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AUTOMORPHISMS OF FILTERS: A SELECTION OF OPEN PROBLEMS

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Given a set X and a filter φ on X, a bijection $f : X \longrightarrow X$ is called an automorphism of φ if, for every subset $A \subseteq X$, $A \in \varphi$ if and only if $f(A) \in \varphi$. We select and discuss some open problems concerning automorphisms of filters on sets, groups and metric spaces.

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Для множини X та фільтра φ на X, біекція $f : X \longrightarrow X$ називається автоморфізмом φ, якщо підмножина $A \subseteq X$ належить фільтру φ тоді і лише тоді, коли $f(A) \in \varphi$. У статті обговорюються деякі відкриті проблеми, що стосуються автоморзмів фільтрів на множинах, групах і метричних просторах.

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1. Introduction

Let X be a set, S_X is the group of all permutations of X, φ and ψ be filters on X. We say that φ and ψ are *isomorphic* if there exists $g \in S_X$ such that, for every $A \subseteq X$,

$$A \in \varphi \Leftrightarrow g(A) \in \psi.$$

A class of all filters on X isomorphic to φ is called a *type* of φ.

Now we endow X with the discrete topology, identify the Stone-Čech compactification βX of X with the set of all ultrafilters on X, and X with the set of all principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters on X. Recall that a family $\{\bar{A} : A \subseteq X\}$ is a base for open sets in βX, where $\bar{A} = \{p \in \beta X : A \in p\}$. Given a filter φ on X, we set $\bar{\varphi} = \bigcap\{\bar{A} : A \in \varphi\}$ and note that, for every non-empty closed subset K of βX, there exists a filter φ on X such that $K = \bar{\varphi}$. We denote $\varphi^* = \bar{\varphi} \cap X^*$.

By the universal property of βX, every mapping $f : X \longrightarrow K$ from X to a compact Hausdorff space K can be uniquely extended to the mapping $f^\beta : \beta X \longrightarrow K$. By f^* we denote the restriction of f^β to X^*.

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2. Automorphisms

Given any $g \in S_X$ and a filter φ on X, we put

$$fix(g) = \{x \in X : g(x) = x\}, \ Fix(\varphi) = \{g \in S_X : fix(g) \in \varphi\},$$

observe that $Fix(\varphi)$ is an invariant subgroup of the group $Aut(\varphi)$ of all automorphisms of φ, and note that $Aut(\varphi)$ is a subgroup of the group $Homeo(\bar{\varphi})$ of all homeomorphisms of $\bar{\varphi}$. If φ is the Fréchet filter on $\omega = \{0, 1, \dots\}$ then $Homeo(\bar{\varphi})$ is the group of all autohomeomorphisms of ω^*. For open questions concerning this group see [4].

We define the reduced automorphism group $Aut^{\sim}(\varphi)$ as the quotient group $Aut'(\varphi)/Fix(\varphi)$ and note that $Aut^{\sim}(\varphi)$ can be identified with equivalence classes the relation $\sim$ on $Aut(\varphi)$ defined by: $g \sim h$ if and only if $fix(g^{-1}h) \in \varphi$. We note also that $Aut^{\sim}(\varphi)$ can be considered as a subgroup of $Homeo(\bar{\varphi})$: for each $A \in Aut^{\sim}(\varphi)$, we pick $g \in A$ and put $f(A) = g^{\beta}|_{\bar{\varphi}}$, $g^{\beta} : \beta X \longrightarrow \beta X$. Then f is an embedding of $Aut^{\sim}(\varphi)$ into $Homeo(\bar{\varphi})$.

If $\bar{\varphi}$ is finite, we partition $\bar{\varphi}$ into subsets $\Phi_1, \dots, \Phi_n$ of ultrafilters of the same type, and note that $Aut^{\sim}(\varphi)$ is isomorphic to $S_{\Phi_1} \times \dots \times S_{\Phi_n}$, $Homeo(\bar{\varphi})$ is a group of all permutations of $\bar{\varphi}$.

Question 1. Given a set X and a group G (a subgroup of S_X), how can one detect whether $G \simeq Aut^{\sim}(\varphi)$ ($G = Aut(\varphi)$) for an appropriate filter φ on X?

Question 2. For which filter φ on X, one can guarantee that $Aut^{\sim}(\varphi) = Homeo(\bar{\varphi})$?

We say that a filter φ on X is *rigid* if $Aut(\varphi) = Fix(\varphi)$.

Remark 1. 1. If $|\bigcap \varphi| > 1$ then φ is not rigid.

2. Each ultrafilter φ on X is rigid. Indeed, given a mapping $g : X \longrightarrow X$, by the 4-set lemma [2, p. 22], there is a partition

$$X = X_0 \bigcup X_1 \bigcup X_2 \bigcup X_3$$

such that $g|_{X_0} \equiv id$, $g(X_i) \bigcap X_i = \varnothing, i \in \{1, 2, 3\}$. Thus, if $g \notin Fix(\varphi)$ then $g \notin Aut(\varphi)$.

3. If all ultrafilters from $\bar{\varphi}$ are of distinct type, then φ is rigid.

4. Suppose that the set Φ_0 of all isolated points of $\bar{\varphi}$ is dense in $\bar{\varphi}$. Then φ is rigid if and only if all ultrafilters from Φ_0 are of distinct types.

5. We partition ω into infinite subsets $\omega = \bigcup_{i \in \omega} W_i$. For each $n > 0$, we pick $p_n \in \omega^*$ such that $W_n \in p_n$ and all ultrafilters $\{p_n : n > 0\}$ are of distinct types. We choose $q \in cl\{p_n : n > 0\} \setminus \{p_n : n > 0\}$ non-isomorphic to each p_n, $n > 0$, and take $p_0 \in \omega^*$, $W_0 \in p_0$ of type q. Let φ be a filter on ω such that $\bar{\varphi} = cl\{p_n : n \in \omega\}$. Then φ is rigid but $\bar{\varphi}$ has two ultrafilters p_0 and q of the same type. In this case $\bar{\varphi}$ is homeomorphic to $\beta\omega$. It is not hard to construct a rigid filter φ on ω such that $\bar{\varphi}$ is homeomorphic to ω^*.

6. For a filter φ on X, we set $\delta(\varphi) = \min\{|\Phi| : \Phi \in \varphi\}$ and denote by $\chi(\varphi)$ the minimal cardinality of a base for φ.

Recall that $\varphi' \subseteq \varphi$ is a *base* for φ if for every $\Phi \in \varphi$ there is $\Phi' \in \varphi'$ such that $\Phi' \subseteq \Phi$.

We assume that $\delta(\varphi) \geq \chi(\varphi) \geq \aleph_0$ and show that φ is not rigid. We choose a base $\{\Phi_\alpha : \alpha < \kappa\}$ of φ of cardinality $\kappa = \chi(\varphi)$. Since $\delta(\varphi) \geq \chi(\varphi) \geq \aleph_0$, we can choose inductively elements $\{x_\alpha, y_\alpha : \alpha < \kappa\}$ of X such that $x_\alpha, y_\alpha \in \Phi_\alpha$ and the subsets $\{x_\alpha : \alpha < \kappa\}$, $y_\alpha : \alpha < \kappa\}$ are disjoint. We define a permutation $g \in S_X$ by the rule: $g(x_\alpha) = y_\alpha$, $g(y_\alpha) = x_\alpha,\ \alpha < \kappa\}$ and $g(x) = x$ for each $x \in X \setminus \bigcup\{x_\alpha, y_\alpha : \alpha < \kappa\}$. By the construction, $g \in Aut(\varphi) \backslash Fix(\varphi)$ so φ is not rigid.

Question 3. Given a filter φ on ω, how can one recognize whether φ is rigid?

We say that a point x of a topological space X is *rigid* if the filter φ_x of neighborhoods of x is rigid. By Remark 1(6), a point x of a compact Hausdorff space X is rigid if and only if x is isolated. We say that X is *rigid* if each point of X is rigid.

Recall that a Hausdorff topological space X with no isolated points is *maximal* if X has an isolated point in any stronger topology on X. Equivalently, X is maximal if, for each $x \in X$, there is only one free ultrafilter converging to x. By Remark 1(2), each maximal space is rigid. It would be interesting to clarify a relationship between rigid spaces and well-known "extremal" topological spaces: submaximal, nodec, irresolvable, etc.

Let G be a group endowed with a topology in which the inversion $x \longmapsto x^{-1}$ is continuous at the identity e. If e is a rigid point then some member of φ_e must contain only elements of order 2. It follows that each rigid topological group contains an open Boolean subgroup. By [3, Theorem 11.3.4], an existence of a maximal topological group is consistent with ZFC.

Question 4. In ZFC, does there exist a non-discrete rigid topological group?

Question 5. Let $(G, \mathcal{T})$ be a topological group such that $\mathcal{T}$ is maximal in the class of all non-discrete regular topologies on G (see [3, Section 11.3]). Is $(G, \mathcal{T})$ rigid?

3. Local automorphisms

For a discrete group G, the Stone-Čech compactification G has a natural structure of a semigroup (see [5, Chapter 4]). Given $p, q \in \beta G$, the product pq is defined by

$$A \in pq \Longleftrightarrow \{q \in G : g^{-1}A \in q\} \in p.$$

The semigroup βG is right topological (for each $q \in \beta G$, the shift $x \longmapsto xq$ is continuous in βG) and G^* is a subsemigroup of βG.

By [7], each topological automorphism of G^* is internal, i.e. there is an automorphism h of G such that $f = g^*$. See also [3, Section 8.2] for more simple proof of this statement.

If an infinite Abelian group G admits a compact group topology then there exists a discontinuous automorphism of βG [6].

Question 6. Does there exist a discontinuous automorphism of $\beta\mathbb{Z}$? of $\mathbb{Z}^*$?

A group G endowed with a topology $\mathcal{T}$ is called left topological if each left shift $x \longmapsto gx$, $g \in G$ is continuous in $\mathcal{T}$. Each left invariant topology $\mathcal{T}$ on G is uniquely determined by the filter τ of neighborhoods, of the identity e, and $\bar{\tau}$ is a subsemigroup of βG.

Let $(G_1, \mathcal{T}_1)$, $(G_2, \mathcal{T}_2)$ be left topological groups. A mapping $f : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ is called a *local homomorphism* if $f(e_{G_1}) = e_{G_2}$ and, for each $x \in G_1$, there exist $U \in \tau_1$ such that $f(xy) = f(x)f(y)$ for each $y \in U$. If f is a bijection such that f and f^{-1} are local homomorphisms, f is called a *local isomorphism*.

By [10, Corollary 8.12], any two countable non-discrete regular left topological groups with countable bases of their topologies are locally isomorphic.

If f is a local automorphism of a left topological group $(G, \mathcal{T})$ then $f^{\beta} |_{\bar{\tau}}$ is a topological automorphism of the semigroup $\bar{\tau}$. On the other hand, if $(G, \mathcal{T})$ is countable non-discrete regular of countable weight and $f : G \longrightarrow G$ is a bijection such that $f^{\beta} |_{\bar{\tau}}$ is a topological automorphism then f is a local automorphism.

The next question has been posed by the first author at the conference "Automorphism Groups of Topological Structures"; Eilat, June 19-24, 2010.

Question 7. Let $(G, \mathcal{T})$ be a countable non-discrete regular left topological group of countable weight and let h be a topological automorphism of $\bar{\tau}$. Does there exist a local automorphism f of $(G, \mathcal{T})$ such that $h = f^{\beta} |_{\bar{\tau}}$?

4. Asymorphisms

For two metric spaces (X_1, d_1) and (X_2, d_2), a bijection $f : X_1 \longrightarrow X_2$ is said to be an *asymorphism* if there are two sequences $(c_n)_{n\in\omega}$ and $(c'_n)_{n\in\omega}$ in ω such that for each $n \in \omega$ and $x, y \in X_1$,

$$d_1(xy) \leq n \Longrightarrow d_2(f(x), f(y)) < c_n,$$

$$d_2(f(x), f(y)) \leq n \Longrightarrow d_1(x, y) < c_n.$$

These morphisms arouse in General Asymptopogy, see [8], [9].

For a metric space (X, d) we denote by $Asy(X, d)$ the group of all asymorphisms of (X, d) onto itself. As to our knowledge, these groups have not been considered at all.

Following [1], by the Cantor macro-cube we mean the set

$$2^{<\mathbb{N}} = \{(x_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} : \exists\, n \in \mathbb{N} \;\; \forall m > n \;\; x_m = 0\}$$

endowed with the ultrametric

$$d((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = max\{n \in \mathbb{N} : x_n \neq y_n\}.$$

Question 8. Which groups are embeddable into $Asy(2^{<\mathbb{N}})$? What about S_ω and $Homeo(\mathbb{Q})$?

Perhaps, instead of too large group $Asy(X)$ it is worth to define some its reduced version similar to $Aut^{\sim}(\varphi)$.

A subset Y of a metric space (X, d) is called *bounded* if $Y \subseteq B_d(x_0, r)$ for some $x_0 \in X$ and $r \in \mathbb{R}^+$, where $B_d(x_0, r) = \{y \in X : d(y, x_0) \leq r\}$.

Let (X, d) be an unbounded metric space. Denote by $X^\sharp$ the subset of βX consisting of all ultrafilters whose members are unbounded. Given two ultrafilters $p, q \in X^\sharp$, we write $p \parallel q$ if there exists $r \in \mathbb{R}^+$ such that $B_d(P, r) \in q$ for each $P \in p$. It is easy to see that $\parallel$ is an equivalence relation on $X^\sharp$. Following [8, Chapter 81], we denote by $\sim$ the smallest by inclusion closed in $X^\sharp \times X^\sharp$ equivalence on $X^\sharp$ such that $\parallel \subseteq \sim$. The quotient-space $\nu(X, d) = X^\sharp / \sim$ is called the *corona* of (X, d) and coincides with the Higson's corona if each bounded closed subset of X is compact.

Let f be an asymorphism of (X, d), p and q be two parallel ultrafilters from $X^\sharp$. Since $f^\beta(p) \parallel f^\beta(q)$, f induces a homeomorphism of $\nu(X, d)$.

For some open questions concerning homeomorphisms of a corona, see [2].

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