$*).*

3. Extended stochastic integrals on spaces of generalized functions

As we saw in Subsection 1.3, one of the main drawbacks of extended stochastic integral (13) consists in its *unboundedness* and, moreover, in *dependence of its domain on* t_1, t_2 (see (14)). This essentially restricts the area of possible applications. A possible solution of this problem—to define stochastic integrals as linear continuous operators acting on spaces of generalized functions (in particular, from $(L^2)\otimes\mathcal{H}$ to a suitable space of generalized functions, see [17]). In this section we introduce such integrals and study some of their properties.

3.1. Extended stochastic integrals on spaces of regular generalized functions

Let $F \in (L^2)^{-\beta}_{-q} \otimes \mathcal{H}$. It follows from representation (22) for elements of $(L^2)^{-\beta}_{-q}$ that F can be presented in the form

$$F(\cdot) = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F^{(m)}_{\cdot} \rangle:, \ F^{(m)}_{\cdot} \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}, \tag{33}$$

with

$$\|F\|^2_{(L^2)^{-\beta}_{-q}\otimes\mathcal{H}} = \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} |F^{(m)}_{\cdot}|^2_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}} < \infty. \tag{34}$$

It is natural to define an extended stochastic integral on $(L^2)^{-\beta}_{-q}\otimes\mathcal{H}$ as a direct generalization of integral (13). Namely, we accept the following definition.

Definition 3.1. For $F \in (L^2)^{-\beta}_{-q} \otimes \mathcal{H}$, $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, we define an extended stochastic integral $\int_{t_1}^{t_2} F(u)\widehat{d}L_u \in (L^2)^{-\beta}_{-q-1}$ by setting

$$\int_{t_1}^{t_2} F(u)\widehat{d}L_u := \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}^{(m)}_{[t_1,t_2)} \rangle :, \tag{35}$$

where the kernels $\widehat{F}^{(m)}_{[t_1,t_2)} \in \mathcal{H}^{(m+1)}_{ext}$ are constructed as in Subsection 1.3 by the kernels $F^{(m)}_{\cdot} \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}$ from decomposition (33) for F.

Since (see (23), (11) and (34))

$$\begin{aligned} &\Big\| \int_{t_1}^{t_2} F(u)\widehat{d}L_u \Big\|^2_{-q-1,-\beta} = \sum_{m=0}^{\infty} ((m+1)!)^{1-\beta} 2^{-(q+1)(m+1)} |\widehat{F}^{(m)}_{[t_1,t_2)}|^2_{ext} \\ &\leq \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} |F^{(m)}_{\cdot}|^2_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}} [2^{-q-1-m}(m+1)^{1-\beta}] \leq c \|F\|^2_{(L^2)^{-\beta}_{-q}\otimes\mathcal{H}}, \end{aligned} \tag{36}$$

where $c := \max_{m \in \mathbb{Z}_+} [2^{-q-1-m}(m+1)^{1-\beta}]$, this definition is well posed and, moreover, the extended stochastic integral

$$\int_{t_1}^{t_2} \circ(u)\widehat{d}L_u : (L^2)^{-\beta}_{-q} \otimes \mathcal{H} \to (L^2)^{-\beta}_{-q-1} \tag{37}$$

is a linear *continuous* operator. It is clear also that integral (37) is an extension of integral (13) (cf. (35) and (12)), and, in particular, is an extension of the Itô stochastic integral. Note that in the case $q = \beta = 0$ $(L^2)^{-0}_{-0} = (L^2)$ and therefore integral (13) can be extended to a linear continuous operator with values in $(L^2)^{-0}_{-1}$ (such an extension was considered in [17]). We remark also that integral (37) can be naturally extended to a linear continuous operator acting from $(L^2)^{-\beta} \otimes \mathcal{H}$ to $(L^2)^{-\beta}$; and, as it follows from calculation (36), in the case $\beta = 1$ the operator $\int_{t_1}^{t_2} \circ(u)\widehat{d}L_u : (L^2)^{-1}_{-q} \otimes \mathcal{H} \to (L^2)^{-1}_{-q}$, defined by (35), is a linear *continuous* one.

Remark 3.2. Sometimes it can be convenient to consider the extended stochastic integral given by (35) as an operator acting from $(L^2)^{-\beta}_{-q} \otimes \mathcal{H}$ to $(L^2)^{-\beta}_{-q}$. In the case $\beta < 1$ this operator will be unbounded, but closable: if we take the set of $F \in (L^2)^{-\beta}_{-q} \otimes \mathcal{H}$ such that $\| \int_{t_1}^{t_2} F(u)\widehat{d}L_u \|_{-q,-\beta} < \infty$ as its domain, then $\int_{t_1}^{t_2} \circ(u)\widehat{d}L_u : (L^2)^{-\beta}_{-q} \otimes \mathcal{H} \to (L^2)^{-\beta}_{-q}$ is a *closed* operator; this can be proved by analogy with the case $q = \beta = 0$, see [16].

It was proved in [16] that extended stochastic integral (13) is the adjoint operator to the Hida stochastic derivative. Let us show that this property holds true for integral (37).

Definition 3.3. Let $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$. For $g \in (L^2)^{\beta}_{q+1}$ we define a Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial_{\cdot} g \in (L^2)^{\beta}_{q} \otimes \mathcal{H}$ by setting

$$1_{[t_1,t_2)}(\cdot)\partial_{\cdot} g := \sum_{n=0}^{\infty} (n+1) :\langle \circ^{\otimes n}, g^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot) \rangle :, \tag{38}$$

where $g^{(n+1)} \in \mathcal{H}^{(n+1)}_{ext}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (5) for g, considered as elements of $\mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}$ (see Subsection 1.4).

Since (see (15) and (20))

$$\begin{aligned}\|1_{[t_1,t_2)}(\cdot)\partial.g\|^2_{(L^2)^\beta_q\otimes\mathcal{H}}&=\sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(n+1)^2|g^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot)|^2_{\mathcal{H}^{(n)}_{ext}\otimes\mathcal{H}}\\ &\le\sum_{n=0}^{\infty}((n+1)!)^{1+\beta}2^{(q+1)(n+1)}|g^{(n+1)}|^2_{ext}[(n+1)^{1-\beta}2^{-q-1-n}]\le c\|g\|^2_{q+1,\beta},\end{aligned}\tag{39}$$

where $c:=\max_{n\in\mathbb{Z}_+}[(n+1)^{1-\beta}2^{-q-1-n}]$, this definition is well posed and, moreover, the Hida stochastic derivative

$$1_{[t_1,t_2)}(\cdot)\partial.:(L^2)^\beta_{q+1}\to(L^2)^\beta_q\otimes\mathcal{H}\tag{40}$$

is a linear *continuous* operator. It is clear also that this derivative is a restriction of derivative (17) onto $(L^2)^\beta_{q+1}$. We note that the restriction of derivative (40) onto $(L^2)^\beta$ can be considered as a linear continuous operator $1_{[t_1,t_2)}(\cdot)\partial.:(L^2)^\beta\to(L^2)^\beta\otimes\mathcal{H}$; and, as it follows from calculation (39), in the case $\beta=1$ the operator $1_{[t_1,t_2)}(\cdot)\partial.:(L^2)^1_q\to(L^2)^1_q\otimes\mathcal{H}$, defined by (38), is a linear *continuous* one.

Remark 3.4. Sometimes it can be convenient to consider the Hida stochastic derivative given by (38) as an operator acting from $(L^2)^\beta_q$ to $(L^2)^\beta_q\otimes\mathcal{H}$. In the case $\beta<1$ this operator will be unbounded, but closable: if we take the set of $g\in(L^2)^\beta_q$ such that $\|1_{[t_1,t_2)}(\cdot)\partial.g\|_{(L^2)^\beta_q\otimes\mathcal{H}}<\infty$ as its domain, then $1_{[t_1,t_2)}(\cdot)\partial.:(L^2)^\beta_q\to(L^2)^\beta_q\otimes\mathcal{H}$ is a *closed* operator; this can be proved by analogy with the case $q=\beta=0$, see [16].

Theorem 3.5. *For arbitrary $t_1,t_2\in[0,+\infty]$, $t_1<t_2$, extended stochastic integral* (37) *and Hida stochastic derivative* (40) *are mutually adjoint operators.*

Proof. By analogy with [16], Subsection 2.2, one can show that for $F\in(L^2)^{-\beta}_{-q}\otimes\mathcal{H}$ and $g\in(L^2)^\beta_{q+1}$

$$\langle\!\langle\int_{t_1}^{t_2}F(u)\widehat{d}L_u,g\rangle\!\rangle=\big(F(\cdot),1_{[t_1,t_2)}(\cdot)\partial.g\big)_{(L^2)\otimes\mathcal{H}},\tag{41}$$

where $(\cdot,\cdot)_{(L^2)\otimes\mathcal{H}}$ denotes the dual pairing generated by the scalar product in $(L^2)\otimes\mathcal{H}$. Since operators (37) and (40) are continuous ones, the result of the theorem follows from (41). □

Remark 3.6. The result of this theorem holds true for the extended stochastic integral $\int_{t_1}^{t_2}\circ(u)\widehat{d}L_u:(L^2)^{-\beta}\otimes\mathcal{H}\to(L^2)^{-\beta}$ and the Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial.:(L^2)^\beta\to(L^2)^\beta\otimes\mathcal{H}$; in the same way as for $\int_{t_1}^{t_2}\circ(u)\widehat{d}L_u:(L^2)^{-\beta}_{-q}\otimes\mathcal{H}\to(L^2)^{-\beta}_{-q}$ and $1_{[t_1,t_2)}(\cdot)\partial.:(L^2)^\beta_q\to(L^2)^\beta_q\otimes\mathcal{H}$ (see Remarks 3.2 and 3.4), in the last case these (unbounded) operators are closed. The proof is quite analogous to the corresponding proof in the case $q=\beta=0$, see Subsection 2.2 in [16]: it is necessary to use (41) and to verify that the domains of $\int_{t_1}^{t_2}\circ(u)\widehat{d}L_u$ and $(1_{[t_1,t_2)}(\cdot)\partial.)^*\circ$ (correspondingly of $1_{[t_1,t_2)}(\cdot)\partial.$ and $\big(\int_{t_1}^{t_2}\circ\widehat{d}L\big)^*_.$) coincide.

3.2. Extended stochastic integrals on spaces of nonregular generalized functions

Let $F\in(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}$. It follows from representation (30) for elements of $(\mathcal{H}_{-\tau})_{-q}$ that F can be represented in the form

$$F(\cdot)=\sum_{m=0}^{\infty}:\langle\circ^{\otimes m},F^{(m)}_{ext,\cdot}\rangle:,\ \ F^{(m)}_{ext,\cdot}\in\mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H},\tag{42}$$

with

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}} = \sum_{m=0}^{\infty} 2^{-qm}|F^{(m)}_{ext,\cdot}|^2_{\mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H}} < \infty. \tag{43}$$

In order to define an extended stochastic integral on $(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}$, we need some preparation. Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes}m} \supset \mathcal{H}^{\widehat{\otimes}m}_{-\tau} \supset \mathcal{H}^{\widehat{\otimes}m} \supset \mathcal{H}^{\widehat{\otimes}m}_{\tau} \supset \mathcal{D}^{\widehat{\otimes}m},\ m\in\mathbb{N}. \tag{44}$$

Since the spaces of test functions in chains (44) and (29) coincide, there exists a family of natural isomorphisms

$$U_m : \mathcal{D}'^{(m)} \to \mathcal{D}'^{\widehat{\otimes}m}$$

such that for all $F^{(m)}_{ext} \in \mathcal{D}'^{(m)}$ and $f^{(m)} \in \mathcal{D}^{\widehat{\otimes}m}$

$$\langle F^{(m)}_{ext}, f^{(m)}\rangle_{ext} = \langle U_m F^{(m)}_{ext}, f^{(m)}\rangle.$$

It is easy to see that the restrictions of U_m onto $\mathcal{H}^{(m)}_{-\tau}$ are isometrical isomorphisms between $\mathcal{H}^{(m)}_{-\tau}$ and $\mathcal{H}^{\widehat{\otimes}m}_{-\tau}$.

Definition 3.7. Let $t_1, t_2 \in [0,+\infty]$, $t_1 < t_2$, and $F \in (\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}$. We define an extended stochastic integral $\int_{t_1}^{t_2} F(u)\overline{d}L_u \in (\mathcal{H}_{-\tau})_{-q}$ by setting

$$\int_{t_1}^{t_2} F(u)\overline{d}L_u := \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}^{(m)}_{ext,[t_1,t_2)}\rangle :, \tag{45}$$

where

$$\widehat{F}^{(m)}_{ext,[t_1,t_2)} := U^{-1}_{m+1}\{Pr[(U_m\otimes 1)F^{(m)}_{ext,\cdot}1_{[t_1,t_2)}(\cdot)]\} \in \mathcal{H}^{(m+1)}_{-\tau}, \tag{46}$$

Pr is the symmetrization operator (more exactly, the orthoprojector acting for each $m\in\mathbb{Z}_+$ from $\mathcal{H}^{\widehat{\otimes}m}_{-\tau}\otimes\mathcal{H}$ to $\mathcal{H}^{\widehat{\otimes}m+1}_{-\tau}$), $F^{(m)}_{ext,\cdot} \in \mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H}$, $m\in\mathbb{Z}_+$, are the kernels from decomposition (42) for F, $U_0 = 1 : \mathbb{R}\to\mathbb{R}$.

Since

$$\begin{aligned}|\widehat{F}^{(m)}_{ext,[t_1,t_2)}|_{\mathcal{H}^{(m+1)}_{-\tau}} &= |Pr[(U_m\otimes 1)F^{(m)}_{ext,\cdot}1_{[t_1,t_2)}(\cdot)]|_{\mathcal{H}^{\widehat{\otimes}m+1}_{-\tau}}\\ &\le |(U_m\otimes 1)F^{(m)}_{ext,\cdot}|_{\mathcal{H}^{\widehat{\otimes}m}_{-\tau}\otimes\mathcal{H}} = |F^{(m)}_{ext,\cdot}|_{\mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H}}\end{aligned}$$

and therefore (see (31) and (43))

$$\begin{aligned}\Big\|\int_{t_1}^{t_2} F(u)\overline{d}L_u\Big\|^2_{-\tau,-q} &= \sum_{m=0}^{\infty} 2^{-q(m+1)}|\widehat{F}^{(m)}_{ext,[t_1,t_2)}|^2_{\mathcal{H}^{(m+1)}_{-\tau}}\\ &\le 2^{-q}\sum_{m=0}^{\infty} 2^{-qm}|F^{(m)}_{ext,\cdot}|^2_{\mathcal{H}^{(m)}_{-\tau}\otimes\mathcal{H}} = 2^{-q}\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}},\end{aligned}$$

this definition is correct and, moreover, the extended stochastic integral

$$\int_{t_1}^{t_2} \circ(u)\overline{d}L_u : (\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H} \to (\mathcal{H}_{-\tau})_{-q} \tag{47}$$

is a linear *continuous* operator.

In what follows, we will show that integral (47) is an extension of integral (13), but at first let us establish an interconnection between integral (47) and the Hida stochastic derivative.

Definition 3.8. Let $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$. For $g \in (\mathcal{H}_\tau)_q$ we define a Hida stochastic derivative $1_{[t_1,t_2)}(\cdot)\partial.g \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}$ by formula (38), where $g^{(n+1)} \in \mathcal{H}_\tau^{\widehat{\otimes} n+1}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (24) for g, considered as elements of $\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}$.

Since (see (25))

$$\|1_{[t_1,t_2)}(\cdot)\partial.g\|^2_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}} = \sum_{n=0}^{\infty} ((n+1)!)^2 2^{qn} |g^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot)|^2_{\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}}$$
$$\leq 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |g^{(n+1)}|^2_\tau \leq 2^{-q} \|g\|^2_{\tau,q},$$

this definition is well posed and, moreover, the Hida stochastic derivative

$$1_{[t_1,t_2)}(\cdot)\partial. : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q \otimes \mathcal{H} \tag{48}$$

is a linear *continuous* operator. Moreover, as it follows from construction of the kernels $g^{(n+1)}(\cdot) \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ from (16) (see Subsection 1.4), this derivative is the restriction of derivative (17) onto $(\mathcal{H}_\tau)_q$. We also note that the restrictions of derivative (48) onto $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ are linear continuous operators $1_{[t_1,t_2)}(\cdot)\partial. : (\mathcal{H}_\tau) \to (\mathcal{H}_\tau) \otimes \mathcal{H}$ and $1_{[t_1,t_2)}(\cdot)\partial. : (\mathcal{D}) \to (\mathcal{D}) \otimes \mathcal{H}$ respectively.

Theorem 3.9. *For arbitrary $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, extended stochastic integral* (47) *and Hida stochastic derivative* (48) *are mutually adjoint operators.*

Proof. Using (45), (24), (32), (46), (38) and (42), for $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ and $g \in (\mathcal{H}_\tau)_q$ we obtain

$$\langle\!\langle \int_{t_1}^{t_2} F(u)\overline{d}L_u, g \rangle\!\rangle = \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}^{(m)}_{ext,[t_1,t_2)} \rangle:, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, g^{(n)} \rangle: \rangle\!\rangle$$
$$= \sum_{m=0}^{\infty} (m+1)! \langle \widehat{F}^{(m)}_{ext,[t_1,t_2)}, g^{(m+1)} \rangle_{ext}$$
$$= \sum_{m=0}^{\infty} (m+1)! \langle (U_m \otimes 1) F^{(m)}_{ext,\cdot} 1_{[t_1,t_2)}(\cdot), g^{(m+1)} \rangle$$
$$= \sum_{m=0}^{\infty} m!(m+1) (F^{(m)}_{ext,\cdot}, g^{(m+1)}(\cdot)1_{[t_1,t_2)}(\cdot))_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}}$$
$$= \Big(\sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F^{(m)}_{ext,\cdot} \rangle:, \sum_{n=0}^{\infty} (n+1) :\langle \circ^{\otimes n}, g^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot) \rangle: \Big)_{(L^2) \otimes \mathcal{H}}$$
$$= (F(\cdot), 1_{[t_1,t_2)}(\cdot)\partial.g)_{(L^2) \otimes \mathcal{H}}.$$

Since operators (47) and (48) are continuous, the statement of the theorem follows from this calculation. □

Corollary 3.10. *Extended stochastic integral* (47) *is an extension of integral* (13).

Proof. The result follows from the above theorem, (18) and the fact that derivative (48) is the restriction of derivative (17) onto $(\mathcal{H}_\tau)_q$. □

Remark 3.11. The statements of Theorem 3.9 and its corollary hold true for $\int_{t_1}^{t_2} \circ(u)\overline{d}L_u : (\mathcal{H}_{-\tau}) \otimes \mathcal{H} \to (\mathcal{H}_{-\tau})$ (correspondingly $\int_{t_1}^{t_2} \circ(u)\overline{d}L_u : (\mathcal{D}') \otimes \mathcal{H} \to (\mathcal{D}')$) and $1_{[t_1,t_2)}(\cdot)\partial_\cdot : (\mathcal{H}_\tau) \to (\mathcal{H}_\tau) \otimes \mathcal{H}$ (correspondingly $1_{[t_1,t_2)}(\cdot)\partial_\cdot : (\mathcal{D}) \to (\mathcal{D}) \otimes \mathcal{H}$).

Finally, as is easy to see now, if $F \in (\mathcal{D}') \otimes \mathcal{H} \cap (L^2)^{-1} \otimes \mathcal{H}$ then $\int_{t_1}^{t_2} F(u)\overline{d}L_u = \int_{t_1}^{t_2} F(u)\widehat{d}L_u \in (\mathcal{D}') \cap (L^2)^{-1}$.

Stochastic derivatives on the spaces of generalized functions will be considered in another paper.

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