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PHILOSOPHY OF LOGIC AND MATHEMATICS IN THE LWÓW SCHOOL OF MATHEMATICS

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The paper is devoted to the presentation and analysis of philosophical views concerning mathematics and logic of some representatives of Lwów school of mathematics.

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У статті проаналізовано філософські погляди у математиці та логіці деяких представників Львівської Математичної Школи.

The aim of this paper is to consider philosophical ideas concerning logic and mathematics developed in Lwów school of mathematics. Views of Hugo Steinhaus (1887–1972), Stefan Banach (1892–1945), Eustachy Żyliński (1889–1954) and Leon Chwistek (1884–1944) will be analyzed. In the case of the first three of them there is no room for doubt that they belonged to this school. There may be some doubts in the case of Chwistek. We have included him into the Lwów school because since 1930 he was the chairman of the chair of mathematical logic at the faculty of mathematics and natural sciences of the Jan Kazimierz University in Lwów – though some part of his scientific career was connected with Kraków, he developed his main philosophical ideas just in Lwów.

Lwów school of mathematics, accepting main ideas of Janiszewski's programme (1917), developed another specialization than the Warsaw school. In Warsaw mainly set theory, topology and mathematical logic were developed. In Lwów functional analysis dominated,

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which was initiated by Stefan Banach (his mathematical talent has been discovered by Steinhaus) and developed by Steinhaus, Stanisław Mazur, Władysław Orlicz, Juliusz Schauder, Stefan Kaczmarz, Stanisław Ulam and Władysław Nikliborc. It did not demand deeper studies of logic and foundations of mathematics as it was the case in Warsaw. Consequently it is rather difficult to find philosophical remarks concerning mathematics in works of Lwów mathematicians. It could be also the result of the fact that logic as such has not been developed in Lwów, though the intellectual atmosphere for it and for the foundations of mathematics was good here (cf. [18]). Only in 1928 it has been decided to form a chair for mathematical logic – its first chairman became Chwistek. Earlier the only Lwów mathematician who worked in logic was Eustachy Żyliński. One should add however that other Lwów mathematicians did not disparage logic and the foundations of mathematics or even casually worked in it – one should mention here Banach and his joint paper with Alfred Tarski on the paradoxical decomposition of sphere [1] or results of Banach and Mazur concerning the computational analysis and constructive methods in mathematics (cf. [13]).

1. Stefan Banach did not avoid to take part in the philosophical life of Lwów and from time to time was active there. In particular Kazimierz Twardowski in his *Dzienniki* [Diary] writes, that Banach took part (on 7th March 1921) in the inaugural meeting of the Section of Epistemology of Polish Philosophical Society (cf. [16, vol. 1, p. 201]) and that he was present at the talk by Zygmunt Zawirski on relations between logic and mathematics held on 26th March 1927 during a meeting of Polish Philosophical Society ([16, vol. 1, p. 300]). On the 1st Congress of Polish Mathematicians held in Lwów in 1927 Banach gave (on 7th September 1927) in the section of mathematical logic a talk "O pojęciu granicy" [On the concept of a limit] ([16, vol. 1, p. 323]). In January 1923 at the meeting of Polish Philosophical Society in Lwów Banach gave a talk on paradoxes connected with the concept of equipollence of certain sets (for example the set of integers and the set of even natural numbers) as well as on problems connected with Banach-Tarski paradox. As source of those paradoxes he indicated infinite sets and the axiom of choice (formally consistent with set theory). According to him a logical system that "would not awake any objections" should be constructed to solve those paradoxes. This remark characterizes the attitude of Lwów mathematicians towards logic. In particular Banach did not see anything wrong for the mathematical practice in the lack of a good logical system. In the Lwów mathematical school the development of mathematics did not require additional studies in logic and the foundations of mathematics.

2. The best way to reconstruct the picture of mathematics cherished in Lwów is to analyze some remarks contained in works of popular character, in particular in works by Steinhaus.

One should tell here first of all about his book *Czem jest a czem nie jest matematyka* [What is and what is not mathematics] [14]. He writes there about various topics, in particular about the definition of mathematics, about its historical development, practical applications, method of mathematics, about differential and integral calculus, about numerical mathematics, about errors in mathematics and about connections of mathematics with the everyday life. From our point of view the most important are his remarks on defining mathematics as a science and his considerations about mathematical methods.

Trying to define mathematics Steinhaus stresses that on the one hand mathematics grew out from some practical needs of human being but on the other it is in fact a theoretical discipline. A characteristic feature of mathematics is its deductive method. He adds that "its axioms and definitions are in a large extent arbitrary" [14, p. 25]. Another feature of mathematics is the usage of symbols.

Logic is treated by Steinhaus with sympathy but not as an independent discipline having its own problems and methods. He treats logic as a tool of deduction. The deductive method determines in a certain sense also the subject of mathematics.

Mathematics is deductive, synthetic and formal. It is deductive since deduction is the only method allowed in it. It is synthetic because axioms, both logical and mathematical, are chosen not logically but with the help of intuition. It is formal because in mathematical argumentation one can take into account only those elements of concepts that have been included in definitions. Logic plays only an utilitarian role towards mathematics providing it with tools.

In the development of mathematics an important role is played also – according to Steinhaus – by aesthetical elements. Though there are no absolute criteria of beauty, in fact the feeling of beauty and the aspiration for it influence more the development of mathematics than the principle of perfect precision.

Steinhaus appreciated very much applied mathematics and applications of mathematics. Unfortunately he did not describe the connections between concepts and objects of mathematics on the one side and the reality on the other. One finds only his short and aphoristical remark: "Między duchem a materią pośredniczy matematyka" [Between spirit and matter mediates mathematics].

3. Eustachy Żyliński worked mainly in number theory, but after 1919 he began to work in algebra, logic and the foundations of mathematics. In particular he proved (cf. [21]; see also [22]) that in the classical propositional logic the only functors that suffice to define all other functors are binegation and Sheffer's disjunction.[1] One finds no separate papers by Żyliński devoted to the philosophy of mathematics and logic. We have only some remarks of philosophical character he made on various occasions.

In a talk (21st May 1921) "O przedmiocie i metodach matematyki współczesnej" [On the subject and method of contemporary mathematics] he identified mathematical theories with the set of consequences of accepted axioms. One should note here his unprecise treatment of logic – Żyliński refers to subjective feeling of certainty and obviousness rather than to formally and in advance described inference rules. He admits an infinite set of consequences of accepted axioms talking about corollaries that can be obtained.

Considering the problem of relations between logic and mathematics he compares it with the relation between "special set theories and a general one". He claims that mathematics is a natural science about certain objects and says that one refers to observation and even experiment in developing particular mathematical theories.

In the paper „Z zagadnień matematyki. II. O podstawach matematyki" [Problems of Mathematics. II. On Foundations of Mathematics] [23] Żyliński says about intuition. He

[1] A proof of this theorem can be found in [12].

argues that intuition can help to construct a proof but stresses that the proof itself cannot refer to intuition. Hence one has here the distinction between the context of discovery and the context of justification. In the first one – intuition is admitted, in the second – not.

He saw the great role played by mathematics in other disciplines as well as generally in culture. In a memorial by Żyliński, Ruziewicz and Banach from 14th April 1924 one reads: "Contemporary mathematics is nothing else as a general theory of strict thinking connected with the feeling of certainty. [...] Being a most general science about relations between objects, mathematics finds applications in every scientific and practical discipline that goes out in a sufficient manner beyond a description, simple induction and literary-artistic methods" (cf. [20, p. 1]).

4. Leon Chwistek (1884–1944) is known mainly for his logical works, in particular for his simplification of Whitehead and Russell's theory of types. His logical investigations however were – as it was the case by some Polish logicians, e.g., by Stanisław Leśniewski – connected with his philosophical ideas concerning logic and mathematics. Moreover, they were in a sense motivated by those ideas. Building semantics he wanted to overcome the philosophical idealism and was against the conception of an absolute truth. He did not content himself with solving particular definite fragmentary problems but – similarly to Leśniewski – attempted to construct a system containing the whole of mathematics.

Chwistek's interests in logic dates from his studies in Göttingen, in particular from the moment he attended the lecture by Poincaré in the spring 1909. Chwistek decided then to unify the ideas of Russell and Poincaré and to reform the theory of logical types by eliminating the non-predicative definitions. He decided to rebuild the system of Whitehead and Russell and did it in the nominalistic way by constructing a simple theory of types rediscovered later by F.P. Ramsey. In [2] and [3] Chwistek formulated a pure theory of logical types – a theory of constructive types. In this theory the nonconstructive objects are rejected but the price for that is the greater formal complication of the system.

Those investigations led Chwistek to the construction of a full theory of expressions and – on the base of it – of the so called rational metamathematics. This should be a system more fundamental than logic and it should enable the reconstruction of a classical logical calculus and of the Cantor's set theory. Moreover, it should fulfil nominalistic assumptions, hence in particular it should be free of any existential axioms, first of all of the reduction axiom and the axiom of choice. All this was based on the assumption that theorems of the system being constructed, and consequently of classical logic and of set theory, refer only to expressions/inscriptions that can be obtained in a finite number of steps by a rule of construction fixed ahead and not to the meaning of those expressions. Moreover, those expressions/inscriptions were understood as physical objects.

Those ideas have been developed by Chwistek later as a part of his philosophy of logic and mathematics, in particular as a part of his ideas concerning the methodology of deductive sciences. He developed them mainly in his book [4] *Granice nauki. Zarys logiki i metodologii nauk ścisłych* from 1935 – English translation *The Limits of Science. Outline of Logic and of the Methodology of the Exact Sciences* appeared in 1948.

According to Chwistek the human knowledge is neither full nor absolute. It cannot be full because statements concerning the whole of objects lead to inconsistencies. It cannot be absolute because there is no absolute reality. In *Limits of Science* he wrote:

> From those considerations it follows that the principle of contradiction excludes a full knowledge that could answer all questions. The aspiration for such a knowledge must – earlier or later – lead to a conflict with a common sense.

And a common sense is according to him – beside the admission of experience as a fundamental source of knowledge and of the necessity of schematization of experienced objects and phenomena – a common factor of all correct cognitive processes. It consists of rejecting all assumptions that cannot be experimentally checked or are inconsistent with experiments or are not based on reliable and certain statements concerning simple facts or cannot be logically reduced to such statements. Both empirical and deductive knowledge are relative. The first is relative because there are various types of experiments corresponding to various realities, and the second – because it depends on the accepted system of concepts. Chwistek says here about rational relativism.

Chwistek accepted the principle of the rationalism of knowledge and was decidedly against irrationalism. Rationalism consists of accepting only two sources of knowledge, namely the experience and strict reasoning. It concerns not only mathematics and exact sciences but experimental sciences and philosophy as well. He wrote (cf. [4]):

> The starting point of our conception of the world should be not metaphysical dregs but simple and clear truths based on experience and strict reasoning.

Consequently he was against irrationalism, metaphysics and idealism in philosophy and mathematics.[2] He sharply criticized Plato, Hegel, Husserl and Bergson. Seeing the defects of positivism he appreciated its epistemological conceptions. Add that Chwistek highly appreciated also dialectical materialism seeing in fact almost no fundamental conflicts between it and positivism. His own epistemological conceptions he described as critical rationalism and set it against dogmatic rationalism.

A way from the difficulties caused by irrationalism and simultaneously a weapon in a struggle against it is formal logic, in particular rational metamathematics founded by him. Chwistek begins his *Limits of Science* writing in the first sentence:

> We are living in a period of unparalleled growth of antirationalism.

And he finishes the Introduction by the sentence:

> History teaches that ultimately victory has always been the destiny of societies who employ the principles of exact reasoning.

He writes also in the Introduction:

[2] It is worth noting here that Chwistek was against irrationalism and idealism not only because they are – in his opinion – incorrect philosophical theories but also because they are the source of human sufferings, social injustice, cruel excesses and wars.

> When this new system [i.e., the system of rational metamathematics – R.M.] is completely worked out, we will be able to say, that we have at our disposal an infallible apparatus which sets off exact thought from other forms of thought.

The epistemological views of Chwistek were close to the neopositivism. He claimed that an object of a scientific knowledge can be only what is or can be given in experience, hence only what can be seen or experienced by senses eventually assisted by instruments. He wrote in [4]:

> Talking about reality we do not think about an ideal object but about those schemes we have to do with in a given case.

Both in science as in the philosophy one should – according to Chwistek – use a constructive method. Though one can apply it in a full form mainly in deductive sciences, it can be used also in empirical sciences and in the philosophy. It is based on the analysis of intuitive concepts used in a given discipline. It enables the separation of primitive notions whose meaning is characterized in axioms. On the basis of axioms one obtains now theorems with the help of laws of (formal) logic.

Only what is given in an experience can be an object of a cognition. There are however various types of experience. In this way we come to the best known original philosophical conception of Chwistek, namely to his theory of the plurality of realities.

In *Granice nauki* he accepted four types of reality corresponding to possible types of experience. Hence we have there the reality of impressions, reality of images, reality of things (reality of everyday life) and physical reality (constructed in exact sciences). He attributed independent existence and full theoretical equality of rights to all particular kinds of reality.

Having presented general methodological and ontological conceptions of Chwistek let us turn now to his views connected directly with the philosophy of mathematics. In fact we have already mentioned some of his views concerning mathematics and lying at the base of his logical conceptions. Now we shall consider his nominalism which found full expression in his philosophy of mathematics.

Chwistek claimed that the object of deductive sciences, hence in particular of mathematics, are expressions being constructed according to accepted rules of construction. Consequently objects of mathematics are not ideal objects such as points, lines, numbers or sets. Objects of mathematics are in fact expressions being physical objects given to us in experience. They can be transformed according to accepted rules. In every given system one accepts such rules as well as some expressions that play the role of axioms and form the base on which one deduces theorems. Rules of transformation and axioms are chosen in such a way that the expressions can be interpreted as descriptions of considered states of things. To be able to apply deductive theories to particular disciplines and generally to getting know particular domains of the reality one should schematize elements of the latter.

Geometry is – according to Chwistek – an experimental discipline. In Chapter VIII of *Limits of Science* [4, p. 170] he wrote:

> Geometry is an experimental science. It consists of measuring segments, angles and surfaces. In such a way it was considered by Egyptians and such it remained in fact till today. What is nowadays commonly considered to be geometry, i.e., about what one writes in handbooks, is a strange mixture of experimental geometry and of geometric metaphysics, that we inherited from Greeks in the form of Euclid's elements.

The development in the 19th century of systems of non-Euclidean geometry of Bolyai, Gauss and Lobatchevsky – that Chwistek considered to be the most important achievement in the exact sciences – rejected in his opinion the Kantian idealism. Those geometries have shown that, e.g., the concept of a line has no objective character but depends on adopted axioms.

Similarly as geometry one should treat also arithmetic, mathematical analysis and other mathematical theories obtaining in this way a nominalistic interpretation of them.

The fate of the philosophical conceptions of Chwistek was similar to the fate of his logical conceptions. The system of rational metamathematics has not been developed by Chwistek in detail. Chwistek went solitarily along his own paths. His investigations were not in the main stream of the development of logic and philosophy of mathematics. Similarly as Leśniewski (cf. [10]), Chwistek worked on his own conceptions and ideas without any collaboration with other logicians, mathematicians or philosophers. Being a professor of Lwów university he had in fact no stronger contacts with Lwów-Warsaw school of philosophy (cf. [17]). His ideas have been often sharply criticized. Philosophical investigations of Chwistek had no systematic character and it seems that they were not treated by himself with full sense of responsibility (cf. Preface to [6, p. VII]). He did not explain many of concepts he used, his conceptions has been "earlier proclaimed than checked" [6]. He did not develop his systems in detail but satisfied himself by sketching them. His works did generally not find an interest by logicians and philosophers (with the exception of his version of type theory). Only after 1945 together with the growing interest in the nominalism in the philosophy of mathematics some of his ideas found a recognition.

5. The above considerations show that in the Lwów school of mathematics no general, comprehensive and homogeneous philosophical concept concerning mathematics and logic has been formulated and developed – the only exception was here Chwistek. One finds there only separate, detached remarks formulated when considering other problems and being in fact a reflection on own research in mathematics. Only Chwistek who was in fact not a mathematician but a (mathematical) logician tried to develop certain comprehensive theory. The dominating feature of his approach were nominalism and constructivism with all their consequences. Unfortunately the style of his work did not allow to develop his concept with all details.

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