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A CHARACTERIZATION OF PERMANENT RADICALS IN COMMUTATIVE LOCALLY PSEUDOCONVEX ALGEBRAS

Wiesław Żelazko

Mathematical Institute of the Polish Academy of Sciences, Śniadeckich 8, Warsaw, Poland

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We show that for a commutative unital locally pseudoconvex algebra A an element $x \in A$ belongs to the radical of every topological algebra extension of A if and only if it has small powers (see definition below). This result extends our previous result [12] obtained for locally convex algebras onto the larger class of locally pseudoconvex algebras.

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Доведено, що елемент x комутативної унітальної локально псевдоопуклої алгебри A належить радикалу довільного тополоrчного розширення алгебри A тоді і лише тоді, коли x має малі степені.

All algebras considered in this paper will be commutative, complex, and unital. The unity of an algebra A will be denoted by e_A, or by e, if it does not lead to a confusion. By a *topological algebra* we shall mean a complex topological vector space equipped with an associative jointly continuous multiplication making of it an algebra. That means that for each neighbourhood U of zero in A there is a neighbourhood V such that

$$V^2 \subset U. \tag{1}$$

The class of topological algebras will be denoted by $\mathcal{T}$. The joint continuity of multiplication implies that the completion of a topological algebra is again a topological algebra. In the sequel we shall assume that all considered topological algebras are complete. For general results on topological algebras the reader is referred to [4], [6], [8] and [10].

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E-mail: zelazko@impan.pl

A topological vector space X is called *locally pseudoconvex* if there is a basis of neighbourhoods of zero consisting of absolutely p-convex sets, $0 < p \le 1$, i.e. sets U such that for any n-tuple $(x_1, \dots, x_n) \subset U$ and for any scalars $\lambda_1, \dots, \lambda_n$ satisfying $\sum_1^n |\lambda_i|^p \le 1$ the element $\sum \lambda_i x_i$ is in U. The number p may depend upon U. In this case the topology of X is given by means of a family $(\|\cdot\|_\alpha)_{\alpha\in\mathfrak{a}}$ of p_α-homogeneous seminorms, $0 < p_\alpha \le 1$, i.e. functions $x \mapsto \|x\|_\alpha$ satisfying

(i) $\|x\|_\alpha \ge 0$ for all x in X and all α in $\mathfrak{a}$,

(ii) $\|x + y\|_\alpha \le \|x\|_\alpha + \|y\|_\alpha$ for all $x, y \in X$ and all $\alpha \in \mathfrak{a}$,

(iii) $\|\lambda x\|_\alpha = |\lambda|^{p_\alpha}\|x\|_\alpha$ for all scalars λ and all α in $\mathfrak{a}$.

If $\|\cdot\|$ is a p-homogeneous seminorm and $0 < r < 1$, then $\|\cdot\|^r$ is a pr-homogeneous seminorm equivalent to the previous one. For two seminorms $\|\cdot\|_1, \|\cdot\|_2$ respectively p_1 and p_2 homogeneous write $\|\cdot\|_1 \preceq \|\cdot\|_2$ if the convergence in $\|\cdot\|_2$ implies the convergence in $\|\cdot\|_1$. In this case there is a positive constant C such that $\|x\|_1^{p_2} \le C\|x\|_2^{p_1}$ for all x in X. It is convenient to assume that a family of seminorms generating the topology of X is saturated, i.e. for a finite subset $\|\cdot\|_1, \dots, \|\cdot\|_n$ of this family there is a seminorm $\|\cdot\|$ such that $\|\cdot\|_i \preceq \|\cdot\|$ for $1 \le i \le n$. As $\|x\|$ we can take $\max\{\|x\|_1^{q_1}, \dots, \|x\|_n^{q_n}\}$, where $q_i = p_1 \dots p_{i-1}p_{i+1} \dots p_n$. For further information on locally pseudoconvex spaces the reader is referred to [2] and [6].

A *locally pseudoconvex algebra* A (shortly: an $\mathcal{LPC}$*-algebra*) is a topological algebra which is a locally pseudoconvex space. The joint continuity of multiplication means that the seminorms giving the topology of A can be choosen so that for each α there is a β such that

$$\|xy\|_\alpha^{r_\beta} \le \|x\|_\beta^{r_\alpha}\|y\|_\beta^{r_\alpha} \quad \text{for all} \quad x, y \in A. \tag{2}$$

The class of $\mathcal{LPC}$-algebras, denoted by $\mathcal{LPC}$, is larger than the class of locally convex algebras ($\mathcal{LC}$-algebras). We shall consider only complete unital algebras with the unity element denoted by e. In this case it can be assumed also that $\|e\|_\alpha = 1$ for all α. An LPC-algebra B is said an *extension* of A if there is a unital topological isomorphosm of A into B. In this case A can be treated as a closed subalgebra of of B containing its unity. The *radical* $\mathsf{rad}A$ of A is the intersection of all its maximal ideals. The radical of A can be also defined by the formula

$$\mathsf{rad}A = \{x \in A : \forall_{a\in A}\ e + ax \in G(A)\}, \tag{3}$$

where $G(A)$ denotes the group of invertible elements in A. For more information on locally pseudoconvex algebras the reader is referred to [6].

The *permanent* $\mathcal{LPC}$*-radical* of A is the maximal subideal of $\mathsf{rad}A$ which is contained in the radical of every $\mathcal{LPC}$-extension B of A. Since the intersection with A of an ideal in an extension of A is an ideal in A, the permanent $\mathcal{LPC}$-radical of A is the intersection of A with the radicals in all $\mathcal{LPC}$-extensions of A. Thus it is a uniquely determined object. It will be denoted by $\mathsf{rad}_{\mathcal{LPC}}(A)$. The permanent radical of A is the maximal subideal of $\mathsf{rad}A$ which is contained in the radical of every topological algebra extension of A. It will be denoted by $\mathsf{rad}_{\mathcal{T}}(A)$. Clearly

$$\mathsf{rad}_{\mathcal{T}}(A) \subset \mathsf{rad}_{\mathcal{LPC}}(A). \tag{4}$$

Let A be a topological algebra. We say that an element x in A has *small powers*, if for each neighbourhood U of zero in A there is a natural n such that the linear span of x^n is contained in U. Denote by $I_s(A)$ the set of all elements of A possessing small powers. Clearly, any nilpotent element has small powers. Let $x \in I_s(A)$ and let a be an arbitrary element of A and U an arbitrary neighbourhood of zero in A. Choose a neighbourhood V of zero such that $VV \subset U$. Since V absorbs all elements of A, there is a positive μ such that μa is in V. There is also a natural k such that $\lambda x^k \in V$ for all scalars λ. Consequently

$$ax^k = \mu a \mu^{-1} x^k \in V^2 \subset U$$

for all a in A, and so

$$Ax^k \subset U. \tag{5}$$

In particular we obtain $\lambda(ax)^k \in U$ for all a in A and all scalars λ, which implies

$$AI_s(A) \subset I_s(A). \tag{6}$$

Take again an arbitrary U and V so that $V + V \subset U$. Let $x, y \in I_s(A)$. By the formula (5) there are natural m and n such that ax^m and ay^n are in V for all a in A. Consequently, for every scalar λ we have

$$\lambda(x+y)^{m+n} = \lambda x^m u + \lambda y^n v \in V + V \subset U$$

for suitable u and v in A. That means that $x + y$ belongs to $I_s(A)$ and so $I_s(A)$ is a linear subspace of A. Thus, by the formula (6), $I_s(A)$ is an ideal in A. The above reasonings are already obtained in [12] and they are given here for the convenience of the reader. We shall need also the following lemmas proved in [12] (here the assumption of completeness of A is essential).

Lemma 1. *Let A be a topological algebra. Then an element x is in $I_s(A)$ if and only if for each sequence $(a_k)_{k=0}^{\infty}$ of elements of A the series $\sum_{k=0}^{\infty} a_k x^k$ is convergent in A.*

This lemma implies

Lemma 2. *The ideal $I_s(A)$ is contained in the radical* $\mathsf{rad}A$.

In fact, put $a_n = a^n$ for an a in A and let $x \in I_s(A)$. Then the sum of the series $\sum_{k=0}^{\infty}(-1)^k a^k x^k$ equals $(e + ax)^{-1}$, and so, by the formula (3) x is in $\mathsf{rad}A$ and our claim follows.

Observe that $I_s(A)$ is contained in the permanent radical of A. This follows from the above lemma and from the fact that if x has small powers, then it also has small powers in any extension of A.

Examples. (a) A topological vector space X is called *locally bounded*, if it has a basis of neighbourhoods of zero consisting of bounded sets. If X has a bounded neigbourhood of zero U, then the sequence $\frac{1}{n}U$ is such a basis and X is locally bounded. The Aoki-Rolewicz theorem ([2], [5]) states that the topology of a locally bounded space can be given by the means of a p-homogeneous norm, $0 < p \leq 1$. Thus a locally bounded space is also locally pseudoconvex. The completion of a locally bounded space is locally bounded. A Kolmogorov theorem ([3]) states that a topological vector space is normed if and only if it is both locally bounded ane locally convex. So there are two natural generalizations of normed spaces: locally bounded spaces and locally convex spaces. An example of locally bounded non-normed space is the space $l_p, 0 < p < 1$ of numerical sequences $x = (\xi_i)_{i=0}^{\infty}$ provided with the p-homogeneous norm

$$\|x\| = \sum_{i=0}^{\infty} |\xi_i|^p. \tag{7}$$

If we treat x as a power series $x = x(t) = \sum_{i=0}^{\infty} \xi_i(x)t^i$, and provide l_p with the convolution multiplication we obtain a topological (locally bounded) algebra, since

$$\|xy\| \leq \|x\|\|y\| \quad \text{for} \quad x, y \in l_p.$$

The class $\mathcal{LB}$ of locally bounded algebras is more general than the class of Banach algebras, but possesses all essential properties of the latter (see [5], [6], [7]). This means that the local boundedness and not local convexity is responsible for most properties of Banach algebras. Clearly $\mathcal{LB} \subset \mathcal{LPC}$. For more information on locally bounded algebras the Reader is referred to [6], [7], [8] and [10].

(b) The intersection

$$A = \bigcap_{0<p\leq 1} l_p$$

provided with norms (7) and the convolution multiplication is an $\mathcal{LPC}$-algebra which is not locally bounded. The relation between (complete) locally bounded and $\mathcal{LPC}$ spaces and algebras is analogous to that between Banach and locally convex spaces and algebras.

(c) If an $\mathcal{LPC}$-algebra has a continuous norm $\|\cdot\|$, then its only elements possessing small powers are nilpotents. Indeed, if $\|\lambda x^n\| < 1$ for all scalars λ, then $\|x^n\| = 0$ and so $x^n = 0$. In particular the only elements of a locally bounded algebra possessing small powers are nilpotents.

(d) The ideal $I_s(A)$ is not necessarily closed. In the Volterra algebra (unitization of $L_1[0,1]$ with convolution multiplication) the set of nilpotent elements is dense in $L_1[0,1]$.

(e) The locally convex algebra $C[0,\infty)$ of all continuous functions on the positive half-line provided with pointwise algebra operations and the compact-open topology (topology of uniform convergence on compact sets) has no continuous norm, but its only element possessing small powers is the zero function.

(f) Let $A = (s)$ be the vector space of all numerical sequences $x = (\xi_i(x))_0^{\infty}$ provided with the topology of pointwise convergence (this space was already considered by Banach in [1] as an example of complete metric linear space). This topology is given by seminorms

$$\|x\|_k = \sum_{i=0}^{k-1} |\xi_i(x)|, \quad k = 1, 2, \ldots.$$

Treating elements of A as power series $x = x(t) = \sum_i \xi_i t^i$ and providing it with Cauchy multiplication we obtain a locally convex algebra, since

$$(8) \qquad \|xy\|_k \leq \|x\|_k \|y\|_k \text{ for all } x, y \in A \text{ and all natural } k.$$

An element x in A is invertible if and only if $\xi_0(x) \neq 0$ and all non-invertible elements of this algebra have small powers.

The following result was presented during the Conference devoted to the 120th anniversary of the birthdate of Stefan Banach.

Theorem 1. *Let A be a commutative complete locally pseudoconvex unital algebra. Then the permanent $\mathcal{LPC}$-radical of A coincides with its permanent radical and also coincides with the ideal $I_s(A)$.*

This theorem extends onto locally pseudoconvex algebras our previous result [3] on locally convex algebras.

Proof. Denote by $\mathcal{S} = (\|\cdot\|_\alpha)_{\alpha\in\mathfrak{a}}$ a saturated family of seminorms giving the topology of A and satisfying relations (2). Let $x \in I_s(A)$ and let B be a $\mathcal{T}$-extension of A. It is clear that x also has small powers in B, and so by the Lamma 4 it belongs to the radical $\mathsf{rad}B$. Consequently, using the formula (4), we get

$$I_s(A) \subset \mathsf{rad}_{\mathcal{T}}(A) \subset \mathsf{rad}_{\mathcal{LPC}}(A).$$

For obtaining our conclusion it is sufficient to show that

$$(9) \qquad \mathsf{rad}_{\mathcal{LPC}}(A) \subset I_s(A).$$

To this end we have to show that if $x_0 \notin I_s(A)$ then $x_0 \notin \mathsf{rad}_{\mathcal{LPC}}(A)$. Equivalently, if an element x_0 does not belong to $I_s(A)$, then there is an extension B of A such that x_0 does not belong to $rad(B)$. Observe first that if some element x_0 does not have small powers, then there is a neighbourhood of zero U such that for each natural n there is a scalar λ_n such that $\lambda_n x_0^n \notin U$. In terms of seminorms, it means that there is a continuous p-homogeneous seminorm $\|\cdot\|$ on A, $0 < p \leq 1$, such that

$$(10) \qquad \|x_0^n\| \neq 0 \text{ for all non-negative integers } n, x_0^0 = e.$$

Without loss of generality we can assume that $\|\cdot\| \in \mathcal{S}$ and that $\|\cdot\|_\alpha \succ \|\cdot\|$ for all $\alpha \in \mathfrak{a}$. Our extension B will be a locally pseudoconvex algebra consisting of power series

$$(11) \qquad x(t) = \sum_{k=0}^{\infty} x_k t^k, x_k \in A,$$

provided with Cauchy multiplication. It will be constructed by means of an infinite matrix $(a_{k,j}), k = 1, 2, \ldots; j = 0, 1, \ldots.$ with positive entries. The first row of this matrix is given by

$$a_{1,j} = \max\{1, \|x_0^j\|^{-1}\}.$$

By the formula (9) it is a correct definition. If we have a sequence $(r_i)_0^\infty$ of positive numbers with $r_0 = 1$, we can find a sequence $(s_i)_0^\infty$ of positive numbers with $s_0 = 1$ such that

$$r_{i+j} \le s_i s_j, \quad i, j \ge 0. \tag{12}$$

The sequence (s_i) is obtained by an induction, setting $s_0 = 1$, and having defined s_i, $i < k$, $s_i > 0$, we put

$$s_k = \max\bigl\{1, r_{2k}^{1/2}, \frac{r_{i+k}}{s_i} : 0 \le i < k\bigr\}.$$

We put now $r_i = a_{1,i}$ we obtain the second row $a_{2,i} = s_i$, and further by an induction: setting in (12) $r_i = a_{n,i}$ we obtain next row defined as $a_{n+1,i} = s_i$. Obtained in this way matrix $(a_{k,j})$ satisfies the following relations

$$a_{k,i+j} \le a_{k+1,i}\, a_{k+1,j},\ a_{k,0} = 1,\ a_{k,j} \ge 1, \quad \text{for} \quad k \ge 1,\ i, j \ge 0. \tag{13}$$

We shall construct now the announced LC-extension B of A such that $x_0 \notin \mathsf{rad}B$, which will finish the proof.

It consists of all formal power series of the form (11) such that

$$\|\mathbf{x}\|_{(\alpha,k,r)} = \sum_{i=0}^{\infty} a_{k,i} \|x_i\|_\alpha^r < \infty, \qquad \text{for all} \quad k \ge 1, \tag{14}$$

where $\alpha \in \mathfrak{a}, \alpha \succeq \alpha_0$, and $0 < r \le 1$. The seminorm defined by this formula is rr_α-homogeneous, where r_α is the exponent of homogeneity of the seminorm $\|\cdot\|_\alpha$. We leave to the Reader the proof that obtained in this way locally pseudoconvex space B is complete. The multiplication in B is the Cauchy multiplication of power series. Denote by $\beta \succ \alpha$ an index such that for all x, y in A

$$\|xy\|_\alpha^{r_\beta} \le \|x\|_\beta^{r_\alpha} \|y\|_\beta^{r_\alpha}. \tag{15}$$

The formulas (13) (14) and (15) imply

$$\|\mathbf{xy}\|_{(\alpha,k,r)}^{r_\beta} = \|\sum_{i=0}^{\infty} \bigl(\sum_{j=0}^{i} x_{i-j} y_j\bigr) t^i \|_{(\alpha,k,r)}^{r_\beta} \le \sum_{i=0}^{\infty} a_{k,i} \| \sum_{j=0}^{i} x_{i-j} y_j \|_\alpha^{rr_\beta} \le$$
$$\le \sum_{i,j} a_{k+1,i-j} a_{k+1,j} \|x_{i-j}\|_\beta^{rr_\alpha} \|y_j\|_\beta^{rr_\alpha} = \|\mathbf{x}\|_{(\beta,k+1,rr_\alpha)} \|\mathbf{y}\|_{(\beta,k+1,rr_\alpha)}.$$

Consequently the multiplication is jointly continuous in B, so that it is a pseudoconvex algebra. The imbedding of A into B is given by identification of an element x in A with the "constant" element $x(t)$ in B for which $x_0 = x$ and $x_i = 0$ for $i > 0$. Clearly it is an isomorphism into. Moreover, for this $x(t)$, we have $\|x(t)\|_{(\alpha,k,r)} = \|x\|_\alpha$ and the isomorphism is topological. Thus B is an extension of A. It remains to be shown that $x_0 \notin \mathsf{rad}B$. Suppose towards contradiction that $x_0 \in \mathsf{rad}B$. Then $e - x_0 t$ is invertible in B (see formula (3)) so that there is an element $z(t) = \sum_i z_i t^i$ with $(e - x_0 t) \sum_{i=0}^\infty z_i t^i = e$. This formula implies

$$(z_0 - e) + \sum_{i=1}^{\infty} (z_{i-1} x_0 - z_i) t^i = 0,$$

which, by an easy induction implies $z_k = x_0^k$ for all $k \geq 0$. But then

$$\|z(t)\|_{(\alpha_0,1,1)} = \sum_{i=0}^{\infty} a_{1,i}\|x_0^i\|_{\alpha_0} = \sum_{i=0}^{\infty} \|x_0^i\|_{\alpha_0}^{-1}\|x_0^i\|_{\alpha_0}$$

and the right hand series is divergent so that $z(t)$ is not in B. The contradiction proves our assertion. □

We say that an element x of a topological algebra is permanerntly singular if it is non removable in every $\mathcal{T}$-extension of A. As a corollary to the Theorem 1 we obtain

Theorem 2. *Let A be a commutative unital $\mathcal{LPC}$-algebra and let x be a permanently singular element of A. Then for every y in $I_s(A)$ the element $x + y$ is also permanently singular (i.e. every $I_s(A)$-perturbation of a permanently singular element is permanently singular).*

Proof. Let B be a $\mathcal{T}$-extensiom of A. Since x is permanently singular, it belongs to some maximal ideal M of B. Since $y \in \mathsf{rad}B$, we have $x + y \in M$ and so $x + y$ is not invertible in B. The conclusion follows. □

Final Remarks. We do not know whether Theorem 1 remains true if we replace there the class $\mathcal{LPC}$ by the class $\mathcal{T}$ of all topological algebras. For proving such a result we should show that if an element x_0 does not belong to $I_s(A)$ for a topological algebra A, then there is an extension B of A such that x_0 is not in $rad(B)$.

As mentioned earlier, Theorem 1 holds true for the (smaller) class $\mathcal{LC}$ of locally convex algebras. For still smaller classes $\mathcal{B}$ (Banach algebras), or $\mathcal{LB}$ we have $\mathsf{rad}_{\mathcal{B}}(A) = \mathsf{rad}A$ (resp. $\mathsf{rad}_{\mathcal{LB}}(A) = \mathsf{rad}A$). This follows from the fact that in these class the elements of radical are defined by the condition $\lim_n \|x^n\|^{1/n} = 0$. For the classes $\mathcal{MLPC}$ and $\mathcal{MLC}$ of multiplicatively pseudoconvex and multiplicatively convex algebras (they are defined by the condition that the seminorms giving their topologies are submultiplicative, i.e. satisfy condition (8)) the radical is defined by the condition $\lim_n \|x^n\|^{1/n} = 0$ for each continuous seminorm $\|\cdot\|$. This again implies that $\mathsf{rad}_{\mathcal{MLC}}(A) = \mathsf{rad}A$ and $\mathsf{rad}_{\mathcal{MLPC}}(A) = \mathsf{rad}A$.

There is, however, an essential difference between classes $\mathcal{B}$ or $\mathcal{LB}$ and classes $\mathcal{MLC}$ and $\mathcal{MLPC}$. If A is in $\mathcal{B}$ or in $\mathcal{LB}$ and $x \in \mathsf{rad}A$, then x is permanently non-invertible. Even more: the ideal $\mathsf{rad}A$ is permanently non-removable, i.e., is contained in a proper ideal of B (but not necessarily in $\mathsf{rad}B$) for every $\mathcal{T}$ extension B of A. This follows from the following result [11] (proved for the class $\mathcal{B}$, but the proof works as well for $\mathcal{LB}$): Let $A \in \mathcal{B}$ (resp. $\mathcal{LB}$). Then each maximal ideal in the Shilov boundary of A consists of joint topological divisors of zero, i.e. there is a net (z_α) of elements of A, $\|z_\alpha\| = 1$, such that $\lim_\alpha \|z_\alpha x\| = 0$ for all x in A. Consequently, such an ideal, as well as each its subideal, is permanently non-removable. Thus the radical $\mathsf{rad}A$ is permanently non removable.

On the other hand, it is constructed in [9] an $\mathcal{MLC}$-algebra A with an element x in $\mathsf{rad}A$ which is invertible in some $\mathcal{LC}$-extension of A.

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