SOLUTIONS WITH STRONG POWER SINGULARITIES TO NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

©2006 *Halyna LOPUSHANSKA*

Ivan Franko Lviv National University,
1 Universitetska Str., Lviv 79000, Ukraine

Received September 7, 2006.

The solvability of a nonlinear boundary value problem for the $2m$-order elliptic equation $Au = \mu|u|^q$ ($q > 0$, $\mu \in L_\infty$) in a bounded domain with given generalized functions with strong power singularities on the frontier and the character of bihaviour of a solution in a vinicity of points of singularity of boundary data is established.

The nature of traces g of solutions to the problem

$$\Delta u = |u|^{q-1}u, \quad x \in \Omega, \qquad u\,|_{\partial\Omega} = g$$

has been investigated in [14] for $q \in (1, q_c)$, where $q_c = \frac{n+1}{n-1}$, in [13] for $q = 2$, in [12] for $q \in [q_c, 2]$, in [4] for $q > q_c$ (including $q > 2$).

The uniqueness solvability of this problem for every g from the space of bounded Borel measures on $\partial\Omega$ has been established if $q \in (1, q_c)$. It also follows from these results that for $q \geq 1 + \frac{2}{n-1}$ the trace-measures of a solutions of this equation may not exist. It proved in [15] that the solvability of semi-linear second order elliptic equation $Lu = f(u)$ (under some hypotheses) in class of positive functions satisfying the boundary condition $u = \infty$ on $\partial\Omega$. The asymptotics of solution $u(x) = O(dist^{-\frac{2}{q-1}}(x, \partial\Omega))$ in a vinicity of the boundary for $f(u) = |u|^q$, $q > 1$ is founded.

It is known (see [5] and references here) that any solution of linear homogeneous equation in Ω has traces from the space $(C^\infty(\partial\Omega))'$ if and only if it belongs to some weighted $L_1(\Omega)$-space.

UDC 517.95; MSC 2000: 35N40, 35J60

In [5–8] the method of research of the boundary value problems for semi-linear elliptic equations in the case of generalized functions giving on the boundary has been proposed. In particular, the solvability of the problem

$$\Delta u = |u|^q, \quad x \in \Omega, \qquad u\,|_{\partial\Omega} = g$$

in some weight L_1-space for arbitrary generalized function $g \in (C^\infty(\partial\Omega))'$ and $q \in (0, q_0)$, where $q_0 \in (0,1)$ and belongs on the order of the singularity of generalized function g, follows from the results of [7].

In [9] the character of power singularities of solutions of a normal boundary value problem for $2m$-order semi-linear elliptic equation in bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ under given generalized functions from $(C^\infty(\partial\Omega))'$ on the boundary has been founded.

In this paper we study the solvability of such problem (in some subspaces of weight L_1-space) under nonlinear boundary conditions and given generalized functions on the boundary. We establish in what sense the solution may be treat. In order to prove the solvability of the problem we use the method of reduction of such generalized boundary value problem to some integro-differential equation in weight L_1-space.

FUNCTIONAL SPACES AND STATEMENT OF PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞-boundary,
$\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$,

$$A(x,D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha, \quad a_\alpha \in C^\infty(\overline{\Omega}),$$

be an elliptic differential expression, $A^*(x,D)$ be a formally adjoint one with respect to $A(x,D)$. The boundary differential expressions

$$B_j(x,D) = \sum_{|\alpha| \leq m_j} b_{j\alpha}(x) D^\alpha, \quad b_{j\alpha} \in C^\infty(S), \quad j = \overline{1,m},$$

are given on S, the system $\{B_j(x,D)\}_{j=1}^m$ is normal and satisfying the Lopatynskyy condition with respect to $A(x,D)$, the system $\{T_j(x,D)\}_{j=1}^m$ of the boundary differential expressions of the orders $\hat{m}_j$ complement the system $\{B_j(x,D)\}_{j=1}^m$ with respect to Dirichlet system of $2m$ order. $\hat{B}_j, \hat{T}_j$ are such boundary differential expressions of the orders $2m - \hat{m}_j - 1, 2m - m_j - 1$, $j = \overline{1,m}$ respectively that the Green formula

$$\int_\Omega (vAu - uA^*v)dx = \sum_{j=1}^m \int_S (\hat{T}_j v B_j u - \hat{B}_j v T_j u) dS, \quad u, v \in C^\infty(\overline{\Omega})$$

hold (see, for example, [4]).

Let $\varepsilon_0 \in (0,1)$ and for all $\varepsilon \in (0,\varepsilon_0)$ surfaces $S_\varepsilon = \{x_\varepsilon = x + \varepsilon\nu(x) : x \in S\}$ parallel with respect to S be also of class C^∞. Here $\nu(x)$ is the unit vector of inner normal to the surface S at a point $x \in S$.

For every fixed point $\hat{x} \in S$ we denote by $\varrho(x,\hat{x})$ $(x \in \overline{\Omega})$ the infinitely differentiable, positive in $\overline{\Omega} \setminus \{\hat{x}\}$, function, having the order of $|x-\hat{x}|$ for $|x-\hat{x}| < \frac{\varepsilon_0}{2}$, $\varrho(\hat{x},\hat{x}) = 0$. We also suppose $\varrho(x,\hat{x}) = 1$ for $|x-\hat{x}| \geq \varepsilon_0$.

Let $\hat{m} = \min\limits_{1\leq j\leq m} \hat{m}_j$. For $k > -\hat{m}-1$, as in [6, sec. 2], we define the functional spaces:

$Z_k(\overline{\Omega},\hat{x}) = \{\varphi \in C^\infty(\overline{\Omega}\setminus\hat{x}) : \varrho^{|\alpha|-k}(\cdot,\hat{x})D^\alpha\varphi \in C(\overline{\Omega})$ for all α if k is nonintegral, for all α, $|\alpha| \neq k$ and $\frac{\varphi}{ln\varrho(\cdot,\hat{x})} \in C(\overline{\Omega})$ if $|\alpha| = k \in N \cup \{0\}\}$,

$Z_k(S,\hat{x}) = \{\varphi \in C(S\setminus\hat{x}) : \varrho^{|\alpha|-k}(\cdot,\hat{x})D^\alpha\varphi \in C(S)$ for all α if k is nonintegral, for all α, $|\alpha| \neq k$ and $\frac{\varphi}{ln\varrho(\cdot,\hat{x})} \in C(S)$ if $|\alpha| = k \in N \cup \{0\}\}$,

$X_k(\overline{\Omega},\hat{x}) = \{\varphi \in Z_{k+2m}(\overline{\Omega},S) : (A^*\varphi)(x) = O(\varrho^k(x,\hat{x}))$ for $|x-\hat{x}| \to 0$, $\hat{T}_j\varphi \in Z_{k+m_j+1}(S)$, $\quad \hat{B}_j\varphi = 0$, $\quad j = \overline{1,m}\}$
(according to [5, 8] this space is nonempty),

$M_k(\Omega,\hat{x}) = \{v \in L_{1,loc}(\Omega) : ||v||_k = \int\limits_\Omega \varrho^k(x,\hat{x}|v(x)|dx < +\infty\}$.

We shall write $D(S)$ instead of $C^\infty(S)$. $\Phi'(S)$ is the space of linear continuous functionals (generalized functions) on the space of basic functions $\Phi(S)$, $<\varphi, F>$ is a value of generalized function $F \in \Phi'(S)$ on a basic function $\varphi \in \Phi(S)$; $s(F) \leq s$ means that the order of the singularity of generalized function $F \in \Phi'(S)$ does not exceed s, that is $\quad <\varphi,F> = \int\limits_S \sum\limits_{|\alpha|\leq p'} D^\alpha\varphi f_\alpha dS$ for every $\varphi \in \Phi(S)$, where $f_\alpha \in L_1(S)$ if $p' \in Z_+$ and $D^\alpha F \in L_1(S)$ for all $\alpha, |\alpha| \leq -p'$, if $p' \in Z_-$, $|\alpha| \leq s$ ([11]); $\Omega_{(0)} = \Omega$, $\Omega_{(j)} = S$ for $j = \overline{1,m}$, $\quad (0) = 0$, $(j) = 1$ for $j = \overline{1,m}$. We note that $Z_k(\overline{\Omega},\hat{x}) \subset C^{[k]}(\overline{\Omega})$, $Z_k(S,\hat{x}) \subset C^{[k]}(S)$, hence

$$D'_s(S) = \{F \in D'(S) : s(F) \leq s\} \subset Z'_k(S,\hat{x}) \qquad \text{for } s \leq [k],$$

$M_k(\Omega,\hat{x})$ is the space of the regular generalized functions on $Z_k(\Omega,\hat{x})$.

For $\mu_j \in L_\infty(\Omega_{(j)})$, $q_j \in (0,+\infty)$, $j = \overline{0,m}$ and $F_j \in Z'_{p_j}(S,\hat{x})$, $s(F_j) \leq s_j < p_j$, $j = \overline{1,m}$ we consider the boundary value problem

$$A(x,D)u(x) = \mu_0(x)|u(x)|^{q_0},\ x \in \Omega, \tag{1}$$

$$B_j(x,D)u(x) = F_j(x) + \mu_j(x)|u(x)|^{q_j},\ x \in S,\ j = \overline{1,m}. \tag{2}$$

Assumptions: (A1) The corresponding linear boundary value problem has only a trivial solution;

(A2) $k > k_0 = \max\{p_0 - 1, -\hat{m} - 1\}$, where $p_0 = \max\limits_{1\le j\le m}(p_j - m_j)$.

Definition. *A function $u \in M_k(\Omega, \hat{x})$ is called the solution of problem (1), (2) in space $M_k(\Omega, \hat{x})$ if for every $\psi \in X_k(\overline{\Omega}, \hat{x})$ we have*

$$|\int\limits_{\Omega} \psi\mu_0|u|^{q_0}dx| < +\infty, \ |\int\limits_{S} \hat{T}_j\psi\mu_j|u|^{q_j}dS| < +\infty, \ 1 \le j \le m, \qquad (3)$$

$$\int\limits_{\Omega} A^*\psi u dx = \int\limits_{\Omega} \psi\mu_0|u|^{q_0}dx + \sum_{j=1}^{m}\int\limits_{S} \hat{T}_j\psi\mu_j|u|^{q_j}dS + \sum_{j=1}^{m} < \hat{T}_j\psi, F_j > . \quad (4)$$

We note that $\hat{T}_j\psi \in Z_{k+m_j+1}(\overline{\Omega}, \hat{x}) \subset Z_{p_j}(\overline{\Omega}, \hat{x})$ for $\psi \in X_k(\overline{\Omega}, \hat{x})$ (and $k > p_0 - 1$), so that for all $\psi \in X_k(\overline{\Omega}, \hat{x})$ the expression $\sum\limits_{j=1}^{m} < \hat{T}_j\psi, F_j >$ is defined.

Remark 1. Since

$|\int\limits_{\Omega_{(j)}} \mu_j|u|^{q_j}d_{(j)}x| \le c_j[\int\limits_{\Omega_{(j)}} \varrho^{-\frac{kq_j}{1-q_j}}(x,\hat{x})d_{(j)}x]^{1-q_j}\cdot||u||_k^{q_j}, j = \overline{0,m},$ for $u \in M_k(\Omega, \hat{x})$, it follows from [13] that for $F_j \in D'(S)$, $s(F_j) \le s_j$, $suppF_j = \{\hat{x}\}$ (that is $F_j = \sum\limits_{|\alpha|\le s_j-1} c_{j\alpha}D^\alpha\delta(x-\hat{x})$, $c_{j\alpha} = const$), in the case $q_j \in (0,1)$, $j = \overline{0,m}$, $\quad s_0 = \max\limits_{1\le j\le m}(s_j - m_j) < 1 + \min\limits_{0\le j\le m}(\frac{n-(j)}{q_j} + (j) - n)$, $\max\{s_0 - 1, -\hat{m} - 1\} < k < \min\limits_{0\le j\le m}(\frac{n-(j)}{q_j} + (j) - n)$ a solution $u \in M_k(\Omega, \hat{x})$ of the problem (1), (2) exists.

In this paper we prove the solvability of problem (1), (2) in

$$C_l^M(\overline{\Omega}, \hat{x}) = \{v \in C(\overline{\Omega} \setminus \hat{x}) : \varrho^{-l}(\cdot, \hat{x})v \in C(\overline{\Omega}) \ (\ ||v||'_{C_l^M(\overline{\Omega},\hat{x})} = ||v||'_{l,\hat{x}} = \sup_{x\in\overline{\Omega}} \varrho^{-l}(x,\hat{x})|v(x)| < +\infty \ )\}, \ l \le 0.$$

$C_l^M(\overline{\Omega}, \hat{x})$ is the subspace of space $M_k(\Omega, \hat{x})$ if $k + l > -n$.

It follows from condition (A2) that $l \le 1 - n + \min\{\hat{m}, -p_0\}$ for $k = -l - n + \varepsilon$, $\varepsilon > 0$. From here $l \le 2m - n$, so that for all $\psi \in X_{-l-n+\varepsilon}(\overline{\Omega}, \hat{x})$ we have $\psi(x) = O(1 + \varrho^{-l-n+\varepsilon+2m}(x, \hat{x}))$ in a vicinity of the point $\hat{x}$, that is ψ is bounded in $\overline{\Omega}$. Further, for every $\psi \in X_{-l-n+\varepsilon}(\overline{\Omega}, \hat{x})$, $u \in C_l^M(\overline{\Omega}, \hat{x})$, $lq_0 > -n$ and $l \le 1 - n + \hat{m}$ the integral $|\int\limits_{\Omega} \psi|u|^{q_0}dx| =$

$$|\int\limits_{\Omega}\psi(x)\varrho^{lq_0}(x,\hat{x})[\varrho^{-l}(x,\hat{x})|u(x)|]^{q_j}dx| \leq [||u||'_{l,\hat{x}}]^{q_0}\int\limits_{\Omega}|\mu_0(x)\psi(x)|\varrho^{lq_0}(x,\hat{x})dx$$

is finite. For $\psi \in X_{-l-n+\varepsilon}(\overline{\Omega},\hat{x})$ and $l \leq 1-n-p_0$ functions $\hat{T}_j\psi$ belong to $Z_{-l-n+\varepsilon+m_j+1}(\overline{\Omega},\hat{x}) \subset Z_{p_j}(\overline{\Omega},\hat{x})$ and are bounded on S, hence for $lq_j > 1-n$ integrals

$$|\int\limits_{S}\hat{T}_j\psi\mu_j|u|^{q_j}dS| \leq [||u||'_{l,\hat{x}}]^{q_j}\int\limits_{S}|(\hat{T}_j\psi)(x)\mu_j(x)|\varrho^{lq_j}(x,\hat{x})dS,\ j=\overline{1,m},$$

are finite.

Furthermore, for $lq_j > (j)-n$ we have $\hat{m}+1-n < 2m-n+(n+lq_0) = 2m+lq_0$, $\hat{m}+1-n < \hat{m}+1-n+(n-1+lq_j) = m_j+lq_j$, hence for $l \leq \hat{m}+1-n$ we obtain $l(q_j-1)+m_j \geq p_j \geq 0$, $j=\overline{1,m}$, $l(q_0-1)+2m \geq 0$.

For $p_0 < 1-2m$ we have $p_j - m_j < 1-2m$, hence, $p_j < 0$, $j=\overline{1,m}$.

Assumption (A3). We suppose
$q_j > 0,\ j=\overline{0,m}$ if $2m \geq n$ and $\hat{m} = \min\limits_{1\leq j\leq m}\hat{m}_j \geq n-1$,
$0 < q_j < \frac{n-(j)}{n-1-\hat{m}},\ j=\overline{0,m}$ if $2m < n$ or $2m \geq n$ and $\hat{m} < n-1$,
$1-2m \leq p_0 < 1-n+\min\{\frac{n-1}{\tilde{q}},\frac{n}{q_0}\}$,
$-\min\{\frac{n-1}{\tilde{q}},\frac{n}{q_0}\} < l \leq 1-n+\min\{\hat{m},-p_0\}$, where $\tilde{q} = \max\limits_{1\leq j\leq m} q_j$.

Thus, under assumption (A3) for all $\psi \in X_{-l-n+\varepsilon}(\overline{\Omega},\hat{x})$, $u \in C_l^M(\overline{\Omega},\hat{x})$ the conditions (3) are fulfilled and the expression $\sum\limits_{j=1}^{m} < \hat{T}_j\psi, F_j >$ is well defined. So, under assumption (A3) function $u \in C_l^M(\overline{\Omega},\hat{x})$, satisfying the condition (4) for every $\psi \in X_{-l-n+\varepsilon}(\overline{\Omega},\hat{x})$, is a solution to problem (1), (2).

EXISTANCE THEOREM

Let $G(x,y) = (G_0(x,y), G_1(x,y),\ldots,G_m(x,y))$ be the Green vector-function of problem (1), (2). Its existence and properties have been established in [1,3]. Under supposition (A1) the function $G_0(x,y)$ is uniquely defined, $G_j(x,y) = \hat{T}_j(y,D)G_j(x,y)$, $j=\overline{1,m}$, for every multi-indexes α,γ and $x \neq y$, the estimates

$$|D_x^\alpha D_y^\gamma G_j(x,y)| \leq C_{j,\alpha,\gamma}(|x-y|^{m_j-|\alpha|-|\gamma|} + \kappa_{m_j-|\alpha|-|\gamma|}|\ln|x-y|| + 1) \quad (5)$$

hold. Here $C_{j,\alpha,\gamma} = const > 0$, $\kappa_s \neq 0$ only for $s=0$, $\kappa_0 = 1$, $m_0 = 2m$. We suppose further that $C_{j,\alpha,\gamma} = C_{j,\alpha}$ for $|\gamma| = 0$, $C_{j,\alpha,\gamma} = C_j$ for $|\alpha| = |\gamma| = 0$.

Let number k satisfies assumption (A2). As in [5, p. 86], using formulas (2.1) from [5, p. 70] , we prove the following theorem.

Theorem 1. *Function $u \in M_k(\Omega, \hat{x})$ is a solution to problem (1), (2) in space $M_k(\Omega, \hat{x})$ if and only if it is a solution in space $M_k(\Omega, \hat{x})$ of the integral equation*

$$u(x) = \int_\Omega G_0(x,y)\mu_0(y)|u(y)|^{q_0}dy+$$

$$+\sum_{j=1}^{m}[< G_j(x,y), F_j(y) > + \int_S G_j(x,y)\mu_j(y)|u(y)|^{q_j}dS], \ x \in \Omega. \quad (6)$$

Obviously, under assumption (A1), (A3) solution $u \in C_l^M(\overline{\Omega}, \hat{x})$ of equation (6) is also a solution to problem (1), (2) in space $M_{-l-n+\varepsilon}(\Omega, \hat{x})$ for every $\varepsilon > 0$.

Lemma 1. *If $F_j \in Z'_{p_j}(S, \hat{x})$, $s(F_j) \le s_j < p_j$, $j = \overline{1,m}$, $l \le -(p_0+n-1)$, then $g = \sum_{j=1}^{m} < G_j(\cdot, y), F_j(y) > \in C_l^M(\overline{\Omega}, \hat{x})$.*

Proof. We shall use lemma 2 from [8]: there exist functions $f_j \in L_2(S)$, natural numbers N_j, $s_j + \frac{n-1}{2} < N_j < p_j + \frac{n-1}{2}$, such that

$$g_j(x) =< G_j(x,y), F_j(y) >= \int_S (1-\Delta_S)^{\frac{N_j}{2}} G_j(x,y) f_{j\gamma}(y)dS, \ j = \overline{1,m},$$

where Δ_S is the Laplace-Beltrami operator with respect to variables $y \in S$.

Let $S^1 = S^1(x,\hat{x}) = \{y \in S : |y - \hat{x}| < \frac{1}{2}|x-\hat{x}|\}$,
$S^2 = S^2(x,\hat{x}) = \{y \in S : |y - x| < \frac{1}{2}|x-\hat{x}|\}$, $S_3 = S \setminus (S^1 \cup S^2)$,
$I_j^i(x,\hat{x}) = \varrho^{-l}(x,\hat{x}) \int_{S^i} |(1-\Delta_S)^{\frac{N_j}{2}} G_j(x,y)||f_j(y)|dS$, $x \in \overline{\Omega}, \hat{x} \in S$,
$i = 1,2,3$, $j = \overline{1,m}$.

If $y \in S^1$, then $|x - y| > \frac{1}{2}|x-\hat{x}|$, hence

$$I_j^1(x,\hat{x}) \le \tilde{c}_{j1}[\varrho^{-l+m_j+\frac{1-n}{2}-N_j}(x,\hat{x}) + 1][\int_{S^1} |f_j(y)|^2 dS]^{\frac{1}{2}}$$

and for $l \le m_j - p_j + 1 - n < m_j + \frac{1-n}{2} - N_j$, $x \in \overline{\Omega}$ it is finite.

For $y \in S^2$ we have $|y-\hat{x}| \ge |x-\hat{x}| - |y-x| > \frac{1}{2}|x-\hat{x}| > |x-y|$, then $I_j^2(x,\hat{x}) \le \tilde{c}_{j2}[\int_{S^2}[|y-x|^{2(-l+m_j+1-n-N_j)}+1]dS]^{\frac{1}{2}} \le c_{j2}[|x-\hat{x}|^{m_j+\frac{1-n}{2}-N_j-l}+1]$, $x \in \overline{\Omega}$.

For $y \in S^3$ we have $\frac{1}{2}|x-\hat{x}| < |y-\hat{x}| < \frac{3}{2}|x-\hat{x}|$ and $|x-y| > \frac{1}{2}|x-\hat{x}|$. Then $I_j^3(x,\hat{x}) \le \tilde{c}_{j3}\max\{|x-\hat{x}|^{-l+\frac{n-1}{2}}, |x-\hat{x}|^{m_j+\frac{1-n}{2}-N_j-l}, 1\}$, $x \in \overline{\Omega}$, where $\tilde{c}_{j1}, \tilde{c}_{j1}, \tilde{c}_{j1}$ are positive constants.

Thus, for every $\hat{x} \in S$, $l \le m_j + 1 - n - p_j$ we obtain

$$||g_j||'_{l,\hat{x}} = \max_{x\in\overline{\Omega}}[I_j^1(x,\hat{x}) + I_j^2(x,\hat{x}) + I_j^3(x,\hat{x})] < +\infty,\ j=\overline{1,m}.$$

It follows from lemma 1 that for l satisfying the assumption (A3) we have $g \in C_l^M(\overline{\Omega}, \hat{x})$.

Let

$$\tilde{G}_{lj}(x,y,\hat{x}) = \varrho^{-l}(x,\hat{x})G_j(x,y)\varrho^{lq_j}(y,\hat{x}),$$

$$R_j = \max_{x\in\overline{\Omega}} \int_{\Omega_{(j)}} |G_j(x,y)|d_{(j)}y, \quad j=\overline{0,m}.$$

$q' = \max\limits_{0\le j\le m} q_j$, $q'' = \min\limits_{0\le j\le m} q_j$, $C_{1l} = ||g||'_{l,\hat{x}}$, $\mu'_j = ||\mu_j||_{L_\infty(\Omega_{(j)})}$, $j=\overline{0,m}$,
$\tilde{a}(q) = 1$ при $q \ge 2$, $\tilde{a}(q) = 2^{2-q}$ if $q\in[1,2)$ ($\tilde{a}(q)\ge 1$ for all $q\ge 1$),
$a_l = 2\tilde{a}(q'')\sum\limits_{j=0}^{m}\mu'_j C'_{lq_j}$, $\hat{a}_l = \min\{\frac{(q'-1)^{q'-1}}{C_{1l}^{q'-1}(q')^{q'}}, \frac{1}{q'}\}$.

Lemma 2. *Let $q_j\in(0,1)$, $j=\overline{0,m}$, $-\min\{\frac{n}{q_0},\frac{n-1}{\bar{q}}\} < l \le 0$. For given $\varepsilon>0$ we can find $\delta=\delta(\varepsilon)>0$ such that for all domains $\omega\subset\overline{\Omega}_{(j)}$, whose measures $m_{(j)}(\omega)$ in $\mathbb{R}^{n-(j)}$ satisfy condition $m_{(j)}(\omega)\le\delta$, we have*

$$I_j = \max_{x\in\overline{\Omega}}\int_{\omega}|\tilde{G}_{lj}(x,y,\hat{x})|d_{(j)}y \le \varepsilon,\ j=\overline{0,m}.$$

For the case $j=0$ the statement of lemma 2 follows from lemma 2 [9], in the case $j\ne 0$ the proof is similar.

Theorem 2. *Under assumption (A1), (A3) and one of the conditions a) $q'\in(0,1)$, b) $q''\ge 1$, without any restriction in the case a and under additional assumption $a_l<\hat{a}_l$ in the case b), there exists the solution $u\in C_l^M(\Omega,\hat{x})$ to problem (1), (2).*

Proof. Let for $v\in C_l^M(\overline{\Omega},\hat{x})$

$$(Pv)(x) = \sum_{j=0}^{m}\int_{\Omega_{(j)}} G_j(x,y)\mu_j(y)|v(y)|^{(q_j)}d_{(j)}y + g(x),\ x\in\Omega.$$

Then we can write equation (6) in the form $u=Pu$. In order to prove its solvability we shall use the Schauder principle in the case a) and the contraction mapping principle in the case b). For $v\in C_l^M(\overline{\Omega},\hat{x})$ we have

$$||Pv||'_{l,\hat{x}} \le \max_{x\in\overline{\Omega}}\varrho^{-l}(x,\hat{x})|\sum_{j=0}^{m}\int_{\Omega_{(j)}} G_j(x,y)\mu_j(y)|v(y)|^{q_j}dS| + ||g||'_{l,\hat{x}} \le$$

$$\mu'_0\max_{\hat{x}\in\overline{\Omega}}\varrho^{-l}(x,\hat{x})\int_{\Omega}|G_0(x,y)|\varrho^{lq_0}(y,\hat{x})(\max_{y\in\overline{\Omega}}\varrho^{-l}(y,\hat{x})|v(y)|)^{q_0}dy +$$

$$\sum_{j=1}^{m}\mu'_j\max_{x\in\overline{\Omega}}\varrho^{-l}(x,\hat{x})\int_S|G_j(x,y)|\varrho^{lq_j}(y,\hat{x})(\max_{y\in S}\varrho^{-l}(y,\hat{x})|v(y)|)^{q_j}dS+C_{1l}\le$$
$$\mu'_0(||v||'_{l,\hat{x}})^{q_0}\max_{x\in\overline{\Omega}}\varrho^{-l}(x,\hat{x})\int_\Omega|G_0(x,y)|\varrho^{lq_0}(y,\hat{x})dy+$$
$$\sum_{j=1}^{m}\mu'_j(||v||'_l)^{q_j}\max_{x\in\overline{\Omega}}\varrho^{-l}(x,\hat{x})\int_S|G_j(x,y)|\varrho^{lq_j}(y,\hat{x})dy+C_{1l}.$$

As in lemma 3 from [8], we prove that for $s>-n$, $|\alpha|<m_j+(j)$, $j=\overline{0,m}$

$$\int_\Omega|D^\alpha_xG_j(x,y)|\varrho^s(x,\hat{x})dx\le C'_{js\alpha}(\varrho^{s+m_j+(j)-|\alpha|}(y,\hat{x})+1),y\in\overline{\Omega},\hat{x}\in S \quad (7)$$

hold, and for $s>1-n$, $|\alpha|<m_j+(j),j=\overline{1,m}$

$$\int_S|D^\alpha_xG_j(x,y)|\varrho^s(y,\hat{x})dS\le C''_{js\alpha}(\varrho^{s+m_j-|\alpha|}(y,\hat{x})+1),y\in S,\hat{x}\in S \quad (8)$$

hold. The constants $C'_{js\alpha}, C"_{js\alpha}$ depend on the constants $C_{j,\alpha,\gamma}$ in estimates (5) for $|\gamma|=0$. Further $C'_{js\alpha}=C'_{js}$, $C''_{js\alpha}=C''_{js}$ for $|\alpha|=0$.

By using estimates (7), (8) for $lq_j>(j)-n$, $j=\overline{0,m}$, we obtain

$$||Pv||'_{l,\hat{x}}\le\sum_{j=0}^{m}\mu'_j(||v||'_{l,\hat{x}})^{q_j}C'_{lq_j}\max_{x\in\overline{\Omega}}\varrho^{-l}(x,\hat{x})(\varrho^{lq_j+m_j}(x,\hat{x})+1)+C_{1l}.$$

Under assumption (A3) and notation $A_{lq_j}=2\mu'_jC'_{lq_j}$, $j=\overline{1,m}$ we have

$$||Pv||'_{l,\hat{x}}\le\sum_{j=0}^{m}A_{lq_j}(||v||'_{l,\hat{x}})^{q_j}+C_{1l}<+\infty,\ v\in C^M_l(\overline{\Omega},\hat{x}). \quad (9)$$

Let $\tilde{M}_{l,C}(\overline{\Omega},\hat{x})=\{v\in C^M_l(\overline{\Omega},\hat{x}):||v||'_{l,\hat{x}}\le C\}$. It is a ball in $C^M_l(\overline{\Omega},\hat{x})$.

Let us prove the existence of such constants $C>0$ that $P:\tilde{M}_{l,C}(\overline{\Omega},\hat{x})\to\tilde{M}_{l,C}(\overline{\Omega},\hat{x})$. It follows from (9) that

$$||Pv||'_{l,\hat{x}}\le\sum_{j=0}^{m}A_{lq_j}C^{q_j}+C_{1l}\qquad\forall v\in\tilde{M}_{l,C}(\overline{\Omega},\hat{x}). \quad (10)$$

If $q_j\in(0,1)$ for all $j=\overline{0,m}$, then for every positive A_{lq_j}, C_{1l} there exists a constant $C_0>0$, such that for every $C>C_0$ we have

$$\sum_{j=0}^{m}A_{lq_j}C^{q_j}+C_{1l}<C. \quad (11)$$

It follows from (10) that for every $C > C_0$ $\quad P : \tilde{M}_{l,C}(\overline{\Omega}, \hat{x}) \to \tilde{M}_{l,C}(\overline{\Omega}, \hat{x})$.

In the case $q'' \geq 1$, $q' > 1$ (11) follows from inequality

$$a_l C^{q'} + C_{1l} < rC, \qquad C \geq 1, \quad r \in (0,1). \tag{12}$$

The existence of $\min\limits_{0 \leq t < +\infty} \hat{f}(t) < -C_{1l}$, where $\hat{f}(t) = a_l t^{q'} - rt$, is enough for (12) (see [10, p. 320]). Let us prove the performance of this condition.

Number $\hat{C} = (\frac{r}{a_l q'})^{\frac{1}{q'-1}}$ is a point of minimum of the function $\hat{f}(t)$. We find

$$\hat{f}(\hat{C}) = \hat{C}(a_l \hat{C}^{q'-1} - r) = \hat{C}(a_l \tfrac{r}{a_l q'} - r) = -\hat{C}r(1 - \tfrac{1}{q'});$$

$$-\hat{C}r(1 - \tfrac{1}{q'}) < -C_{1l} \Leftrightarrow \hat{C} > \tfrac{C_{1l} q'}{r(q'-1)} \Leftrightarrow a_l < \tfrac{r^{q'}(q'-1)^{q'-1}}{C_{1l}^{q'-1}(q')^{q'}}.$$

The condition $\hat{C} \geq 1$ is fulfilled for $a_l \leq \frac{r}{q'} < \frac{1}{q'}$. By using the arbitrariness of number $r \in (0,1)$, under the hypotheses of theorem, we obtain the existence $C \geq 1$, such that (11) is fulfilled and $P : \tilde{M}_{l,C}(\overline{\Omega}, \hat{x}) \to \tilde{M}_{l,C}(\overline{\Omega}, \hat{x})$.

For arbitrary $v_1, v_2 \in C_l^M(\overline{\Omega}, \hat{x})$ we consider $\quad ||Pv_1 - Pv_2||'_{l,\hat{x}} =$

$$\max_{x \in \overline{\Omega}} \varrho^{-l}(x, \hat{x}) \sum_{j=0}^{m} \Big| \int\limits_{\Omega_{(j)}} G_j(x,y)\mu_j(y)[|v_1(y)|^{q_j} - |v_2(y)|^{q_j}] d_{(j)}y \Big|.$$

For all $q_j \in (0,1]$, $v_1, v_2 \in C_l^M(\overline{\Omega}, \hat{x})$ we obtain $||Pv_1 - Pv_2||'_{l,\hat{x}} \leq \sum\limits_{j=0}^{m} A_{lq_j}[||v_1 - v_2||'_{l,\hat{x}}]^{q_j}$. Hence, the operator P is continuous on $C_l^M(\overline{\Omega}, \hat{x})$, it is contracted on $C_l^M(\overline{\Omega}, \hat{x})$ for all $q_j = 1$ and $\sum\limits_{j=0}^{m} \mu'_j < \frac{1}{2C'}$, where C' is some constant.

For all $q_j \geq 1$, $v_1, v_2 \in \tilde{M}_{l,C}(\overline{\Omega}, \hat{x})$ we have $\quad ||Pv_1 - Pv_2||'_{l,\hat{x}} \leq$

$$\sum_{j=0}^{m} \mu'_j \int\limits_{\Omega_{(j)}} |G_{lj}(x,y,\hat{x})| |[\varrho^{-l}(y,\hat{x})|v_1(y)|]^{q_j} - [\varrho^{-l}(y,\hat{x})|v_1(y)|]^{q_j}| d_{(j)}y.$$

By using above estimates of the integrals, it is possible to obtain that

$$||Pv_1 - Pv_2||'_{l,\hat{x}} \leq a_l \max\{(||v_1||'_{l,\hat{x}})^{q_j-1}, (||v_2||'_{l,\hat{x}})^{q_j-1}\} \cdot ||v_1 - v_2||'_{l,\hat{x}} \leq \tag{13}$$

$$a_l C^{q'-1} ||v_1 - v_2||'_{l,\hat{x}} \quad \forall v_1, v_2 \in \tilde{M}_{l,C}(\overline{\Omega}, \hat{x}).$$

According to choice of number C we have $a_l C^{q'-1} = \frac{r}{q'} < 1$. Hence, under hypotheses of the theorem operator P is contracted on $\tilde{M}_{l,C}(\overline{\Omega}, \hat{x})$.

In the case $q' < 1$ we prove the compactness of operator P on $\tilde{M}_{l,C}(\overline{\Omega}, \hat{x})$. It follows from (10) that set $||Pv||'_{l,\hat{x}}$ is uniformly bounded on $\tilde{M}_{l,C}(\overline{\Omega}, \hat{x})$. In order to prove the equicontinuous of set $\{Pv : v \in \tilde{M}_{l,C}(\overline{\Omega}, \hat{x})\}$ on $\tilde{M}_l(\overline{\Omega}, \hat{x})$, it is sufficient to prove that for given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that for all $x \in \overline{\Omega}$, $z \in \mathbb{R}^n$, $|z| \leq \delta$, $x + z \in \overline{\Omega}$, $v \in C_l^M(\overline{\Omega}, \hat{x})$ the inequality $||(Pv)(\cdot + z) - (Pv)(\cdot)||'_{l,\hat{x}} \leq \varepsilon$ holds.

For $x \in \Omega$, $z \in \mathbb{R}^n$, $v \in C_l^M(\overline{\Omega}, \hat{x})$ we have

$$||(Pv)(x+z) - (Pv)(x)||'_{l,\hat{x}} =$$
$$\max_{x\in\overline{\Omega}} |\varrho^{-l}(x+z,\hat{x})(Pv)(x+z) - \varrho^{-l}(x,\hat{x})(Pv)(x)| \leq$$
$$\max_{x\in\overline{\Omega}}\{|\varrho^{-l}(x+z,\hat{x})g(x+z) - \varrho^{-l}(x,\hat{x})g(x)| +$$
$$\mu'_0 \int_\Omega |\varrho^{-l}(x+z,\hat{x})G_0(x+z,y) - \varrho^{-l}(x,\hat{x})G_0(x,y)||v(y)|^{q_0}dy +$$
$$\sum_{j=1}^m \mu'_j \int_S |\varrho^{-l}(x+z,\hat{x})G_j(x+z,y) - \varrho^{-l}(x,\hat{x})G_j(x,y)||v(y)|^{q_j}dS\} \leq$$
$$J_1(z) + J_2(z) + J_3(z),$$

where $J_1(z) = \max_{x\in\overline{\Omega}} |\varrho^{-l}(x+z,\hat{x})g(x+z) - \varrho^{-l}(x,\hat{x})g(x)|$,

$J_2(z) = C^{q_0}\mu'_0 \max_{x\in\overline{\Omega}} \int_\Omega |\tilde{G}_{l0}(x+z,y,\hat{x}) - \tilde{G}_{l0}(x,y,\hat{x})|dy$,

$J_3(z) = \sum_{j=1}^m C^{q_j}\mu'_j \max_{x\in\overline{\Omega}} \int_S |\tilde{G}_{lj}(x+z,y,\hat{x}) - \tilde{G}_{lj}(x,y,\hat{x})|dS.$

It follows from uniform continuity of function $\varrho^{-l}(x,\hat{x})g(x)\ (x \in \overline{\Omega})$ that for given $\varepsilon > 0$ we can find $\delta' = \delta'(\varepsilon) > 0$ such that for all $x \in \overline{\Omega}, z \in \mathbb{R}^n, |z| \leq \delta'$, $x + z \in \overline{\Omega}$, $\hat{x} \in S$ inequality $J_1(z) \leq \frac{\varepsilon}{2}$ holds.

Let $\eta \in (0, \varepsilon_0)$, Ω_η be a subset of Ω with boundary S_η. By lemma 2 for given $\varepsilon > 0$ we can find $\delta_0 = \delta_0(\varepsilon) > 0$, $\eta_0 \in (0, \varepsilon_0)$ such that $m(\Omega\setminus\Omega_{\eta_0}) \leq \delta_0$ and for all $z \in \mathbb{R}^n$

$$I_{21}(z) = \mu'_0 C^q \max_{x\in\overline{\Omega}} [\int_{\Omega\setminus\Omega_{\eta_0}} |\tilde{G}_{l0}(x,y,\hat{x})|dy +$$
$$\int_{\Omega\setminus\Omega_{\eta_0}} |\tilde{G}_{l0}(x+z,y,\hat{x})|dy] \leq \frac{\varepsilon}{12(m+1)}.$$

Further by ω^z we denote the displacement of set ω by vector z. Obviously, $m(\omega^z) = m(\omega)$. We choose $\eta_1 < \min\{\frac{1}{2}\eta_0, (\frac{\delta_0}{\sigma_n})^{\frac{1}{n}}\}$, where σ_n is the area of the unit sphere in $\mathbb{R}^n$.

For $x \in \overline{\Omega}_{\frac{1}{2}\eta_0}$ we define sets $\omega_{\eta_1}(x) = \{\xi \in \Omega_{\eta_0} : |\xi - x| < \eta_1\}$. We have $m(\omega_{\eta_1}(x)) = \sigma_n\eta_1^n < \delta_0$. Then by lemma 2

$$\mu'_0 C^q \max_{x\in\overline{\Omega}} \int_{\omega_{\eta_1}(x)} |\tilde{G}_{l0}(x,y,\hat{x})|dy \leq \frac{\varepsilon}{24(m+1)}.$$

We choose $\delta_1 = \min\{\sigma_n\eta_1^n, \delta_0, \frac{1}{2}\eta_1\}$. If $x \in \overline{\Omega}_{\frac{1}{2}\eta_0}$, $z \in \mathbb{R}^n$, $|z| \leq \delta_1 (< \frac{1}{4}\eta_0)$ then $x + z \in \overline{\Omega}_{\frac{1}{4}\eta_0} \subset \Omega$, $\omega_{\eta_1}^{-z}(x) \subset \Omega$. So, by lemma 2

$$I_{22}(z) = \mu_0' C^q \max_{x\in\overline{\Omega}_{\frac{\eta_0}{2}}} [\int_{\omega_{\eta_1}(x)} |\tilde{G}_{l0}(x,y,\hat{x})|dy + \int_{\omega_{\eta_1}(x)} |\tilde{G}_{l0}(x+z,y,\hat{x})|dy] =$$

$$= \mu_0' C^q \max_{x\in\overline{\Omega}_{\frac{\eta_0}{2}}} [\int_{\omega_{\eta_1}(x)} |\tilde{G}_{l0}(x,y,\hat{x})|dy + \int_{\omega_{\eta_1}^{-z}(x)} |\tilde{G}_{l0}(x,y,\hat{x})|dy] \le \tfrac{\varepsilon}{12(m+1)}.$$

For $x \in \overline{\Omega}_{\frac{1}{2}\eta_0}$, $y \in \overline{\Omega_{\eta_0} \setminus \omega_{\eta_1}(x)}$, $z \in \mathbb{R}^n$, $|z| \le \delta_1 (< \frac{1}{4}\eta_0)$ we have $x+z \in \overline{\Omega}_{\frac{1}{4}\eta_0} \subset \Omega$, $|y-x| \ge \eta_1$, $|y-(x+z)| \ge |y-x|-|z| \ge \eta_1 - \delta_1 \ge \frac{1}{4}\eta_0 \ge \frac{1}{2}\eta_1$, hence, $y \ne x$ and $y \ne x+z$. By uniform continuity of function $\tilde{G}_{l0}(x,y,\hat{x})$ on closed set $V = \overline{\Omega_{\frac{1}{4}\eta_0}} \times \overline{(\Omega_{\eta_0} \setminus \omega_{\eta_1}(x))} \times S$ we obtain: for given $\varepsilon > 0$ we can find $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_1]$ such that for all $(x,y,\hat{x}) \in V_1 = \overline{\Omega}_{\frac{1}{2}\eta_0} \times \overline{(\Omega_{\eta_0} \setminus \omega_{\eta_1}(x))} \times S \subset V$, $z \in \mathbb{R}^n$, $|z| \le \delta_2$ the inequality $|\tilde{G}_{l0}(x+z,y,\hat{x}) - \tilde{G}_{l0}(x,y,\hat{x})| \le \frac{\varepsilon}{12\mu_0' C^q m(\Omega)}$ holds, hence

$$I_{23}(z) = \mu_0' C^q \max_{x\in\overline{\Omega}_{\frac{\eta_0}{2}}} \int_{\Omega_{\eta_0}\setminus\omega_{\eta_1}(x)} |\tilde{G}_{l0}(x+z,y,\hat{x}) - \tilde{G}_{l0}(x,y,\hat{x})|dy \le \tfrac{\varepsilon}{12(m+1)}.$$

Thus, for given $\varepsilon > 0$ we can find δ_2 such that for $z \in \mathbb{R}^n, |z| \le \delta_2$

$$\max_{x\in\overline{\Omega}_{\frac{\eta_0}{2}}} \int_{\Omega_{\eta_0}} |\tilde{G}_{l0}(x+z,y,\hat{x}) - \tilde{G}_{l0}(x,y,\hat{x})|dy \le I_{22}(z) + I_{23}(z) \le \tfrac{\varepsilon}{6(m+1)}.$$

For $x \in \overline{\Omega \setminus \Omega_{\frac{\eta_0}{2}}}$, $y \in \overline{\Omega}_{\eta_0}$, $z \in \mathbb{R}^n$, $|z| \le \delta_1$, $x+z \in \overline{\Omega}$ we have $y \ne x$, $y \ne x+z$, and also $|y-\hat{x}| \ge \frac{1}{2}\eta_0 \ge \eta_1$. Then by uniform continuity of the function $\tilde{G}_{l0}(x,y,\hat{x})$ on closed set $V' = \overline{(\Omega \setminus \Omega_{\frac{3}{4}\eta_0})} \times \overline{\Omega_{\eta_0}} \times S$ we obtain: for given $\varepsilon > 0$ we can find $\delta_3 \in (0, \delta_1]$ such that for all $(x,y,\hat{x}) \in V_1' = \overline{(\Omega \setminus \Omega_{\frac{1}{2}\eta_0})} \times \overline{\Omega_{\eta_0}} \times S \subset V'$, $z \in \overline{\Omega}$, $|z| \le \delta_3$, $x+z \in \overline{\Omega}$ the inequality

$$I_{24}(z) = \mu_0' C^q \max_{x\in\overline{\Omega\setminus\Omega_{\frac{\eta_0}{2}}}} \int_{\Omega_{\eta_0}} |\tilde{G}_{l0}(x+z,y,\hat{x}) - \tilde{G}_{l0}(x,y,\hat{x})|dy \le \tfrac{\varepsilon}{6(m+1)}$$

holds. So, for $z \in \mathbb{R}^n$, $|z| \le \delta'' = \min\{\delta_2, \delta_3\}$ we have

$$J_2(z) \le \max\{I_{21}(z), I_{22} + I_{23}(z), I_{24}(z)\} \le \tfrac{\varepsilon}{6(m+1)}$$

Let us estimate $J_3(z)$, conserving previous construction. Let $S_{\eta_2}(\hat{x}) = \omega_{\eta_2}(\hat{x}) \cap S$. By lemma 2 for given $\varepsilon > 0$ we can find $\eta_2 = \eta_2(\varepsilon) > 0$, $\delta_4 = \delta_4(\varepsilon) > 0$ such that for all $x \in \overline{\Omega_{\eta_0}}$, $z \in \mathbb{R}^n, |z| \le \delta_4$, $\hat{x} \in S$

$$J_{31}(z) = \max_{x\in\overline{\Omega}_{\eta_0}} \sum_{j=1}^{m} \mu_j' C^{q_j} \int_{S_{\eta_2}(\hat{x})} [|\tilde{G}_{lj}(x+z,y,\hat{x})| + |\tilde{G}_{lj}(x,y,\hat{x})|]dS \le \tfrac{m\varepsilon}{24(m+1)}.$$

The function $\tilde{G}_{lj}(x,y,\hat{x})$ is continuous in $\overline{\Omega_{\frac{\eta_0}{2}}} \times (S \setminus S_{\eta_2}(\hat{x})) \times S$ and for given $\varepsilon > 0$ we can find $\delta_5 = \delta_5(\varepsilon) > 0$ such that for all $x \in \overline{\Omega_{\eta_0}}$, $z \in \mathbb{R}^n, |z| \le \delta_5$, $x+z \in \overline{\Omega}$, $\hat{x} \in S$

$$J_{32}(z) = \max_{x\in\overline{\Omega}_{\eta_0}} \sum_{j=1}^{m} \mu_j' C^{q_j} \int_{S_{\eta_2}(\hat{x})} |\tilde{G}_{lj}(x+z,y,\hat{x}) - \tilde{G}_{lj}(x,y,\hat{x})|dS \le \tfrac{m\varepsilon}{24(m+1)}.$$

For $x \in \overline{\Omega \setminus \Omega_{\eta_0}}$, $z \in \mathbb{R}^n, |z| \le \delta_5$, $x + z \in \overline{\Omega}$ we have

$$J_{33}(z) = \max_{x \in \overline{\Omega \setminus \Omega_{\eta_0}}} \sum_{j=1}^{m} \mu_j' C^{q_j} \int_S |\tilde{G}_{lj}(x+z, y, \hat{x}) - \tilde{G}_{lj}(x, y, \hat{x})| dS \le$$

$$\le \max_{x \in \overline{\Omega \setminus \Omega_{\eta_0}}} \sum_{j=1}^{m} \mu_j' C^{q_j} \{ \int_{S_{\eta_2}(\hat{x}) \cup S_{\eta_2}(x)} [|\tilde{G}_{lj}(x+z, y, \hat{x})| + |\tilde{G}_{lj}(x, y, \hat{x})|] dS +$$

$$+ \int_{S \setminus (S_{\eta_2}(\hat{x}) \cup S_{\eta_2}(x))} |\tilde{G}_{lj}(x+z, y, \hat{x}) - \tilde{G}_{lj}(x, y, \hat{x})| dS \}.$$

By continuity of $\tilde{G}_{lj}(x, y, \hat{x})$ in $V_1 = \overline{\Omega \setminus \Omega_{\eta_0}} \times (S \setminus (S_{\eta_2}(\hat{x}) \cup S_{\eta_2}(x))) \times S$ and by lemma 2 we obtain: for given $\varepsilon > 0$ we can find $\delta_6 = \delta_6(\varepsilon) \in (0, \delta_4)$ such that for all $x \in \overline{\Omega \setminus \Omega_{\eta_0}}$, $z \in \mathbb{R}^n, |z| \le \delta_6$, $x + z \in \overline{\Omega}$, $y \in S$ the inequality

$$J_{33}(z) \le \left(\frac{\varepsilon}{12} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} \right) \frac{m}{m+1} = \frac{m\varepsilon}{4(m+1)}$$

holds. So, for $\delta < \delta'' = \min\{\delta_4, \delta_5, \delta_6\}$ we have

$$J_3(z) \le \max\{J_{31}(z) + J_{32}, J_{33}(z)\} \le \frac{m\varepsilon}{12(m+1)} + \frac{m\varepsilon}{4(m+1)} = \frac{m\varepsilon}{3(m+1)}.$$

Finally, for given $\varepsilon > 0$ we can find $\delta = \min\{\delta', \delta''\}$ such that for all $z \in \overline{\Omega}, |z| \le \delta$

$$||(Pv)(\cdot + z) - (Pv)(\cdot)||'_{l,\hat{x}} \le J_1(z) + J_2(z) + J_3(z) < \varepsilon.$$

So, the realization of the Schauder principle's conditions in the case $q \in (0, 1)$ is proved.

Remark 2. In the case of the linear boundary conditions ($\mu_j = 0, j = \overline{1, m}$) assumption (A3) goes into such one:

$q_0 > 0$ if $2m \ge n$ and $\hat{m} \ge n - 1$,
$0 < q_0 < \frac{n}{n-1-\hat{m}}$ if $2m < n$ or $2m \ge n$ and $\hat{m} < n - 1$,
$1 - 2m \le p_0 < 1 - n + \frac{n}{q_0}$, $\quad -\frac{n}{q_0} < l \le 1 - n + \min\{\hat{m}, -p_0\}$.

The case of the linear boundary conditions and $F_j \in D'(S)$, $s(F_j) \le s_j$, $j = \overline{1, m}$ has been discussed in [9].

[1] *Березанский Ю.М., Ройтберг Я.А.* Теорема о гомеоморфизмах и функция Грина для общих эллиптических граничных задач // Укр. мат. журн., 1967, **19**, № 5. – С. 3–32.

[2] *Жидик У.В., Лопушанська Г.П.* Точкові особливості та крайові значення розв'язків квазілінійного еліптичного рівняння // Вісник Нац. ун-ту „Львівська політехніка". – Сер. фіз.-мат. наук. – 2005, № 540.

[3] *Красовский Ю.П.* Свойства функций Грина и обобщенные решения эллиптических граничных задач // Изв. АН СССР. Сер. мат. – 1969, **33**, № 1. – С. 109–137.

[4] *Лионс Ж.-Л., Мадженес Э.* Неоднородные граничные задачи и их приложения. – М.: Мир, 1971. – 372 с.

[5] *Лопушанська Г.П.* Крайові задачі у просторі узагальнених функцій D': Монографія.- Львів: Вид. центр ЛНУ імені Івана Франка, 2002. – 287 с.

[6] *Лопушанська Г.П.* Задача Діріхле для квазілінійних еліптичних рівнянь у просторі розподілів // Вісник Львівського ун-ту. Сер. мех.-мат. – 1990. – Вип. 35. – С. 26–31.

[7] *Лопушанська Г.П., Жидик У.В.* Про узагальнені граничні значення розв'язків квазілінійного еліптичного рівняння 2-го порядку // Вісник Львів. ун-ту. Сер. мех.-мат. – 2001. – Вип. 59. – С. 126–138.

[8] *Лопушанська Г.П.* Узагальнені крайові задачі для лінійних та напівлінійних еліптичних рівнянь // Укр. матем. вісник. – 2005, **2**, № 3. – С. 377–394.

[9] *Лопушанська Г.П.* Крайові значення із $(C^\infty)'$ розв'язків напівлінійних еліптичних рівнянь // Нелин. гран. задачи. – 2006, **16**. – С. 173–185.

[10] *Функциональный анализ* // Под общей ред. С.Г. Крейна. СМБ. – М.: Наука, 1972. – 544 с.

[11] *Шилов Г.Е.* Математический анализ. Второй спецкурс. – М.: Наука, 1965. – 328 с.

[12] *Dynkin E.B., Kuznetsov S.E.* Trace on the boundary for solutions of nonlinear equations // Trans. Amer. Math. Soc. – 1998, **350**. – P. 4499–4519.

[13] *Le Gall J.-F.* The Brounian snake and the solutions of $\Delta u = u^2$ in a domain // Probab. Theory Related Fields. – 1995, **102**. – P. 393–432.

[14] *Gmira A., Veron L.* Boundary singularities of solutions of some nonlinear elliptic equations //Indiana Math. J. – 1991. – **64**. – P. 271–324.

[15] *Kondrat'ev V.A. and Nikishkin V.A.* Asymptotics near the boundary, of a solution of a singular boundary-value problem for a semilinear elliptic equation // Diff. uravn., 1990, **26**, № 3. – P. 465–468.

[16] *Marcus M., Veron L.* Removable singularities and boundary traces // J. Math. Pures Appl. – 2001, **80**, № 1. – P. 879–900.

РОЗВ'ЯЗКИ ІЗ СИЛЬНИМИ СТЕПЕНЕВИМИ ОСОБЛИВОСТЯМИ НЕЛІНІЙНИХ ЕЛІПТИЧНИХ КРАЙОВИХ ЗАДАЧ

Галина ЛОПУШАНСЬКА

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів 79000

Встановлено розв'язність нелінійної крайової задачі для еліптичного рівняння $Au = \mu|u|^q$ ($q > 0$, $\mu \in L_\infty$) порядку $2m$ в обмеженій області при заданих на її межі узагальнених функціях із сильними степеневими особливостями та знайдено характер поведінки розв'язку поблизу цих особливостей.