ON NORMALITY OF THE STRONG-WEAK TOPOLOGY

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Received July 17, 2006.

We consider the problem of normality of functional spaces endowed with the strong-weak topology and also of spaces of equivariant maps. Some open problems are also formulated.

1. INTRODUCTION

The strong Whitney topology on the functional spaces has numerous applications in differential topology. Its counterpart for C^0-maps (i.e. continuous maps; see [2] for various properties of this topology) is also widely used in the topology of infinite-dimensional manifolds [18].

In his textbook [7], M. Hirsch formulated the following problem. Is the space of continuous functions defined on locally compact non-compact metrizable space normal in the strong Whitney topology? Note that the related problem of paracompactness of the Whitney C^∞-topology or his extension of the Schwartz $\mathcal{D}$-topology on $C^\infty(M, N)$ when M is an open manifold was formulated by Michor [13].

The Hirsch problem was first solved in the negative by the authors [5] (see also [6]), Neves [14,15], and Wegenkittl [19]. These results were considerably improved by Serrano [17] and van Douwen [4].

The proofs of nonnormality theorems are based on embedding of the van Douwen nonnormal space [3], which itself is the box product of metrizable spaces, into the functional spaces. The proof in [14] is based on nonstandard analysis. Recall that the *box product* of an indexed family $(X_\alpha)_{\alpha\in\Gamma}$ is their

UDC 515.12; MSC 2000: 54C35, 54D15

cartesian product $\prod_{\alpha\in\Gamma} X_\alpha$ endowed with the topology whose base consists of the sets of the form $\prod_{\alpha\in\Gamma} U_\alpha$ (boxes), where every U_α is open in X_α, for every $\alpha \in \Gamma$.

Lately, the technique applied in [5] was also used in another situations. In this way it was shown that the spaces of foliations on manifolds [20] and the spaces of preference relations [8] are not normal.

In this note we consider the so-called strong-weak topology on the sets of differentiable maps of differentiable manifolds introduced in [10]. The main result is another theorem on nonnormality of the obtained function space. We also formulate some open problems on function spaces endowed with the Whitney topology.

2. DEFINITIONS AND PRELIMINARIES

For any m-tuple $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$ we let $|\alpha| = \alpha_1 + \dots + \alpha_m$.

Let M, N be differentiable manifolds. Let U be an open subset of $\mathbb{R}^m$, K be a compact subset of U, and $f = (f_1, \dots, f_n)\colon U \to \mathbb{R}^n$ be a C^r-map, $0 < r < \infty$. We define

$$\|f\|_K^r = \sup_{x\in K}\left\{\left|\frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}\right| \mid 1 \le i \le n,\ 0 \le |\alpha| \le r\right\}.$$

Given C^r-manifolds M, N, we denote by $C^r(M,N)$ the set of all C^r-maps from M to N. Let $f \in C^r(M,N)$, (U,φ) and (V,ψ) be charts in M and N respectively, $K \subset U$ be a compact subset such that $f(K) \subset V$, and $\varepsilon > 0$. Define

$$O^r(f; K, (U,\varphi), (V,\psi), \varepsilon) = \{g \in C^r(M,N) \mid g(K) \subset V,$$
$$\|\psi\circ g\circ\varphi^{-1} - \psi\circ f\circ\varphi^{-1}\|^r_{\varphi(K)} < \varepsilon\}.$$

The following notions are introduced in [10].

Definition 2.1. *A phase map is a continuous map from a C^r-manifold to a topological space.*

Definition 2.2. *Let $\varrho\colon M \to \Omega$ be a phase map. The strong-weak topology on $C^r(M,N)$ with respect to the phase map ϱ is the topology which as a basis has the family of all neighborhoods of the form*

$$\bigcap_{i\in\Lambda} O^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i),$$

where the family $\{K_i \mid i \in \Lambda\}$ is locally finite in Ω.

The obtained topological space is denoted by $C^r_{\mathrm{SW}[\varrho]}(M,N)$.

If a Lie group G acts on C^r-manifolds M, N, we denote by $C^{r,G}(M,N)$ the set of all G-equivariant maps. We endow $C^{r,G}(M,N)$ with the strong-weak topology with respect to the projection map $M \to M/G$ onto the orbit space.

3. RESULT

Theorem 3.1. *Suppose that the image of M under the phase map ϱ is not precompact (i.e., is not contained in a compact subset of Ω) and* $\dim M > 0$, $\dim N > 0$. *Then the space $C^r_{\mathrm{SW}[\varrho]}(M,N)$ is not normal.*

Proof. One can easily see that there exists a countable discrete family $\{K_i\}$ of compact subsets of M satisfying the following properties:

1. every K_i is diffeomorphic to the m-dimensional disc, where $m = \dim M$;
2. the family $\{\varrho(K_i)\}$ is locally finite.

Without loss of generality, one may assume that there exists a map $f \in C^r_{\mathrm{SW}[\varrho]}(M,N)$ with the following property: for every i, the set $f(K_i)$ is contained in a submanifold with boundary L_i of N diffeomorphic to the unit disc in $\mathbb{R}^{\dim N}$. Consider the set

$$R = \{g \in C^r_{\mathrm{SW}[\varrho]}(M,N) \mid g_{|(M \setminus \bigcup_{i=1}^{\infty} K_i)} = f_{|(M \setminus \bigcup_{i=1}^{\infty} K_i)} \text{ and } g(K_i) \subset L_i \text{ for every } i\}.$$

It easily follows from the definition of the strong Whitney topology that the set R is closed in $f \in C^r_{\mathrm{SW}[\varrho]}(M,N)$. Moreover, we see that the set R is homeomorphic to the box product $\square_{i=1}^{\infty} C^r(K_i, \partial K_i; L_i)$, where $C^r(K_i, \partial K_i; L_i)$ denotes the set of all C^r-maps $g\colon K_i \to L_i$ satisfying the property $g|\partial K_i = f|\partial K_i$. Note that, if we identify the set L_i with the standard unit ball in $\mathbb{R}^{\dim N}$, then the set $C^r(K_i, \partial K_i; L_i)$ is a convex subset in the Fréchet space $C^r(K_i, \mathbb{R}^{\dim N})$, which is known to be homeomorphic to the separable Hilbert space ℓ^2. Moreover, since $C^r(K_i, \partial K_i; L_i)$ is infinite-dimensional, it itself is homeomorphic to ℓ^2 (see, e.g. [1]). Like in [5], we conclude that the van Douwen nonnormal space $(\omega + 1)\square(\square^{\omega}\omega^{\omega})$ can be closely embedded into $\square_{i=1}^{\infty} C^r(K_i, \partial K_i; L_i)$ and therefore in $C^r_{\mathrm{SW}[\varrho]}(M,N)$, whence the conclusion of the theorem follows.

The proof of the following result is based on similar arguments.

Theorem 3.2. *Suppose that a Lie group G acts on a manifold M such that the orbit space M/G is not compact and* $\dim M/G > 0$. *Then the space* $C^{r,G}_{\mathrm{SW}[\varrho]}(M,N)$ *is not normal.*

4. REMARKS AND OPEN QUESTIONS

4.1. Coarse counterpart of the Whitney topology. Recently, I. Belins'ka considered a counterpart of the strong Whitney topology in the asymptotic topology. This topology is defined on the classes of coarse maps between proper metric spaces.

Let us recall briefly the necessary definitions. A metric space X is said to be *proper* if every its closed ball is compact. Given two proper metric spaces, X and Y, we introduce topology on the space $C(X,Y)$ of continuous functions from X to Y. Let $\mathcal{E}$ denote the set of continuous functions $\varepsilon\colon X \to (0,\infty)$ with the property: for every $C > 0$ there exists $r > 0$ such that $d(x,x_0) > r$ implies $\varepsilon(x) > C$ (here x_0 is a base point of X). Given $\varepsilon \in \mathcal{E}$ and $f \in C(X,Y)$, let

$$O(f;\varepsilon) = \left\{ g \in C(X,Y) \mid \frac{\rho(f(x),g(x))}{\varepsilon(x)} \to 0 \ \text{ as } \ x \to \infty \right\}$$

(here ρ denotes the metric on Y; also „$\varphi(x) \to 0$ as $x \to 0$" means that for every $\delta > 0$ there exists a compact subspace $K \subset X$ such that $|\varphi(x)| < \delta$ whenever $x \in X/K$).

The family $O(f,\varepsilon)$, where $f \in C(X,Y)$ and $\varepsilon \in \mathcal{E}$, forms a base of a topology on $C(X,Y)$.

Recall that a map $f\colon X \to Y$ is *coarse* if (i) f is coarse uniform, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \varepsilon$ implies $\varrho(f(x),f(y)) < \delta$ for every $x,y \in X$ and (ii) f is coarsely proper, i.e. the preimage $f^{-1}(A)$ of every bounded set A in Y is bounded in X.

Question 4.1. Are spaces of equivalence classes of coarse maps between proper metric spaces normal? Here, two maps are equivalent if the distance between them is finite.

4.2. Another function spaces. In [11], the transversal Whitney topology modulo a foliation $\mathcal{F}$ on the target manifold N is defined on the set $C^\infty(M,N)$.

It is proved in [11] that the obtained topological space (it is denoted by $C^\infty(M,N;\mathcal{F})$ is a Baire space.

Question 4.2. Is the space $C^\infty(M,N;\mathcal{F})$ normal?

One can also ask similar question about the spaces of Hölder maps [12].

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ПРО НОРМАЛЬНІСТЬ СИЛЬНО-СЛАБКОЇ ТОПОЛОГІЇ

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Розглянуто проблему нормальності функціональних просторів, наділених сильно-слабкою топологією Уітні і також просторів еквіваріантних відображень. Сформульовано також деякі відкриті проблеми.