ASYMPTOTICAL SOLUTIONS TO SINGULARLY PERTURBED SYSTEMS OF DIFFERENTIAL EQUATIONS WITH DEGENERATIONS AND IMPULSES

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In present paper we consider a problem of constructing of asymptotical solutions to a singularly perturbed non-linear system with degenerations and pulses at the fixed moments of time. Here we propose an algorithm of constructing of asymptotical approximation to the solutions and prove a theorem on asymptotical estimation for constructed m-approximation of this solution.

INTRODUCTION

Study of differential equations with degenerative matrix at derivatives is an actual and important problem of the up-to-date theory of differential equations [6, 9, 12, 15, 16]. Different aspects of the problem were considered in a number of papers. Indeed, in [6] there were investigated the linear singular perturbed differential equations with degenerations by means of method of transformation method of two matrices to a canonical form. In [12] it was proposed an operator method for analysis of the same problem. Here it would be mentioned that the main attention in papers devoted to the studying of degenerative singular perturbed differential equations was paid to development of different algorithms of constructing of (formal) asymptotical solutions of such differential equations and to their decomposition [7].

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The papers [8, 11] deal with a problem of constructing of asymptotical solutions for linear degenerative singular perturbed systems of differential equations with pulses [2] and establishment of asymptotical estimation for the constructed approximative (asymptotical) solution. Similar problems have a great practical significance because such differential equations appear as mathematical models in case when different physical processes and phenomenon are examined [13, 17]. It should be noted, that by studying systems of differential equations with pulses we have necessarily to take into account that such systems are essentially non-linear even in a case of linear differential equations and have some so-called specific properties caused by presents of pulse conditions [3, 14].

This paper deals with a problem of constructing of asymptotical solutions for the non-linear degenerative singular perturbed system of differential equations with pulses at the fixed moments of time. We develop an algorithm of constructing of asymptotic solution on the basis of methods [10, 18] and establish an estimation for constructed m-approximated solution.

FORMULATION OF PROBLEM AND MAIN ASSUMPTIONS

We consider a singularly perturbed non-linear system

$$\varepsilon B(t)\frac{dx}{dt} = f(t,x) \tag{1}$$

with pulses

$$\Delta x|_{t=t_j} = x(t_j+0) - x(t_j-0) = I_j(x(t_j-0)), \quad j \in \mathbb{N}, \tag{2}$$

at the fixed moments of time t_j, $j \in \mathbb{N}$, where $t_{j+1} - t_j \geq \delta > 0$, $j \in \mathbb{N}$, $t_1 > t_0$, and δ is some number. Here $x(t,\varepsilon)$ and $f(t,x)$ are n-dimension vectors; $B(t)$ is the degenerated matrix of order $n \times n$. The problem (1), (2) is supposed to have a solution.

By a solution of the problem (1), (2) we mean a infinity differentiable vector-function $x(t,\varepsilon)$ (for all $t \neq t_j$, $t \in \mathbb{N}$) that has the first type break at the points of pulses t_j, $j \in N$, and is left-hand continuous at these points.

Suppose that the following assumptions holds:

1^0. Elements of matrix $B(t)$ are infinitely differentiable for all $t \in [t_0, T]$.

2^0. The vector-function $f(x,t)$ has continuous partial derivations for all arguments of any power on every intervals $[t_j, t_{j+1})$, $j \in \mathbb{N}$.

4^0. The determinant $\det B(t) \equiv 0$ for all $t \in [t_0; T]$.

5^0. The unperturbed problem (equation (1) when $\varepsilon = 0$, i.e.

$$f(t, z) = 0 \tag{3}$$

under condition (2)) has a solution $z = \varphi(t)$, which is the infinitely differentiable vector-function everywhere except points t_j, $j \in \mathbb{N}$.

6^0. The determinant $\Psi(t) = \left(\frac{\partial f}{\partial x}\right)|_{x=\varphi(t)}$ is not zero.

7^0. The vector-functions $I_j(x)$, $j \in \mathbb{N}$, are infinitely differentiable for all $t \in [t_0, T]$.

8^0. The matrix $\Phi_j(t) = \left(\frac{\partial I_j}{\partial x}\right)|_{x=\varphi(t)}$, $j \in \mathbb{N}$, satisfies the following condition:

$$\det(\Phi_j(t_j) - E) \neq 0, \quad j \in \mathbb{N}, \quad \text{for} \quad \text{all} \quad t \in [t_0; T].$$

9^0. The bundle of matrices $L(t, \lambda) = \Psi(t) - \lambda B(t)$ has an unique s-multiple „finite“ elementary divisor and an unique p-multiple „infinite“ elementary divisor (for all $t \in [t_0, T]$) and $s + p = n$.

Let us develop an algorithm for constructing of approximative (asymptotical) solution of the problem (1), (2) and give its justification.

CONSTRUCTION OF ASYMPTOTICAL SOLUTION

The solution of the problem (1), (2) we represent as sum of two vectors

$$x(t, \varepsilon) = u(t, \varepsilon) + \Pi x(t, \tau, \varepsilon), \tag{4}$$

every of which is supposed to be realized by asymptotical series on small parameter ε as follows:

$$u(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(t), \tag{5}$$

$$\Pi x(\tau, \varepsilon) = \sum_{t_j \leq t} \Pi x(\tau_j, \varepsilon), \quad \tau = (\tau_1, \tau_2, \tau_3, \ldots), \tag{6}$$

where

$$\Pi x(\tau_j, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \Pi_k x(\tau_j), \quad \tau_j = \frac{t - t_j}{\varepsilon}, \quad j \in \mathbb{N}. \tag{7}$$

Series (5) are the regular part and series (6) are the singular part of the asymptotical solution (4). Boundary vector-functions $\Pi_k x(\tau_j)$, $k = 0, 1, \ldots$, are supposed to be defined for $\tau_j \geq 0$ for all $j \in \mathbb{N}$.

Firstly we describe an algorithm of constructing of the asymptotical solution (4). Substituting the expression (4) into equation (1) we have

$$\varepsilon B(t)\left(\frac{du(t,\varepsilon)}{dt}+\varepsilon^{-1}\frac{d\Pi x(t,\tau,\varepsilon)}{d\tau}\right)=f(t,u(t,\varepsilon)+\Pi x(t,\tau,\varepsilon)).$$

Without restriction of generality we will consider the given system of differential equations at any point $t_j \in \{t_1, t_2, t_3, \ldots\}$. Terms of the singular and the regular parts of asymptotic are defined as solutions of the following systems of differential equations:

$$\varepsilon B(t)\frac{du(t,\varepsilon)}{dt}=f(t,u(t,\varepsilon)), \tag{8}$$

$$B(t)\frac{d\Pi x(\tau_j,\varepsilon)}{d\tau_j}=\Pi F(\tau,\varepsilon), \tag{9}$$

where $\Pi F(\tau,\varepsilon)=f(t,u(t,\varepsilon)+\Pi x(\tau,\varepsilon))-f(t,u(t,\varepsilon))$.

Let us decompose vector-functions $f(t,u(t,\varepsilon))$ and $\Pi F(\tau_j,\varepsilon)$ into series on small parameter ε. We have:

$$f(t,u(t,\varepsilon))=f(t,u_0(t))+\sum_{k=1}^{m}\varepsilon^k[\Psi(t)u_k(t)+G_k(t)]+O(\varepsilon^{m+1}), \tag{10}$$

$$\Pi F(\tau_j,\varepsilon)=f(t_j,u_0(t_j)+\Pi_0 x(\tau_j))-f(t_j,u_0(t_j))+$$
$$+\sum_{k=1}^{m}\varepsilon^k[\Psi(\tau_j)\Pi_k x(\tau_j)+F_k(\tau_j)]+O(\varepsilon^{m+1}). \tag{11}$$

where $\Psi(t)$ denotes the Jacobi matrix $\left(\frac{\partial f}{\partial u}\right)$ calculated at the point $(t,u_0(t))$; $\Psi(\tau_j)$ denotes the Jacobi matrix $\left(\frac{\partial f}{\partial u}\right)$ calculated at the point $(t_j,u_0(t_j)+\Pi_0 x(\tau_j))$. Vector-functions $G_k(t)$, $F_k(\tau_j)$ in (10), (11) are infinity differentiable functions and can be recursively determined by $u_i(t)$, $\Pi_i x(\tau_j)$, $i=1,\ldots,k-1$.

Examine the system (8). At $\varepsilon=0$ we obtain equation $f(t,u_0(t))=0$ that by assumption 5^0 has a unique solution

$$u_0(t)=\varphi(t). \tag{12}$$

By equating coefficients at the same powers of ε in both parts of (8) we obtain terms of regular part of asymptotic in exact form:

$$u_k(t)=\Psi^{-1}(t)\left(B(t)\frac{d}{dt}u_{k-1}(t)-G_k(t)\right),\quad k=0,1,\ldots. \tag{13}$$

Thus, the regular part of asymptotic (4) is found.

Let us describe an algorithm of constructing of boundary functions $\Pi x(\tau_j, \varepsilon)$, $j \in \mathbb{N}$. Firstly we equate coefficients at the same powers of small parameter ε in both parts of relation (9).

For ε^0 we have

$$B(t)\frac{d\Pi_0 x(\tau_j)}{d\tau_j} = f(t_j, u_0(t_j) + \Pi_0 x(\tau_j)) - f(t_j, u_0(t_j)), \quad j \in \mathbb{N}. \tag{14}$$

It is easy to see, that the equation (14) has only a trivial solution $\Pi_0 x(\tau_j) = 0$. From (9), by equating the coefficients at ε^k and taking into consideration (11) we obtain:

$$B(t)\frac{d\Pi_k x(\tau_j)}{d\tau_j} = \Psi(\tau_j)\Pi_k x(\tau_j) + T_k(\tau_j), \tag{15}$$

where

$$T_k(\tau_j) = F_k(\tau_j) - B(t)\frac{d\Pi_{k-1} x(\tau_j)}{d\tau_j}, \quad k \in \mathbb{N}.$$

The system of equations (15) is degenerative, because $\det B(t) \equiv 0$ for all $t \in [t_0; T]$. Therefore, some additional conditions must be fulfilled to obtain the system (15) as well-posed problem.

By assumption 9^0, bundle of matrices $L(t, \lambda) = \Psi(t) - \lambda B(t)$ has an unique s-multiple „finite" elementary divisor and unique p-multiple „infinite" elementary divisor for all $t \in [t_0, T]$. In this case due to [1, 4, 12, 15] there exist non-degenerative matrices $P(t, \tau_j)$ and $Q(t, \tau_j)$, which transform matrix $L(t, \tau_j, \lambda)$ to a diagonal matrix of the following form:

$$P(t, \tau_j)(\Psi(\tau_j) - \lambda B(t))Q(t, \tau_j) = S(t, \tau_j) - \lambda H, \tag{16}$$

where $S(t, \tau_j) = \mathrm{diag}(M_s(t, \tau_j), E_p), H = \mathrm{diag}(E_s, J_p)$. Here $M_s(t, \tau_j)$ is some $(s \times s)$-matrix, which can be obtained in exact form; E_s, E_p are the unit matrices which have dimension s and p correspondingly; J_p is the p-dimensional Jordan matrix. Fulfilling a substitution

$$\Pi_k x(\tau_j) = Q(t, \tau_j)\Pi_k y(\tau_j) \tag{17}$$

and decomposing vector $\Pi_k y(\tau_j)$ according to structure of matrices $S(t, \tau_j)$ and H, i.e. applying representation $\Pi_k y(\tau_j) = \mathrm{colon}(\Pi_k y_1(\tau_j), \Pi_k y_2(\tau_j))$, we obtain from the system of differential equations (15) the following system:

$$\frac{d\Pi_k y_1(\tau_j)}{d\tau_j} = M_s(t, \tau_j)\Pi_k y_1(\tau_j) + R_{k,1}(t, \tau_j), \tag{18}$$

$$J_p \frac{d\Pi_k y_2(\tau_j)}{d\tau_j} = \Pi_k y_2(\tau_j) + R_{k,2}(t,\tau_j), \tag{19}$$

where $R_k(t,\tau_j) = P(t,\tau_j)T_k(\tau_j) = \mathrm{colon}(R_{k,1}(t,\tau_j), R_{k,2}(t,\tau_j))$, $k \in \mathbb{N}$, vectors $R_{k,1}(t,\tau_j)$ and $R_{k,2}(t,\tau_j)$ have s and $(n-s)$ dimension respectively.

Let us study a question on solvability of an algebra-differential system (18), (19). Firstly we consider the system (19). In respect that vectors $\Pi_k y_2(\tau_j)$ and $R_{k,2}(t,\tau_j)$ have the following form

$$\Pi_k y_2(\tau_j) = \mathrm{colon}(\Pi_k y_{s+1}(\tau_j), \Pi_k y_{s+2}(\tau_j), \dots, \Pi_k y_n(\tau_j)),$$

$$R_{k,2}(t,\tau_j) = \mathrm{colon}(r_{k,s+1}(t,\tau_j), r_{k,s+2}(t,\tau_j), \dots, r_{k,n}(t,\tau_j)),$$

and according to structure of matrix J_p, we can write algebraic-differential system (19) in the coordinate form as follow:

$$\frac{d}{d\tau_j}\begin{pmatrix} \Pi_k y_{s+2}(\tau_j) \\ \Pi_k y_{s+3}(\tau_j) \\ \dots \\ \Pi_k y_n(\tau_j) \\ 0 \end{pmatrix} = \begin{pmatrix} \Pi_k y_{s+1}(\tau_j) \\ \Pi_k y_{s+2}(\tau_j) \\ \dots \\ \Pi_k y_{n-1}(\tau_j) \\ \Pi_k y_n(\tau_j) \end{pmatrix} + \begin{pmatrix} r_{k,s+1}(t,\tau_j) \\ r_{k,s+2}(t,\tau_j) \\ \dots \\ r_{k,n-1}(t,\tau_j) \\ r_{k,n}(t,\tau_j) \end{pmatrix}. \tag{20}$$

The solution of system (20) can be easily got by recursion. Indeed, solving the last equation of the system (20) we obtain:

$$\Pi_k y_n(\tau_j) = -r_{k,n}(t,\tau_j). \tag{21}$$

From the other equations that are algebraic in respect to unknown functions we get:

$$\Pi_k y_{s+i}(\tau_j) = -r_{k,s+i}(t,\tau_j) - \sum_{k=1}^{p-i} \frac{d^k}{d\tau_j^k} r_{k,s+i+k}(t,\tau_j), \quad i = 1,\dots,p-1. \tag{22}$$

Thus, the problem of constructing boundary functions $\Pi_k y_2(\tau_j)$, $k \in \mathbb{N}$, is solved.

Since the system (18) is a linear system of s ordinary differential equations in respect to s unknown functions, matrix $M_s(t,\tau_j)$ and vectors $R_{k,1}(t,\tau_j)$ are continuous for all $t \in [t_j, t_{j+1})$, $j \in \mathbb{N}$, we can state that the system (18) under given initial data has the unique solution for all $t \in [t_j, t_{j+1})$, $j \in \mathbb{N}$.

In order to find the unique solution of the system (18) we need to determine its initial data. Here we describe a procedure of definition of initial data for the system (18). We can calculate initial values $\Pi_k y_1(0)$, $k \in \mathbb{N}$, for system

(18) using formula (17). For that we find firstly initial data $\Pi_k x(0)$, $k \in \mathbb{N}$, for the system (15).

By expanding vector-functions $I_j(x(t,\varepsilon))$ into series on small parameter ε we obtain:

$$I_j(x(t,\varepsilon)) = \sum_{k=0}^{\infty} \varepsilon^k [\Phi_j(t_j)(u_k(t) + \Pi_k x(\tau_j)) + J_k(t,\tau_j)], \quad j \in \mathbb{N}, \qquad (23)$$

where matrix $\Phi_j(t_j) = (\frac{\partial I_j}{\partial u})$ is calculated at points $(t_j, u_0(t_j))$, $j \in \mathbb{N}$; vectors $J_k(t,\tau_j)$ are determined recursively by vectors $u_i(t)$ and $\Pi_i x(\tau_j)$, $i = 0, \ldots, k-1$. In particular, $J_1(t,\tau_j) \equiv 0$.

Taking into consideration condition (2) and formula (23), for system of differential equations (15) we find (when $k = 1$):

$$\Delta x|_{t=t_j} + \Pi_1 x(0) = \Phi_j(t_j)[u_1(t_j) + \Pi_1 x(0)].$$

Whence we obtain initial data for solution $\Pi_1 x(\tau_j)$ in the following form

$$\Pi_1 x(0) = (\Phi_j(t_j) - E)^{-1}[\Delta x|_{t=t_j} - \Phi_j(t_j) u_1(t_j)], \quad j \in \mathbb{N}. \qquad (24)$$

Analogously to previous step we find initial values $\Pi_k x(0)$ when $k = 2, 3, \ldots$. We have:

$$\Pi_k x(0) = (\Phi_j(t_j) - E)^{-1}[\Delta x|_{t=t_j} - \Phi_j(t_j) u_k(t_j) - J_k(t_j, 0)]. \qquad (25)$$

To determine initial values $\Pi_k y_1(0)$, $k \in \mathbb{N}$, we use conditions (24), (25) and formula (17). With this mater we decompose vector $\Pi_k x(0)$ and matrix $Q(t,\tau_j)$ in accordance to structure of vector $\Pi_k y(\tau_j)$. Thus we obtain:

$$\Pi_k x(0) = \operatorname{colon}(\Pi_k x_1(0), \Pi_k x_2(0)) \qquad (26)$$

and

$$Q(t,\tau_j) = \begin{pmatrix} Q_1(t,\tau_j) & Q_2(t,\tau_j) \\ Q_3(t,\tau_j) & Q_4(t,\tau_j) \end{pmatrix}, \qquad (27)$$

where vectors $\Pi_k x_1(0)$ and $\Pi_k x_2(0)$ have s and $n-s$ dimension respectively, and matrices $Q_1(t,\tau_j)$, $Q_2(t,\tau_j)$, $Q_3(t,\tau_j)$, $Q_4(t,\tau_j)$ have dimensions $s \times s, (n-s) \times s, s \times (n-s), (n-s) \times (n-s)$ respectively.

Then, from (17) for $t = t_j$ we have system of equations

$$\Pi_k x_1(0) = Q_1(t_j, 0)\Pi_k y_1(0) + Q_2(t_j, 0)\Pi_k y_2(0),$$
$$\Pi_k x_2(0) = Q_3(t_j, 0)\Pi_k y_1(0) + Q_4(t_j, 0)\Pi_k y_2(0),$$

whence we find initial values $\Pi_k y_1(0)$, $k \in N$. As a result we obtain:

$$\Pi_k y_1(0) = (Q_1 - Q_2 Q_4^{-1} Q_3)^{-1} [\Pi_k x_1(0) - Q_2 Q_4^{-1} \Pi_k x_2(0)]. \tag{28}$$

Thus, terms $u_k(t)$, $k = 0, 1, \ldots$, of the regular part of asymptotic expansion (4) are determined by formulas (12), (13), and terms $\Pi_k x(\tau_j)$, $k = 0, 1, \ldots$, of the singular part of the asymptotic expansion (4) are determined by formulas (21), (22) and formulas for solution of Cauchy problem (18), (28).

Thus the problem of constructing of asymptotic solution to the problem (1), (2) is solved.

BOUNDARY FUNCTIONS ESTIMATION

Let us show that $\Pi_k x(\tau_j)$, $k = 0, 1, \ldots$, are indeed boundary functions (under some special conditions).

Lemma 1. *If eigen-values $\lambda_i(t, \tau_j)$, $i = 1, \ldots, s$, of matrix $M_s(t, \tau_j)$ satisfy the following condition*

$$\operatorname{Re} \lambda_i(t, \tau_j) \le -\gamma < 0, \quad i = 1, \ldots, s, \tag{29}$$

then there exist such constants $C_k > 0$, $\alpha_k > 0$, that functions $\Pi_k x(\tau_j)$ satisfy inequality

$$\|\Pi_k x(\tau_j)\| \le C_k e^{-\alpha_k \tau_j}, \quad k = 0, 1, \ldots, \tag{30}$$

for $\tau_j > 0$.

Proof. By (17) vector-functions $\Pi_k x(\tau_j)$, $k = 0, 1, \ldots$, are composed from vectors $\Pi_k y_1(\tau_j)$ and $\Pi_k y_2(\tau_j)$, $k = 0, 1, \ldots$. Therefore, formulas

$$\|\Pi_k y_1(\tau_j)\| \le M_{k,1} e^{-\beta_k \tau_j}, \quad \tau_j > 0$$

and

$$\|\Pi_k y_2(\tau_j)\| \le M_{k,2} e^{-\omega_k \tau_j}, \quad \tau_j > 0,$$

with some positive values $M_{k,1}$, $M_{k,2}$, β_k, ω_k, $k \in \mathbb{N}$, directly imply the formula (30).

It is easy to see that vector-function $\Pi_0 x(\tau_j) \equiv 0$ satisfies the inequality (30). Therefore, vector-functions $\Pi_0 y_1(\tau_j) \equiv 0$ and $\Pi_0 y_2(\tau_j) \equiv 0$ are also satisfied the inequality (30).

Let us prove the validity of inequality (30) using the method of mathematic induction. Examine the system of equation (18) when $k = 1$. We have:

$$\frac{d\Pi_1 y_1(\tau_j)}{d\tau_j} = M_s(t, \tau_j) \Pi_1 y_1(\tau_j) + R_{1,1}(t, \tau_j). \tag{31}$$

Assumption 2^0 implies that matrix $M_s(t,\tau_j)$ and vector $R_1(t,\tau_j)$ are continuous in some neighborhood of the point $(t_j, \Pi_k y_1(0))$. Therefore, system (31) satisfies conditions of the Cauchy theorem of existences and uniqueness of solution of ordinary differential equations.

Let by $X(\tau_j)$ be a fundamental matrix of system (31). This matrix satisfies the homogeneous matrix differential equation

$$\frac{d(\tau_j)}{d\tau_j} = M_s(t,\tau_j)X(\tau_j). \tag{32}$$

Solution of system (31) can be represented as follows

$$\Pi_1 y_1(\tau_j) = X(\tau_j)X^{-1}(0)\Pi_1 y_1(0) + \int_0^{\tau_j} X(\tau_j)X^{-1}(s)R_{1,1}(t,s)ds\,, \tag{33}$$

where $\Pi_1 y_1(0)$ are initial values, which can be found from formula (28) when $k = 1$. If eigen-values $\lambda_i(t,\tau_j)$, $i = 1,\dots,s$, of matrix $M_s(t,\tau_j)$ in (18) satisfies the inequality (29) then due to [18, p. 69] the following inequalities for matrix $X(\tau_j)$ are hold

$$\|X(\tau_j)\| \le Le^{-\gamma\tau_j}, \quad \gamma > 0, \quad \tau_j > 0, \tag{34}$$

$$\|X(\tau_j)X^{-1}(s)\| \le Le^{-\gamma(\tau_j - s)}, \quad \gamma > 0, \quad \tau_j > 0, \tag{35}$$

where $L > 0$ is a constant.

Remark that the method [1, 4, 5, 16] of constructing of matrices $P(t,\tau_j)$ and $Q(t,\tau_j)$ provides their continuously. Therefore these matrices have bounded norms for all $t \in [t_0;T]$, i.e.,

$$\|P(t,\tau_j)\| \le M_1, \quad \|Q(t,\tau_j)\| \le M_2,$$

where $M_1 > 0$ and $M_2 > 0$ are some constants. Therefore, for the vector-function $R_1(t,\tau_j)$ we have the following estimation:

$$\|R_1(t,\tau_j)\| \le \|P(t,\tau_j)F_1(\tau_j)\| \le Ke^{-\omega_1\tau_j}, \quad \tau_j > 0,$$

where K, ω_1 are some positive constants.

Boundedness of vector norms $F_k(\tau_j)$, $k \in N$, follows by assumption 2^0 and equality (11). In respect to structure of vector

$$R_1(t,\tau_j) = (R_{1,1}(t,\tau_j), R_{1,2}(t,\tau_j)),$$

we obtain, that for the vectors $R_{1,1}(t,\tau_j)$ and $R_{1,2}(t,\tau_j)$ are valid analogical inequalities

$$\|R_{1,1}(t,\tau_j)\| \le K_1 e^{-\omega_1\tau_j}, \quad \|R_{1,2}(t,\tau_j)\| \le K_2 e^{-\omega_1\tau_j}, \quad \tau_j > 0, \tag{36}$$

where ω_1, K_1, K_2 are certain positive constants.

Then, according to inequalities (34)–(36), for the function (33) we find:

$$\|\Pi_1 y_1(\tau_j)\| \le \|X(\tau_j)X^{-1}(0)\Pi_1 y_1(0)\| + \left\| \int_0^{\tau_j} X(\tau_j)X^{-1}(s)R_{1,1}(t,s)ds \right\| \le$$

$$\le LC_0 e^{-\gamma\,\tau_j} + LK_1 e^{-\gamma\,\tau_j} \int_0^{\tau_j} e^{s(\gamma-\omega_1)} ds \le$$

$$\le LC_0 e^{-\gamma\,\tau_j} + \frac{LK_1}{\gamma-\omega_1} e^{-\gamma\,\tau_j} + \frac{LK_1}{\gamma-\omega_1} e^{-\omega_1\tau_j}, \quad \gamma > \omega_1.$$

Here symbol C_0 denotes norm of vector $\Pi_1 y_1(0)$. Boundedness of the vector $\Pi_1 y_1(0)$ follows from assumption 7^0 and equality (28).

Thus, we obtain the following estimation for the vector $\Pi_1 y_1(\tau_j)$:

$$\|\Pi_1 y_1(\tau_j)\| \le C_{1,1} e^{-\beta_1 \tau_j}, \quad \beta_1 > 0, \quad \tau_j > 0, \quad j \in \mathbb{N}, \tag{37}$$

where $C_{1,1} = \max\{LC_0, LK_1(\gamma-\omega_1)^{-1}\}$, β_1 are some constants.

Taking into account equalities (21), (22), by means of which vector-functions $\Pi_1 y_2(\tau_j)$ are determined, we obtain:

$$\|\Pi_1 y_2(\tau_j)\| = \max_{1\le i\le n-s-1} \left| r_{1,s+i}(t,\tau_j) + \sum_{k=1}^{n-s-i} \frac{d^k}{d\tau_j^k} r_{1,s+i+k}(t,\tau_j) \right|,$$

where $r_{2,l}(t,\tau_j)$, $l = s+1,\dots,n$, are coordinates of vector $R_{1,2}(t,\tau_j)$. From the second inequality in (36), for $\Pi_1 y_2(\tau_j)$ we have the following estimation:

$$\|\Pi_1 y_2(\tau_j)\| \le K_2 e^{-\omega_1\tau_j} \sum_{k=0}^{n-s-i} \omega_1^k \le C_{2,1} e^{-\omega_1\tau_j},$$

where

$$C_{2,1} = K_2 \sum_{k=0}^{n-s-1} \omega_1^k, \quad \omega_1 > 0.$$

Thus, taking into account to the given inequalities for the vector-functions $\Pi_1 y_1(\tau_j)$, $\Pi_1 y_2(\tau_j)$, and estimation $\|Q(t,\tau_j)\| \le M_2$, we obtain the following inequality for vector-function $\Pi_1 x(\tau_j)$:

$$\|\Pi_1 x(\tau_j)\| \le C_1 e^{-\alpha_1\tau_j}, \quad \tau_j > 0,$$

where $C_1 = \max(M_2 C_{1,1}; M_2 C_{2,1})$, $\alpha_1 = \min(\beta_1; \omega_1)$.

At the second step, by $k = 2$, for functions $\Pi_2 y_1(\tau_j)$ we have the following integral equation:

$$\Pi_2 y_1(\tau_j) = X(\tau_j)X^{-1}(0)\Pi_2 y_1(0) + \int_0^{\tau_j} X(\tau_j)X^{-1}(s)R_{2,1}(t,s)ds \,,$$

where

$$R_2(t,\tau_j) = P(t,\tau_j)\left(F_2(\tau_j) - B_1(t)\frac{d}{d\tau_j}\Pi_1 x(\tau_j)\right).$$

Let us estimate vector $R_2(t,\tau_j)$. We have:

$$\|R_2(t,\tau_j)\| \le \|P(t,\tau_j)\|\left(\|F_2(\tau_j)\| + \|B_1(t)\| \cdot \left\|\frac{d}{d\tau_j}\Pi_1 x(\tau_j)\right\|\right) \le$$

$$\le \widetilde{c}_1 e^{-\omega_1 \tau_j} + \widetilde{c}_2 \alpha_1 e^{-\alpha_1 \tau_j} \le \widetilde{c} e^{-\omega_2 \tau_j},$$

where $\widetilde{c}$, $\widetilde{c}_1$, $\widetilde{c}_2$, ω_2 are some positive constants. Boundedness of matrix $B_1(t)$ follows from assumption 1^0. Thus, analogously to mentioned above estimation for vectors $\Pi_1 y_1(\tau_j)$ and $\Pi_1 y_2(\tau_j)$, we can obtain the inequalities for vectors $\Pi_2 y_1(\tau_j)$ and $\Pi_2 y_2(\tau_j)$ in the following form:

$$\|\Pi_2 y_1(\tau_j)\| \le C_{2,1}\, e^{-\beta_2 \tau_j}, \quad \beta_2 > 0, \quad \tau_j > 0,$$

$$\|\Pi_2 y_2(\tau_j)\| \le C_{2,2}\, e^{-\omega_2 \tau_j}, \quad \omega_2 > 0, \quad \tau_j > 0.$$

Thus,

$$\|\Pi_2 x(\tau_j)\| \le C_2 e^{-\alpha_2 \tau_j}, \quad \alpha_2 > 0, \quad \tau_j > 0.$$

Here C_2, $C_{2,1}$, $C_{2,2}$ are some positive constants.

Assume that inequality (37) holds for function $\Pi_k y_1(\tau_j)$ for all $k \in \mathbb{N}$. Let us show that this inequality holds also for $\Pi_{k+1} y_1(\tau_j)$.

Let us replace the system of differential equations (18) for functions $\Pi_{k+1} y_1(\tau_j)$ by equivalent system of integral equations:

$$\Pi_{k+1} y_1(\tau_j) = -X(\tau_j)X^{-1}(0)\Pi_{k+1} y_1(0) + \int_0^{\tau_j} X(\tau_j)X^{-1}(s)R_{k+1,1}(t,s)ds,$$

where

$$\Pi_{k+1} y_1(0) = (Q_1 - Q_2 Q_4^{-1} Q_3)^{-1}[\Pi_{k+1} x_1(0) - Q_2 Q_4^{-1}\Pi_{k+1} x_2(0)],$$

and vectors $\Pi_{k+1} x_1(0)$ and $\Pi_{k+1} x_2(0)$ are defined according to formula (25).

Functions $F_{k+1}(\tau_j)$, as mentioned above polynomially depend on $\Pi_i x(\tau_j)$, $i = 1, \dots, k$. Therefore for norm of vector $R_{k+1}(t, \tau_j)$ we have the following estimation:

$$\|R_{k+1}(t,\tau_j)\| \le \|P(t,\tau_j)\| \left(\|F_{k+1}(\tau_j)\| + \left\| \sum_{i=1}^{k+1} B_i(t) \frac{d\Pi_{k+1-i}x(\tau_j)}{d\tau_j} \right\| \right) \le$$

$$\le \|P(t,\tau_j)\| \left(\|F_{k+1}(\tau_j)\| + \sum_{i=1}^{k+1} \|B_i(t)\| \cdot \left\| \frac{d\Pi_{k+1-i}x(\tau_j)}{d\tau_j} \right\| \right) \le$$

$$\le \widetilde{c}e^{-\alpha_{k+1}\tau_j} + \widetilde{C}_k \sum_{i=1}^{k+1} e^{-\alpha_i \tau_j},$$

where $\widetilde{C}_k = c_1\alpha_k + c_2\alpha_{k-1} + \dots + c_k\alpha_1$.

Hence we obtain the inequality

$$\|R_{k+1}(t,\tau_j)\| \le c_{k+1}e^{-\omega_{k+1}\tau_j}, \quad \omega_{k+1} > 0, \quad \tau_j > 0, \tag{38}$$

where c_{k+1}, $\omega_{k+1} > 0$ are some constants.

So, taking into account inequalities (34), (35) and (38), we have the following inequalities for functions $\Pi_k y_1(\tau_j)$, $k \in \mathbb{N}$:

$$\|\Pi_{k+1}y_1(\tau_j)\| \le C_{k+1}e^{-\beta_{k+1}\tau_j}, \quad \beta_{k+1} > 0, \quad \tau_j > 0, \tag{39}$$

where $\beta_{k+1} > 0$ is constant. Thus, by method of mathematical induction, the following inequality is valid

$$\|\Pi_k y_1(\tau_j)\| \le C_k e^{-\beta_k \tau_j}, \quad \tau_j > 0, \quad j \in \mathbb{N}, \quad k \in \mathbb{N}, \tag{40}$$

where $C_k > 0$, $k \in N$, are some constants.

Let us establish an estimation for vector-functions $\Pi_k y_2(\tau_j)$, $k \in \mathbb{N}$. Using equalities (21), (22) for norm of vector-function $\Pi_k y_2(\tau_j)$, $k \in \mathbb{N}$, we obtain:

$$\|\Pi_k y_2(\tau_j)\| = \max_{1 \le i \le n-s-1} \left| r_{k,s+i}(t,\tau_j) + \sum_{k=1}^{n-s-i} \frac{d^k}{d\tau_j^k} r_{k,s+i+k}(t,\tau_j) \right|, \quad k \in \mathbb{N}.$$

So, according to (38), we get:

$$\|\Pi_k y_2(\tau_j)\| \le c_k e^{-\omega_k \tau_j}, \quad \omega_k > 0, \quad \tau_j > 0, \quad k,j \in \mathbb{N}. \tag{41}$$

Thus, taking into account estimations (39), (41) for vector-functions $\Pi_k y_1(\tau_j)$ and $\Pi_k y_2(\tau_j)$, $k,j \in \mathbb{N}$, we conclude that for vector-functions

$\Pi_k y(\tau_j) = \mathrm{colon}(\Pi_k y_1(\tau_j), \Pi_k y_2(\tau_j))$, $k, j \in \mathbb{N}$, the following inequality is valid

$$\|\Pi_k y(\tau_j)\| \le \widetilde{C}_k e^{-\alpha_k \tau_j}, \quad \alpha_k > 0, \quad \tau_j > 0, \quad k, j \in \mathbb{N}.$$

where $\widetilde{C}_k > 0$ is constant.

Finally, by formula (17) we obtain

$$\|\Pi_k x(\tau_j)\| \le C_k e^{-\alpha_k \tau_j}, \quad \alpha_k > 0, \quad \tau_j > 0, \quad k, j \in \mathbb{N}.$$

Thus, inequality (30) is proved for all $k \in \mathbb{N}$. Lemma 1 is completed.

ASYMPTOTIC ESTIMATION FOR SOLUTION OF THE PROBLEM (1), (2)

Let us show that constructed above asymptotical solution of problem (1), (2) is the asymptotical expansion for exact solution for all $t \in [t_0, T]$.

Let us denote

$$x_m(t, \varepsilon) = \sum_{k=0}^{m} \varepsilon^k \left(u_k(t) + \sum_{t \le t_j} \Pi_k x(\tau_j) \right). \tag{42}$$

Theorem. *Let the assumptions* 1^0 – 10^0 *and conditions of lemma 1 be true. Then series (4) are the asymptotical series for solution* $x(t, \varepsilon)$ *of the problem (1), (2) for all* $t \in [t_0; T]$, *i.e., for all* $m = 0, 1, \dots,$ *there exist such constants* M_m, *that the following estimation is true*

$$\|x(t, \varepsilon) - X_m(t, \varepsilon)\| \le M_m \varepsilon^m, \quad t \in [t_0, T]. \tag{43}$$

Proof. Denote

$$z(t, \varepsilon) = x(t, \varepsilon) - x_m(t, \varepsilon). \tag{44}$$

At first let us clarify the order of exactness by which function $x_m(t, \varepsilon)$ satisfies the condition (2). We have:

$$\Delta z|_{t=t_j} = \Delta x(t, \varepsilon)|_{t=t_j} - \Delta x_m(t, \varepsilon)|_{t=t_j} = I_j(x(t, \varepsilon))|_{t=t_j} - I_j(x_m(t, \varepsilon))|_{t=t_j}.$$

By (23), values $I_j(x(t, \varepsilon))$ can be written as follows:

$$I_j(x(t, \varepsilon)) = \sum_{k=0}^{m} \varepsilon^k [\Phi(t_j)(u_k(t) + \Pi_k x(\tau_j)) + J_k(t, \tau_j)] + O(\varepsilon^{m+1}).$$

So, in respect to (13), (12), (22), (23), (28) the next inequality follows:

$$\|I_j(x(t,\varepsilon)) - I_j(x_m(t,\varepsilon))\|_{t=t_j} \leq C\varepsilon^{m+1}.$$

Now let us show that defined by formula (42) function $x_m(t,\varepsilon)$ satisfies the equation (1) with accuracy $O(\varepsilon^{m+1})$, i.e.,

$$\varepsilon B(t,\varepsilon)\frac{dx_m(t,\varepsilon)}{dt} = f(t, x_m(t,\varepsilon)) + O(\varepsilon^{m+1}).$$

For that let us consider differential expressions

$$L[x(t,\varepsilon)] = \varepsilon B(t,\varepsilon)\frac{dx(t,\varepsilon)}{dt} - f(t, x(t,\varepsilon)),$$

$$L_m[x_m(t,\varepsilon)] = \varepsilon B(t,\varepsilon)\frac{dx_m(t,\varepsilon)}{dt} - f(t, x_m(t,\varepsilon))$$

and their difference

$$L[x(t,\varepsilon)] - L_m[x_m(t,\varepsilon)] = \varepsilon B(t,\varepsilon)\frac{d}{dt}[x(t,\varepsilon) - x_m(t,\varepsilon)] -$$

$$-(f(t, x(t,\varepsilon)) - f(t, x_m(t,\varepsilon))).$$

In respect to (10) and to relation $x(t,\varepsilon) = x_m(t,\varepsilon) + O(\varepsilon^{m+1})$, $L[x(t,\varepsilon)] \equiv 0$, we obtain

$$L_m[x_m(t,\varepsilon)] = O(\varepsilon^{m+1}).$$

Let us estimate $\|x(t,\varepsilon) - x_m(t,\varepsilon)\|$. We have:

$$\varepsilon \left\| B(t,\varepsilon)\left(\frac{dx(t,\varepsilon)}{dt} - \frac{dx_m(t,\varepsilon)}{dt}\right)\right\| =$$

$$= \|f(t, x(t,\varepsilon)) - f(t, x_m(t,\varepsilon)) + O(\varepsilon^{m+1})\|,$$

whence

$$\varepsilon\|B(t,\varepsilon)\| \left\|\frac{dx(t,\varepsilon)}{dt} - \frac{dx_m(t,\varepsilon)}{dt}\right\| \leq$$

$$\leq \left\|\frac{\partial f}{\partial x}\right\| \cdot \|x(t,\varepsilon) - x_m(t,\varepsilon)\| + K\varepsilon^{m+1}, \tag{45}$$

where the Jacobi matrix $\left(\frac{\partial f}{\partial x}\right)$ is calculated at point $(t, x + \theta(x - x_m))$, $0 < \theta < 1$; $K > 0$ is some constant.

Denote

$$g(t,\varepsilon) = \|x(t,\varepsilon) - x_m(t,\varepsilon)\|, \quad K_1 = \left\|\frac{\partial f}{\partial x}\right\| \|B(t,\varepsilon)\|^{-1}, \quad \beta = \frac{K}{\|B(t,\varepsilon)\|}.$$

In that way inequality (45) can be written as

$$\varepsilon\frac{dg(t,\varepsilon)}{dt} \le K_1 g(t,\varepsilon) + \beta\varepsilon^{m+1}. \tag{46}$$

Integrating (46) we have

$$g(t,\varepsilon) \le \varepsilon^{-1}L\int_{t_j}^{t} g(s,\varepsilon)ds + \omega(t),$$

where

$$\omega(t) = \varepsilon^m\beta(t-t_j) + \varepsilon^{-1}g(t_j,\varepsilon).$$

Then we obtain [4]:

$$g(t,\varepsilon) \le \varepsilon^{-1}L\int_{t_j}^{t} \omega'(s)e^{\varepsilon^{-1}L(t-s)}ds,$$

whence

$$g(t,\varepsilon) \le M\varepsilon^m.$$

Thus, solution $x(t,\varepsilon)$ of the problem (1), (2) satisfies the asymptotic estimation for all $t \in [t_j, t_{j+1})$:

$$\|x(t,\varepsilon) - x_m(t,\varepsilon)\| \le M\varepsilon^m. \tag{47}$$

Taking into account that solution $x(t,\varepsilon)$ is continuous from left at points of pulses influences we have that inequality (47) is valid for all $t \in [t_0, T]$. Consequently estimation (43) is true. Therefore the theorem is proved.

CONCLUSION

This paper deals with the problem of constructing of asymptotical solution for degenerative singular perturbed non-linear system of differential equations (1) with pulses (2). Algorithm of constructing asymptotical solution is proposed. It is also proved the asymptotic estimation for constructed asymptotic solution of the problem (1), (2).

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АСИМТОТИЧНІ РОЗВ'ЯЗКИ СИНГУЛЯРНО ЗБУРЕНИХ СИСТЕМ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ВИРОДЖЕННЯМИ ТА ІМПУЛЬСАМИ

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Досліджено задачу про побудову асимптотичних розв'язків сингулярно збуреної нелінійної виродженої системи диференціальних рівнянь з імпульсною дією у фіксовані моменти часу. Запропоновано алгоритм побудови асимптотичного розв'язку цієї задачі і доведено асимптотичну оцінку для нього.