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GELFAND-HILLE TYPE THEOREMS FOR ORDERED UNITAL TOPOLOGICAL ALGEBRAS

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Gelfand-Hille type theorems for topological algebras and for ordered topological algebras are considered. It is shown that if $a \in A$ is Abel bounded in an ordered topological algebra (ordering is defined by a closed normal algebra cone), then a is Cesàro bounded. We find conditions under which the identity element of an ordered topological algebra A is the unique element $a \in A$ with spectrum $\sigma(a) = \{1\}$.

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Розглядаються теореми типу Ґельфанда-Гілле для (впорядкованих) топологічних алгебр. Показано, що кожен обмежений за Абелем елемент $a \in A$ впорядкованої топологічної алгебри A є обмежений по Чезаро. Знайдено умови, за яких єдиним елементом $a \in A$ з одиничним спектром $\sigma(a) = \{1\}$ є одиниця впорядкованої топологічної алгебри A.

1. Introduction

There are several papers written about the different boundedness conditions for ordered Banach algebras. When one looks at the proofs more carefully, it is possible to observe that the existence of the norm is not always necessary and that many results hold also in more general case. The main source for this paper is [2], where several results and ideas of this paper can be found for Banach algebra case. This paper is an attempt to generalize first the notions of different kinds of boundedness for a topological algebra without using the norm.

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The second and the main goal of this paper is to show that results, known for ordered Banach algebras, hold also in general case and that several proofs do not depend on the topology obtained with the norm. Some results remain true also without the partial ordering. Nevertheless, there are some results, which had to be presented with a bit different conditions.

2. Results for general topological algebras

By a topological algebra we mean a topological vector space over $\mathbb{C}$ in which the multiplication is separately continuous. Through the whole paper, let A be a topological algebra with unit e_A and a zero element θ_A. Let $\mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$. Recall that the *spectrum* of an element $a \in A$ is defined as the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \text{ is not invertible in } A\}.$$

We will say that an element $a \in A$ is

a) *power bounded* if for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that $a^n \in \lambda_O O$ for all $n \in \mathbb{N}$.

b) *Cesàro bounded* if for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that

$$M_n(a) := \frac{e_A + a + \cdots + a^n}{n+1} \in \lambda_O O$$

for all $n \in \mathbb{N}$.

c) *Abel bounded* if

$$\sum_{k=0}^{\infty} \mu^k a^k \tag{1}$$

exists in A for every $\mu \in (0,1)$ and for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that

$$(1-\mu) \sum_{k=0}^{\infty} \mu^k a^k \in \lambda_O O$$

for all $\mu \in (0,1)$.

d) *uniformly Abel bounded* if for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that

$$(1-\mu) \sum_{k=0}^{n} \mu^k a^k \in \lambda_O O$$

for all $\mu \in (0,1)$ and all $n \in \mathbb{N}$.

e) (N)-*Abel bounded* (for some $N \in \mathbb{N}$) if (1) exists in A for every $\mu \in (0,1)$ and for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that

$$(1-\mu)^N \sum_{k=0}^{\infty} \mu^k a^k \in \lambda_O O$$

for all $\mu \in (0,1)$.

We start with a generalization of Theorem 2.4 from [2, p. 44].

Theorem 2.1. *Let A be a topological algebra with jointly continuous multiplication and let $a \in A$ be such that a^N is Abel bounded for some $N \in \mathbb{N}$. Then a is Abel bounded.*

Proof. Let $a \in A$ and let $N \in \mathbb{N}$ be such that a^N is Abel bounded. Take any neighbourhood O of zero in A. Then there exist balanced neighbourhoods U and V of zero in A such that $UU \subseteq O$ and

$$\underbrace{V + \cdots + V}_{N \text{ summands}} \subseteq U.$$

Moreover, there exist $\lambda_0, \ldots, \lambda_{N-1} \in \mathbb{R}^+$ such that

$$e_A \in \lambda_0 V, \quad \mu a \in \lambda_1 V, \quad (\mu a)^2 \in \lambda_2 V, \quad \ldots, \quad (\mu a)^{N-1} \in \lambda_{N-1} V$$

for all $\mu \in (0,1)$, because V is balanced. Let $\lambda := \max\{\lambda_0, \ldots, \lambda_{N-1}\}$. Then

$$\begin{aligned} e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1} &\in \lambda_0 V + \lambda_1 V + \lambda_2 V + \cdots + \lambda_{N-1} V = \\ &= \lambda\Big(\frac{\lambda_0}{\lambda} V\Big) + \lambda\Big(\frac{\lambda_1}{\lambda} V\Big) + \cdots + \lambda\Big(\frac{\lambda_{N-1}}{\lambda} V\Big) \subseteq \lambda(\underbrace{V + \cdots + V}_{N \text{ summands}}) \subseteq \lambda U \end{aligned}$$

for all $\mu \in (0,1)$. As $\mu^N \in (0,1)$ and a^N is Abel bounded, there exists $\nu_U \in \mathbb{R}^+$ such that

$$(1-\mu^N) \sum_{k=0}^{\infty} \mu^{Nk} a^{Nk} \in \nu_U U.$$

Since

$$\begin{aligned} \sum_{k=0}^{N(m+1)-1} \mu^k a^k &= \sum_{k=0}^{m} (\mu a)^{Nk} + \sum_{k=0}^{m} (\mu a)^{Nk+1} + \cdots + \sum_{k=0}^{m} (\mu a)^{Nk+(N-1)} = \\ &= (e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1}) \sum_{k=0}^{m} \mu^{Nk} a^{Nk} \end{aligned}$$

for every $m \in \mathbb{N}$ and all $\mu \in (0,1)$, then

$$\sum_{k=0}^{\infty} \mu^k a^k = \lim_{m \to \infty} \sum_{k=0}^{N(m+1)-1} \mu^k a^k = (e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1}) \sum_{k=0}^{\infty} \mu^{Nk} a^{Nk}.$$

Hence (1) exists in A for each $\mu \in (0,1)$. Now

$$\begin{aligned} (1-\mu) \sum_{k=0}^{\infty} \mu^k a^k &= (e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1}) \frac{1-\mu}{1-\mu^N} (1-\mu^N) \sum_{k=0}^{\infty} \mu^{Nk} a^{Nk} \in \\ &\lambda U \frac{1-\mu}{1-\mu^N} \nu_U U = (\lambda \nu_U) U \Big(\frac{1-\mu}{1-\mu^N} U\Big) \subseteq (\lambda \nu_U) UU \subseteq (\lambda \nu_U) O, \end{aligned}$$

because $\frac{1-\mu}{1-\mu^N} \in (0,1)$. Taking $\lambda_O := \lambda \nu_U \in \mathbb{R}^+$, we obtain that $(1-\mu)\sum_{k=0}^{\infty} \mu^k a^k \in \lambda_O O$ for all $\mu \in (0,1)$. Hence, a is Abel bounded. □

Using analoguous argumentation, we can easily prove the following corollary (generalizing Corollary 2.5 from [2, p. 44]).

Corollary 2.2. *Let A be a topological algebra with jointly continuous multiplication and let $a \in A$ be such that a^N is uniformly Abel bounded for some $N \in \mathbb{N}$. Then a is uniformly Abel bounded.*

Next result gives a generalization of Theorem 2.6 of [2, p. 44].

Proposition 2.3. *Let A be a topological algebra with jointly continuous multiplication and with continuous inversion. If $a \in A$ is Abel bounded and $\sigma(a) \subseteq [0, \infty)$, then a^N is Abel bounded for all $N \in \mathbb{N}$.*

Proof. Let $a \in A$ be Abel bounded and $\sigma(a) \in [0, \infty)$. Let $\nu \in (0, 1)$. Moreover, let W be a neighbourhood of zero in A. Fix an arbitrary $N \in \mathbb{N}$. Then there exists $\mu \in (0, 1)$ such that $\mu^N = \nu$ and a balanced neighbourhood O of zero in A such that $NOO \subseteq W$.

Let $p_N : A \to A$ be defined by $p_N(b) = b + b^2 + \cdots + b^{N-1}$ for every $b \in A$. Since $\sigma(a) \subset [0, \infty)$, then, by Spectral Mapping Theorem (see, for example, [1], Proposition 1.7.3.), we have $\sigma(p_N(\mu a)) = p_N(\mu\sigma(a)) \subset [0, \infty)$ for all $\mu \in (0, 1)$. Therefore, $-1 \notin \sigma(p_N(\mu a))$. Hence, $e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1} = p_N(\mu a) - (-1)e_A$ is invertible in A for all $\mu \in [0, 1])$. By assumption, the inversion in A is continuous. Thus, the map $F : [0, 1] \to A$, defined by

$$F(\mu) = (e_A + \mu a + \cdots + (\mu a)^{N-1})^{-1}$$

is continuous. Hence, $F([0, 1])$ is a compact subset in A, because $[0, 1]$ is compact in $\mathbb{R}$. Therefore, $F([0, 1])$ is bounded in A (see, for example, [3, p. 147], Proposition 7). Consequently, there is a positive number ρ such that $(e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1})^{-1} \in \rho O$ for all $\mu \in (0, 1)$.

By the assumptions, we know that (1) exists in A for every $\mu \in (0, 1)$ and for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that $(1 - \mu) \sum_{k=0}^{\infty} \mu^k a^k \in \lambda_O O$ for all $\mu \in (0, 1)$.

As it was shown in the proof of Theorem 1, for every $m \in \mathbb{N}$ and each $\mu \in (0, 1)$ we get

$$\sum_{k=0}^{N(m+1)-1} \mu^k a^k = (e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1}) \sum_{k=0}^{m} \mu^{Nk} a^{Nk}.$$

Hence,

$$\sum_{k=0}^{m} \mu^{Nk} a^{Nk} = (e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1})^{-1} \sum_{k=0}^{N(m+1)-1} \mu^k a^k.$$

Thus,

$$\sum_{k=0}^{\infty} \mu^{Nk} a^{Nk} = \lim_{m\to\infty} \sum_{k=0}^{m} \mu^{Nk} a^{Nk} = (e_A + \mu a + (\mu a)^2 + \cdots + (\mu a)^{N-1})^{-1} \sum_{k=0}^{\infty} \mu^k a^k$$

for each $\mu \in (0,1)$. Therefore, the sum $\sum_{k=0}^{\infty} \nu^k (a^N)^k = \sum_{k=0}^{\infty} \mu^{Nk} a^{Nk}$ exists in A for each $\nu \in (0,1)$. It is easy to see that $(e_A - \mu a) \sum_{k=0}^{\infty} \mu^k a^k = e_A$ and $(e_A - (\mu a)^N) \sum_{k=0}^{\infty} (\mu a)^{Nk} = e_A$. Moreover, $e_A - (\mu a)^N = (e_A - \mu a)(e_A + \mu a + \cdots + (\mu a)^{N-1})$. Hence,

$$\sum_{k=0}^{\infty} (\mu a)^{Nk} = (e_A - (\mu a)^N)^{-1} = (e_A + \mu a + \cdots + (\mu a)^{N-1})^{-1} \sum_{k=0}^{\infty} \mu^k a^k.$$

Finally,

$$\begin{aligned}(1-\nu)\sum_{k=0}^{\infty} \nu^k (a^N)^k &= (1-\mu^N)\sum_{k=0}^{\infty} (\mu a)^{Nk} = \\ &= (1+\mu+\mu^2+\cdots+\mu^{N-1})(e_A + \mu a + \cdots + (\mu a)^{N-1})^{-1}(1-\mu)\sum_{k=0}^{\infty} \mu^k a^k \in \\ &\in \rho\lambda_O N\Big(\frac{1+\mu+\mu^2+\cdots+\mu^{N-1}}{N} OO\Big) \subseteq \rho\lambda_O (NOO) \subseteq \rho\lambda_O W\end{aligned}$$

for all $\nu \in (0,1)$. Thus, taking $\lambda_W := \rho\lambda_O$, we see that a^N is Abel bounded. Since $N \in \mathbb{N}$ was chosen arbitrarily, a^N is Abel bounded for all $N \in \mathbb{N}$. □

3. Results for ordered topological algebras

Let A be an algebra. An algebra cone $C \subseteq A$ is a subset of A which satisfies the following conditions:

1) $C + C \subseteq C$;

2) $\lambda C \subseteq C$ for every $\lambda \in \mathbb{R}^+ \cup \{0\}$;

3) $C \cdot C \subseteq C$;

4) $e_A \in C$.

An algebra cone is called

a) *proper* if $C \cap (-C) = \{\theta_A\}$ and

b) *inverse closed* if for every invertible element $a \in A$ (with inverse $a^{-1} \in A$) the inclusion $a \in C$ implies $a^{-1} \in C$.

Every algebra cone C induces a partial order $\leq_C$ on A as follows:

$$\text{for } a, b \in A \text{ we say that } a \leq_C b \text{ if and only if } b - a \in C.$$

Next, we generalize the definitions for the classes of algebra cones of Banach algebras to the case of general topological algebras.

Let A be a topological algebra and $C \subseteq A$ an algebra cone. We will endowe C with a subspace topology induced by the topology of A. An algebra cone, is called

c) *normal* if for every neighborhood O of zero there exists a real number $\alpha \geq 1$ such that $\{a \in A : \theta_A \leq_C a \leq_C b\} \subset \alpha \cdot O$ for every $b \in O$;

d) *closed* if C is a closed subset in the topology of A.

We will denote by $(A, \leq_C)$ an algebra A with an order $\leq_C$ induced by an algebra cone C.

The following result generalizes Theorem 2.1 of [2, p. 42].

Theorem 3.1. *Let $(A, \leq_C)$ be an ordered topological algebra, where C is a closed normal algebra cone. If $a \in C$ is Abel bounded, then a is Cesàro bounded.*

Proof. Fix an arbitrary neighbourhood U of zero in A. Then there exists a balanced neighbourhood O of zero such that $O \subseteq U$. By assumption, a is Abel bounded. Hence, there exists $\nu_O \in \mathbb{R}^+$ such that

$$(1-\mu)\sum_{k=0}^{\infty} \mu^k a^k \in \nu_O O$$

for all $\mu \in (0,1)$. Since $a \in C$ and C is an algebra cone, we get

$$x_n := (1-\mu)\sum_{k=0}^{n} \mu^k a^k \in C$$

for every $n \in \mathbb{N}$. Since the sum (1) exists in A, then the sequence (x_n) converges in A and the limit x of the sequence (x_n) also belongs to C, because C is closed, i.e.,

$$x := (1-\mu)\sum_{k=0}^{\infty} \mu^k a^k \in C.$$

Similarily, we see that, for every fixed $n_0 \in \mathbb{N}$, we have

$$y_n := (1-\mu)\sum_{k=n_0}^{n} \mu^k a^k \in C$$

for every $n \in \mathbb{N}$ with $n > n_0$. Taking again the limit, we get

$$x - x_{n_0} = (1-\mu)\sum_{k=n_0+1}^{\infty} \mu^k a^k \in C$$

for every $n_0 \in \mathbb{N}$. Now, we have obtained that

$$\theta_A \leq_C x_n \leq_C x \quad \text{and} \quad x \in \nu_O O$$

for every $n \in \mathbb{N}$. Fix now an arbitrary $n \in \mathbb{N}$. Since μ is arbitrary in $(0,1)$, we get that everything remains true also for $\mu = \frac{n}{n+1}$. In this case

$$\Big(1 - \frac{n}{n+1}\Big)\sum_{k=0}^{n}\Big(\frac{n}{n+1}\Big)^k a^k = \frac{1}{n+1}\sum_{k=0}^{n}\Big(\frac{n}{n+1}\Big)^k a^k$$

and

$$\Big(\frac{n}{n+1}\Big)^n M_n(a) = \frac{1}{n+1}\Big(\frac{n}{n+1}\Big)^n \sum_{k=0}^{n} a^k.$$

Since $a \in C$ and C is an algebra cone, we have

$$M_n(a), \Big(\frac{n}{n+1}\Big)^n M_n(a), \frac{1}{n+1}\sum_{k=0}^{n}\Big[\Big(\frac{n}{n+1}\Big)^k - \Big(\frac{n}{n+1}\Big)^n\Big]a^k \in C.$$

As

$$\frac{1}{n+1}\sum_{k=0}^{n}\Big[\Big(\frac{n}{n+1}\Big)^k - \Big(\frac{n}{n+1}\Big)^n\Big]a^k = x_n - \Big(\frac{n}{n+1}\Big)^n M_n(a),$$

then

$$\theta_A \leq_C \Big(\frac{n}{n+1}\Big)^n M_n(a) \leq_C x_n \leq_C x \quad \text{and} \quad x_n \in \nu_O O.$$

Because $\nu_O O$ is also a neighbourhood of zero and C is a normal cone, it follows that there exists a real number $\alpha \geq 1$ such that

$$\Big(\frac{n}{n+1}\Big)^n M_n(a) \in \alpha(\nu_O O).$$

Therefore,

$$M_n(a) \in \Big(1 - \frac{1}{n+1}\Big)^{-n} \alpha\nu_O O = \Big(1 - \frac{1}{n+1}\Big)^{-(n+1)} \frac{n}{n+1} \cdot \alpha\nu_O O \subset e\alpha\nu_0 O$$

because O is balanced and

$$0 < \Big(1 - \frac{1}{n+1}\Big)^{-(n+1)} \leq e$$

for all $n \in \mathbb{N}$, where e is the Euler number. Taking $\lambda_U := e\alpha\nu_O \in \mathbb{R}^+$, we see that $M_n(a) \in \lambda_U O \subseteq \lambda_U U$. Since $n \in \mathbb{N}$ was chosen arbitrarily, we have $M_n(a) \in \lambda_U U$ for all $n \in \mathbb{N}$. Thus, a is Cesàro bounded. □

Now we generalize Theorem 2.7 from [2, p. 45].

Proposition 3.2. *Let $(A, \leq_C)$ be an ordered topological algebra with jointly continuous multiplication and with continuous inversion, where C is a closed proper algebra cone. Let $a \in A$ be such that $\sigma(a) \subseteq [0, \infty)$. Then the following are equivalent:*

a) $a = e_A$;

b) *there exist $L, N \in \mathbb{N}$ such that a^L is Abel bounded and $a^N \geq_C e_A$.*

Proof. The implication a)$\Rightarrow$ b) is obvious (we take $L = N = 1$).

Suppose now, that b) holds. Then (by Theorem 2.1) a is Abel bounded because a^L is Abel bounded. By Proposition 2.3, we see that a^N is also Abel bounded. Hence,

$$\sum_{k=0}^{\infty} \mu^k (a^N)^k$$

exists in A for every $\mu \in (0,1)$ and for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that

$$(1-\mu)S(a^N,\mu) := (1-\mu)\sum_{k=0}^{\infty}\mu^k(a^N)^k \in \lambda_O O$$

for all $\mu \in (0,1)$. Therefore,

$$\lim_{\mu\to 1^-}(1-\mu)^2 S(a^N,\mu) = \theta_A.$$

Since

$$e_A - \mu a^N = \mu\Big(\frac{1-\mu}{\mu}e_A - (a^N - e_A)\Big),$$

we have

$$\sum_{k=0}^{\infty}\mu^k(a^N)^k = (e_A - \mu a^N)^{-1} = \frac{1}{\mu}\Big(\frac{1-\mu}{\mu}e_A - (a^N - e_A)\Big)^{-1}.$$

Hence,

$$\begin{aligned}(1-\mu)^2 S(a^N,\mu) &= \frac{(1-\mu)^2}{\mu}\Big(\frac{1-\mu}{\mu}e_A - (a^N - e_A)\Big)^{-1} = \\ &= (1-\mu)\sum_{k=0}^{\infty}\Big(\frac{\mu}{1-\mu}\Big)^k(a^N - e_A)^k = \\ &= \mu(a^N - e_A) + (1-\mu)e_A + (1-\mu)\sum_{k=2}^{\infty}\Big(\frac{\mu}{1-\mu}\Big)^k(a^N - e_A)^k.\end{aligned}$$

As $e_A \in C$ and $a^N \geq_C e_A$, then $a^N, a^N - e_A, (a^N)^k \in C$ for all $k \in \mathbb{N}$ and

$$\sum_{k=2}^{m}\Big(\frac{\mu}{1-\mu}\Big)^k(a^N - e_A)^k \in C$$

for all $m \in \mathbb{N}$ with $m \geq 2$. Because C is closed, we get

$$\sum_{k=2}^{\infty}\Big(\frac{\mu}{1-\mu}\Big)^k(a^N - e_A)^k \in C.$$

Taking this into account, we have that

$$(1-\mu)e_A + (1-\mu)\sum_{k=2}^{\infty}\Big(\frac{\mu}{1-\mu}\Big)^k(a^N - e_A)^k \in C$$

for all $\mu \in (0,1)$. Hence,

$$\lim_{\mu\to 1^-}\Big[(1-\mu)e_A + (1-\mu)\sum_{k=2}^{\infty}\Big(\frac{\mu}{1-\mu}\Big)^k(a^N - e_A)^k\Big] = -(a^N - e_A).$$

Again, as C is closed, then $-(a^N - e_A) \in C$.

So, we have obtained that $a^N - e_A \in C$ and $-(a^N - e_A) \in C$. Consequently, $a^N - e_A \in C \cap (-C) = \{\theta_A\}$, since C is a proper cone. Therefore, $a^N = e_A$.

In case $N = 1$, we have $a - e_A = \theta_A$ and our problem is solved. Suppose now that $N \geq 2$. Then

$$\theta_A = a^N - e_A = (a - e_A)(a^{N-1} + \cdots + a + e_A).$$

Using again the map $p_N : A \to A$, defined in the proof of Proposition 2.3, we obtain that

$$a^{N-1} + \cdots + a + e_A = p_N(a) - (-1)e_A$$

is invertible in A, because $\sigma(p_N(a)) \in [0, \infty)$. Hence, $(a^{N-1} + \cdots + a + e_A)^{-1}$ exists in A. Thus,

$$a - e_A = \theta_A(a^{N-1} + \cdots + a + e_A)^{-1} = \theta_A$$

and we have again obtained $a = e_A$. □

Next, we give a version of the Theorem 3.1 from [2, p. 47].

Theorem 3.3. *Let $(A, \leq_C)$ be a topological algebra, where C is a closed proper algebra cone. If $a \in A$ and $N \in \mathbb{N}$ are such that $a \geq_C e_A$ and a is (N)-Abel bounded, then $(a - e_A)^N = \theta_A$.*

Proof. Let $a \in A$ and $N \in \mathbb{N}$ be such that a is (N)-Abel bounded. Then (1) exists in A for every $\mu \in (0, 1)$ and for every neighbourhood O of zero in A there exists $\lambda_O \in \mathbb{R}^+$ such that

$$(1-\mu)^N \sum_{k=0}^{\infty} \mu^k a^k \in \lambda_O O$$

for all $\mu \in (0, 1)$.

Similarily as in the proof of Proposition 3.2, we have

$$(1-\mu)^N \sum_{k=0}^{\infty} \mu^k a^k = \mu^N \Big(\frac{1-\mu}{\mu}\Big)^N \frac{1}{\mu} \Big(\frac{1-\mu}{\mu} e_A - (a - e_A)\Big)^{-1} =$$

$$= \mu^{N-1} \Big(\frac{1-\mu}{\mu}\Big)^N \sum_{k=0}^{\infty} \Big(\frac{\mu}{1-\mu}\Big)^{k+1} (a - e_A)^k.$$

Hence,

$$S := \mu^{N-1} \left(\frac{1-\mu}{\mu}\right)^N \sum_{k=0}^{\infty} \left(\frac{\mu}{1-\mu}\right)^{k+1} (a - e_A)^k \in \lambda_O O$$

for all $\mu \in (0, 1)$, which implies that S is bounded in A. Thus,

$$\lim_{\mu \to 1^-} \frac{1-\mu}{\mu} S = \theta_A.$$

Now,

$$\frac{1-\mu}{\mu} S = \mu^{N-1} \sum_{k=0}^{\infty} \Big(\frac{\mu}{1-\mu}\Big)^{k-N} (a - e_A)^k =$$

$$= \mu^{N-1} \Big(\Big(\frac{1-\mu}{\mu}\Big)^N e_A + \Big(\frac{1-\mu}{\mu}\Big)^{N-1} (a - e_A) + \cdots + (a - e_A)^N\Big) + \mu^{N-1} \sum_{k=N+1}^{\infty} \Big(\frac{\mu}{1-\mu}\Big)^{k-N} (a - e_A)^k.$$

Since $a \geq_C e_A$, then $a - e_A \in C$, which implies that

$$S_m := \mu^{N-1}\left(\frac{1-\mu}{\mu}\right)^N \sum_{k=0}^{m}\left(\frac{\mu}{1-\mu}\right)^{k+1}(a-e_A)^k \in C$$

for every $m \in \mathbb{N}$ and every $\mu \in (0,1)$. Because C is closed, then

$$S = \lim_{m\to\infty} S_m \in C$$

and also

$$\mu^{N-1}\sum_{k=N+1}^{\infty}\left(\frac{\mu}{1-\mu}\right)^{k-N}(a-e_A)^k = \frac{1-\mu}{\mu}(S - S_N) \in C \tag{2}$$

for every $\mu \in (0,1)$. Now the equality

$$\theta_A = \lim_{\mu\to 1^-}\frac{1-\mu}{\mu}S = (a-e_A)^N + \lim_{\mu\to 1^-}\mu^{N-1}\sum_{k=N+1}^{\infty}\left(\frac{\mu}{1-\mu}\right)^{k-N}(a-e_A)^k$$

implies

$$\lim_{\mu\to 1^-}\mu^{N-1}\sum_{k=N+1}^{\infty}\left(\frac{\mu}{1-\mu}\right)^{k-N}(a-e_A)^k = -(a-e_A)^N.$$

Since C is closed, (2) yields $-(a-e_A)^N \in C$. Thus,

$$(a-e_A)^N \in C \cap (-C) = \{\theta_A\},$$

because C is a proper cone. Consequently, $(a-e_A)^N = \theta_A$. □

Next result is a non-normed version of Theorem 3.2 from [2, p. 48].

Proposition 3.4. *Let $(A, \leq_C)$ be an ordered topological algebra, where C is a closed normal algebra cone. If there exists $N \in \mathbb{N}$ such that $a \in C$ is (N)-Abel bounded, then*

$$\lim_{n\to\infty}\frac{M_n(a)}{n^N} = \theta_A. \tag{3}$$

Proof. Let $N \in \mathbb{N}$ be such that $a \in C$ is (N)-Abel bounded. Then the sum (1) exists in A for every $\mu \in (0,1)$. Thus,

$$\lim_{m\to\infty}\sum_{k=n+1}^{m}\mu^k a^k = \sum_{k=n+1}^{\infty}\mu^k a^k$$

belongs to A for every $\mu \in (0,1)$ and every $n \in \mathbb{N}$. Since $a \in C$ and C is a cone, then

$$(1-\mu)^N\sum_{k=0}^{m}\mu^k a^k,\ \ (1-\mu)^N\sum_{k=n+1}^{m}\mu^k a^k,\ \ (1-\mu)^N\sum_{k=0}^{n}(\mu^k-\mu^n)a^k \in C$$

for every $\mu \in (0,1)$ and every $m, n \in \mathbb{N}$ with $n < m$. Therefore,

$$(1-\mu)^N\sum_{k=0}^{\infty}\mu^k a^k,\ \ (1-\mu)^N\sum_{k=n+1}^{\infty}\mu^k a^k \in C$$

for every $n \in \mathbb{N}$, because C is closed. Thus, we have obtained

$$\theta_A \leq_C (1-\mu)^N \mu^n (n+1) M_n(a) = (1-\mu)^N \mu^n \sum_{k=0}^{n} a^k \leq_C (1-\mu)^N \sum_{k=0}^{n} \mu^k a^k \leq_C (1-\mu)^N \sum_{k=0}^{\infty} \mu^k a^k$$

for every $\mu \in (0,1)$ and every $n \in \mathbb{N}$.

Let O be any neighbourhood of zero in A. Then there exists a balanced neighbourhood U of zero in A such that $U \subseteq O$. Moreover, let $\alpha \geq 1$ be the constant from the normality condition for the cone C.

Since a is (N)-Abel bounded, there exists $\lambda_U \in \mathbb{R}^+$ such that

$$(1-\mu)^N \sum_{k=0}^{\infty} \mu^k a^k \in \lambda_U U$$

for all $\mu \in (0,1)$. By assumption, C is normal. Therefore,

$$(1-\mu)^N \mu^n (n+1) M_n(a) \in \alpha \lambda_U U$$

for all $n \in \mathbb{N}$ and all $\mu \in (0,1)$. If we take $\mu := \frac{n}{n+1}$, then

$$M_n(a) \in \alpha \lambda_U (n+1)^{N-1} \Big(1 + \frac{1}{n}\Big)^n U \subseteq \alpha \lambda_U (n+1)^{N-1} e U,$$

because $(1 + \frac{1}{n})^n \leq e$ for every $n \in \mathbb{N}$ and U is balanced. Taking $\lambda_O := \alpha \lambda_U e^2$, gives

$$\frac{M_n(a)}{n^N} \in \alpha \lambda_U e \frac{1}{n} \Big(1 + \frac{1}{n}\Big)^{N-1} U \subseteq \frac{1}{n} \alpha \lambda_U e^2 U \subseteq \frac{1}{n} \lambda_O U \subset U \subset O$$

for all $n > \max\{\lambda_O, (e^{\frac{1}{N-1}} - 1)^{-1}\}$. Hence, (3) is true. □

The last result is a generalization of Theorem 4.1 from [2, p. 49].

Theorem 3.5. *Let $(A, \leq_C)$ be an ordered topological algebra with continuous inversion, where C is a closed proper and inverse closed algebra cone. If $a \in A$ is such that $\sigma(a) = \{1\}$, then the following are equivalent:*

a) $a = e_A$;

b) $a^N \in C$ *for some* $N \in \mathbb{N}$.

Proof. It is obvious that a) implies b).

Suppose, now, that there exists $N \in \mathbb{N}$ such that $a^N \in C$. Since C is an algebra cone, $\frac{a^{Nk}}{\lambda^{k+1}} \in C$ for every $\lambda \in \mathbb{R}^+$ and every $k \in \mathbb{N}$. Therefore,

$$S_m := \sum_{k=0}^{m} \frac{a^{Nk}}{\lambda^{k+1}} \in C$$

for every $\lambda \in \mathbb{R}^+$ and for every $m \in \mathbb{N}$. If $\sigma(a) = \{1\}$ and $\lambda > 1$, then the sequence (S_m) converges and its limit

$$S = \sum_{k=0}^{\infty} \frac{a^{Nk}}{\lambda^{k+1}} = \lim_{m \to \infty} S_m \in C,$$

because C is closed. Hence, $S = (\lambda e_A - a^N)^{-1} \in C$. Since C is inverse closed, then also $\lambda e_A - a^N \in C$ for every $\lambda > 1$. Using again the fact that C is closed, we obtain that

$$e_A - a^N = \lim_{\lambda \to 1^+} (\lambda e_A - a^N) \in C.$$

From $\sigma(a) = \{1\}$ it follows (by the Spectral Mapping Theorem) that $\sigma(a^N) = \{1\}$, thus a^N is invertible in A. Hence, $a^{-N} = (a^N)^{-1} \in C$, because C is inverse closed. Following the steps above, we can show that

$$(\lambda e_A - a^{-N})^{-1} = \sum_{k=0}^{\infty} \frac{a^{-Nk}}{\lambda^{k+1}} \in C$$

for every $\lambda > 1$. Taking the limit $\lambda \to 1^+$, we obtain that $e_A - a^{-N} \in C$, as well. Therefore,

$$-(e_A - a^N) = a^N - e_A = a^N(e_A - a^{-N}) \in C.$$

Since also $e_A - a^N \in C$ and C is proper, we get $a^N - e_A = \theta_A$.

In case $N = 1$, we have $a = e_A$. In case $N \geq 2$, we write

$$a^N - e_A = (a - e_A)(a^{N-1} + \cdots + e_A) = \theta_A$$

and use the same argumentation as in the proof of Proposition 3.2 to see that $(a^{N-1} + \cdots + e_A)$ is invertible, which implies that $a - e_A = \theta_A$, i.e., $a = e_A$. $\square$

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