Математичний Вісник
Наукового товариства
імені Шевченка
2013. — Т.10

Mathematical Bulletin
of the Shevchenko
Scientific Society
2013. — V.10

ON PROJECTIVE CLASSIFICATION OF ALGEBRAIC CURVES

Nadiia Konovenko[1], Valentin Lychagin[2]

[1]*Department of Mathematics, ONAFT, Odessa, Ukraine*

[2]*Department of Mathematics, University of Tromsø, Norway*

N. Konovenko, V. Lychagin, *On projective classification of algebraic curves*, Math. Bull. Shevchenko Sci. Soc. **10** (2013), 51–64.

Algebra of projective differential invariants and description of projective classes of algebraic plane curves are given.

Н. Коновенко, В.Личагін. *Про проективну класифікацію алгебраїчних кривих* // Мат. вісник НТШ. — 2013. — Т.10. — С. 51–64.

Досліджено алгебру проективних диференціальних інваріантів та описано деякі класи проективних плоских кривих.

1. Introduction

In this paper we continue investigation of projective differential invariants for curves in the complex projective plane. Many of these results are classical and mainly they go back to Halphen's dissertation [2]. The case of real curves was considered in [5], here we analyze the case of complex and algebraic.

Curves under consideration are smooth and complex but they possibly might have singularities in a projective sense. For example, singular, from the projective point of view, are points on a curve, where tangent line has second order contact, i.e. inflection or *flex points.* Another example of singularities are provided by points, where osculating quadric has 5-th order contact, so-called *Monge points* [5].

We give a detail description of $\mathbf{SL}_3(\mathbb{C})$ -orbits of the projective action on the jets of plane curves up to 5-th order and classify all possible projective singularities.

The level of 5-th jets taken for the only reason: starting from 6-jets regular orbits have trivial stabilizers, and from the level of 7-jets first differential invariants come up. It worth

2010 *Mathematics Subject Classification:* 53A20, 53A55 *УДК:* 517.956.4

Key words and phrases: differential invariant, jet, Study derivation, plane curves

E-mail: konovenko@ukr.net

to mention, that the orbit classification gives projective classification, or projective normal forms for plane curves, up to 6-th order jets.

To find the full algebra of projective differential invariants we introduce the Study derivation. As opposed to the real case, this derivation has order 7 in complex case (order 5 in the real case).

The Study derivation is a projective invariant and it allows to produce new projective invariants from the known ones. Moreover, as it follows from the general result [6], the field of rational projective differential invariants is generated by the projective curvature [5] and the Study derivation. This field separates regular orbits.

The rest of the paper is devoted to cubics and repeats the known result of Weierstrass that projective classes of regular cubics can be described by one parameter. We give an explicit formula to find this parameter.

2. Jet bundle structure

Let $\mathbb{C}\mathbf{P}^2$ be the complex projective plane and let $\mathbf{J}^k$ be the manifold of k-jets of non-parameterized curves on the plane.

We shall denote by $[L]_a^k \in \mathbf{J}$ the k-jet of curve $L \subset \mathbb{C}\mathbf{P}^2$ at the point $a \in L$. Let $\pi_{k,l} : \mathbf{J}^k \to \mathbf{J}^l$, $\pi_{k,l} : [L]_a^k \mapsto [L]_a^l$, $k > l$, be the natural projections.

The structure of jet-manifolds can be described as follows:

- $\mathbf{J}^0 = \mathbb{C}\mathbf{P}^2$;
- Fibres $\pi_{1,0}^{-1}(a), a \in \mathbb{C}\mathbf{P}^2$, of the projection $\pi_{1,0}$ are projectivizations of the tangent planes $\mathbb{P}(\mathbf{T}_a\mathbf{P}^2) = \mathbb{C}\mathbf{P}^1$, $\pi_{2,1} : \mathbf{J}^1 \overset{\mathbb{C}\mathbf{P}^1}{\to} \mathbb{C}\mathbf{P}^2$;
- Fibres $\pi_{k,k-1}^{-1}\left([L]_a^{k-1}\right)$, when $k \geq 2$, are affine lines, and the vector spaces associated with them are $\mathbf{S}^k\tau_a^* \otimes \nu_a$, where $\tau_a^* = T_a^*L$ is the cotangent space, and $\nu_a = T_a\mathbf{P}^2/T_aL$ is the normal space to a curve at the point $a \in L$.

Let (x, u) be an affine chart on the plane. Denote by $(x, u, u_1, ..., u_k)$ the natural coordinates in the space of k-jets, where

$$u_i\left([L]_a^k\right) = \frac{\partial^i h}{\partial x^i}(b),$$

if $L = L_h \overset{\text{def}}{=} \{u = h(x)\}$ is a graph of function h in a neighborhood of point $a = (b, h(b))$.

In these coordinates the affine action is given by translations along tensors

$$\theta = \frac{\lambda}{k!} dx^k \otimes \overline{\partial}_u \in \mathbf{S}^k\tau_a^* \otimes \nu_a,$$

and has the form $(x, u, u_1, ..., u_{k-1}, u_k) \mapsto (x, u, u_1, ..., u_{k-1}, u_k+\lambda)$, where $\overline{\partial}_u = \partial_u \mod T_aL$.

Finally, any curve $L \subset \mathbb{C}\mathbf{P}^2$ determines curves $L^{(k)} \subset \mathbf{J}^k$, so-called k-th prolongations of L, which are formed by points $[L]_a^k$, where point a runs over curve L.

The action of projective group $\mathbf{SL}_3(\mathbb{C})$ can be also prolonged in manifolds $\mathbf{J}^k$ in the natural way:

$$\varphi^{(k)} : [L]_a^k \longmapsto [\varphi(L)_a^k$$

where $\varphi \in \mathbf{SL}_3(\mathbb{C})$ is a projective transformation.

3. Special classes of plane curves

We use special classes of curves, model curves, to construct tensor invariants. The construction is based on the following observation. Assume that we have a class $\mathfrak{M}$ of plane curves which is invariant under projective transformations and such that for any point $x_k \in \mathbf{J}^k$ there is a unique curve $L = L(x_k) \in \mathfrak{M}$ with k-jet x_k, i.e. such that $x_k = [L]_a^k$, where $a = \pi_k(x_k)$.

Then $(k+1)$-jets $x_{k+1} = [L(x_k)]_a^{k+1}$ can be taken as basic points in the affine line $\pi_{k+1,k}^{-1}(x_k)$ and the corresponding section $\mathfrak{m} : \mathbf{J}^k \to \mathbf{J}^{k+1}$ we'll consider as the zero section in the line bundle $\pi_{k+1,k} : \mathbf{J}^{k+1} \to \mathbf{J}^k$.

Let now $L \subset \mathbb{C}\mathbf{P}^2$ be an arbitrary curve. Then curves $L^{(k+1)} \subset \mathbf{J}^{k+1}$ and $\mathfrak{m}\left(L^{(k)}\right) \subset \mathbf{J}^{k+1}$ differs on element $\Theta_L \in \mathbf{S}^{k+1}T_L^* \otimes \nu_L$. The last tensor is a projective differential invariant of order $(k+1)$ in the sense that $\varphi^*\left(\Theta_{\varphi(L)}\right) = \Theta_L$,for arbitrary projective transformation φ.

Let's now realize this scheme for different classes of projective curves.

3.1. Straight Lines

Let $\mathfrak{M}$ be now the class of straight lines. Then, for any point $x_1 \in \mathbf{J}^1$ one can find a unique straight line $L(x_1)$, such that $x_1 = [L(x_1)]_a^1$. The above construction gives projective differential invariant of order 2

$$\Theta_{2L} \in \mathbf{S}^2 T_L^* \otimes \nu_L.$$

It is easy to check, that, if $L = L_h$ is the graph of function $u = h(x)$ in the affine coordinates, then the restriction of tensor Θ_2 on this curve has the form: $\Theta_{2L} = h''(x)\ \frac{dx^2}{2!} \otimes \overline{\partial}_u$. Let $\Theta_2 = u_2\ \frac{dx^2}{2!} \otimes \overline{\partial}_u$. Then $\Theta_{2L} = \Theta_2|_{L_h^{(2)}}$. Denote by $\Pi_2 \subset \mathbf{J}^2$ a submanifold, where $\Theta_2 = 0$.

Then the points $\Pi_2(L) = \Pi_2 \cap L^{(2)}$ are precisely *inflection* or *flex points* on the curve, i.e. points where tangent lines have 2-rd order contact with the curve.

3.2. Quadrics

Let $\mathfrak{M}$ be now the class of quadrics. Taking derivatives of a general quadric and eliminating its coefficients we get the Monge equation: $9\,u_5 {u_2}^2 + 40\,{u_3}^3 - 45\,u_2 u_4 u_3 = 0$, or

$$u_5 = \frac{5u_3u_4}{u_2} - \frac{40}{9}\frac{u_3^3}{u_2^2}.$$

Therefore, for any point $x_4 \in \mathbf{J}^4 \setminus \pi_{4,1}^{-1}(\Pi_2)$ there is a unique quadric $Q(x_4)$ such that $[Q(x_4)]_a^4 = x_4$.

Follow the above observation, we get projective differential invariant $\Theta_{5L} \in \mathbf{S}^5 T_L^* \otimes \nu_L$, where

$$\Theta_{5L} = \left(h^{(5)} - 5\frac{h^{(3)}h^{(4)}}{h^{(2)}} + \frac{40}{9}\frac{\left(h^{(3)}\right)^3}{\left(h^{(2)}\right)^2}\right)\frac{dx^5}{5!} \otimes \overline{\partial}_u, \quad \text{or} \quad \Theta_5 = \left(u_5 - \frac{5u_3u_4}{u_2} + \frac{40}{9}\frac{u_3^3}{u_2^2}\right)\frac{dx^5}{5!} \otimes \overline{\partial}_u$$

in the domain $\mathbf{J}^5 \setminus \pi_{5,2}^{-1}(\Pi_2)$.

Denote by $\Pi_5 \subset \mathbf{J}^5 \setminus \pi_{5,2}^{-1}(\Pi_2)$ the submanifold, where $\Theta_5 = 0$. Then the points $\Pi_5(L) = \Pi_5 \cap L^{(5)}$ will be called *Monge points.* In other words, the Monge points are exactly the points where osculating quadrics have 5-th order contact with the curve.

3.3. Cubics

Let $\mathfrak{M}$ be now the class of cubics on the projective plane.

Taking derivatives of general cubic up to order 9 and eliminating its coefficients we arrive at equation (see, for example, [8]): $u_2P_7u_9 + P_8 = 0$, where P_8 is a polynomial of degree 10 and order 8, $P_7 = 7\,(60)^{-3}\ \det(M_7)$ is a a polynomial of degree 8 and order 7, and

$$M_7 = \begin{Vmatrix} 120\,u_3 & 30\,u_4 & 6\,u_5 & u_6 & u_7/7 \\ 360\,u_2 & 120\,u_3 & 30\,u_4 & 6\,u_5 & u_6 \\ -180\,{u_2}^2 & 0 & 20\,{u_3}^2 & 10\,u_3u_4 & 2\,u_3u_5 + 5{u_4}^2/4 \\ 0 & 180\,{u_2}^2 & 120\,u_3u_2 & 30\,u_4u_2 + 20\,{u_3}^2 & 6\,u_5u_2 + 10\,u_3u_4 \\ 0 & 0 & 180\,{u_2}^2 & 180\,u_3u_2 & 60\,{u_3}^2 + 45\,u_4u_2 \end{Vmatrix}$$

more explicitly

$$\begin{aligned} P_7 = &- 33600\,u_2u_3^6u_4 - 810\,u_2^5u_3u_4u_7 + 1134\,u_2^5u_3u_5u_6 - 756\,u_2^4u_3^2u_5^2 + 13230\,u_2^4u_3u_4^2u_5- \\ &- 2835\,u_2^5u_4u_5^2 - 12600\,u_2^3u_3^3u_4u_5 - 189\,u_2^6u_6^2 - 7875\,u_2^3u_3^2u_4^3 + 720\,u_2^4u_3^3u_7 - 4725\,u_2^4u_4^4 + \\ &+ 11200\,u_3^8 + 1890\,u_2^5u_4^2u_6 + 6720\,u_2^2u_3^5u_5 + 31500\,u_2^2u_3^4u_4^2 - 3150\,u_2^4u_3^2u_4u_6 + 162\,u_2^6u_5u_7. \end{aligned}$$

Therefore, $u_9 = -\frac{P_8}{u_2P_7}$ for cubics. In other words, for any point $x_8 \in \mathbf{J}^8 \backslash \left(\pi_{8,2}^{-1}(\Pi_2) \cup \pi_{8,7}^{-1}(\Pi_7)\right)$, where $\Pi_7 = P_7^{-1}(0) \subset \mathbf{J}^7$, there is a unique cubic $Q(x_8)$ such that $[Q(x_x)]_a^8 = x_8$.

Therefore, as above, for any curve L we have projective differential invariant

$$\Theta_{9L} \in \mathbf{S}^9T_L^* \otimes \nu_L, \quad \text{where} \quad \Theta_9 = \left(u_9 + \frac{P_8}{u_2P_7}\right)\frac{dx^9}{9!} \otimes \overline{\partial}_u$$

in jet coordinates in the domain $\mathbf{J}^9 \setminus \left(\pi_{9,2}^{-1}(\Pi_2) \cup \pi_{9,7}^{-1}(\Pi_7)\right)$.

Denote by $\Pi_9 \subset \mathbf{J}^9 \setminus \left(\pi_{9,2}^{-1}(\Pi_2) \cup \pi_{9,7}^{-1}(\Pi_7)\right)$ the submanifold, where $\Theta_9 = 0$.

Then the points

$$\Pi_9(L) = \Pi_9 \cap L^{(9)}$$

will be called *the Monge cubic points.* Those are the points where osculating cubics have 9-th order contact with the curve.

4. Projective orbits in jet spaces

4.1. Orbits in 2-jet space

At first, let's remark that the action of the projective group on the manifold of 1-jets is transitive.

It is easy to check that the stabilizer $\mathbf{St}_1 \subset \mathbf{SL}_3(\mathbb{C})$ of point $(0,0,0) \in \mathbf{J}^1$ is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \quad \text{with} \quad a_{11}a_{22}a_{33} = 1.$$

The action of $\mathbf{St}_1$ on the fibre of projection $\pi_{2,1} : \mathbf{J}^2 \to \mathbf{J}^1$ has the form:

$$A^{(2)} : (0,0,0,u_2) \longmapsto \left(0,0,0,a_{11}^{-3}\,u_2\right).$$

Therefore, there is the only one open regular orbit $\Pi_{20} = \mathbf{J}^2 \setminus \Pi_2$ and the singular orbit Π_2:

$$\mathbf{J}^2 = \Pi_{20} \cup \Pi_2 .$$

Elements $p_{20} = (0,0,0,1) \in \Pi_{20}, \quad p_2 = (0,0,0,0) \in \Pi_2$ can be taken as representatives of these orbits.

4.2. Orbits in 3-jet space

Consider the action of the stabilizer of point $(0,0,0,1)$ from the regular orbit Π_{20} on the fibre of projection $\pi_{3,2} : \mathbf{J}^3 \to \mathbf{J}^2$.

This stabilizer $\mathbf{St}_{2,0}$ of the regular point p_{20} is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} .$$

Their action in the fibre of projection $\pi_{2,1} : \mathbf{J}^3 \to \mathbf{J}^2$ is affine :

$$A^{(3)} : (0,0,0,1,u_3) \longmapsto (0,0,0,1, \alpha_A \; u_3 + \beta_A) ,$$

where $\alpha_A = a_{33} a_{11}^{-1}$, $\beta_A = 3 \left(a_{11} a_{31} - a_{12} a_{33} \right) a_{11}^{-2}$. Therefore, $\Pi_{30} = \pi_{3,2}^{-1} \left(\Pi_{20} \right)$ is the open regular orbit.

The stabilizer $\mathbf{St}_2$ of the singular point $(0,0,0,0)$ is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

which act in the following way: $A^{(3)} : (0,0,0,0,u_3) \mapsto \left(0,0,0,0, \frac{a_{33}}{a_{11}^4} u_3 \right)$. Therefore, the preimage $\pi_{3,2}^{-1} \left(\Pi_2 \right)$ of the singular orbit is a union of two orbits $\Pi_{32} = \{ (x, u, u_1, 0, 0) \}$ and $\Pi_{31} = \{ (0,0,0,0,\lambda) \, , \lambda \neq 0 \}$.

Therefore, the space of 3-jets has the following decomposition of $\mathbf{SL}_3 \left(\mathbb{C} \right)$-action:

$$\mathbf{J}^3 = \Pi_{30} \cup \Pi_{31} \cup \Pi_{32} ,$$

where Π_{30} is the regular open orbit. The points

$$p_{30} = (0,0,0,1,0) \in \Pi_{30}, \; p_{31} = (0,0,0,0,1) \in \Pi_{31}, \; p_{32} = (0,0,0,0,0) \in \Pi_{32}$$

can be taken as representatives of these orbits.

4.3. Orbits in 4-jet space

At first, we consider the action of the stabilizer $\mathbf{St}_{3,0}$ of the regular point $(0,0,0,1,0)$ on the fibre of the projection $\pi_{4,3} : \mathbf{J}^4 \to \mathbf{J}^3$. This stabilizer is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{11} a_{31} a_{33}^{-1} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

with the following affine action:

$$A^{(4)}:(0,0,0,1,0,u_4)\longmapsto\left(0,0,0,1,0,a_{33}^2a_{11}^{-2}\ u_4+\left(6a_{33}a_{32}-3a_{31}^2\right)a_{11}^{-2}\right).$$

Therefore, $\Pi_{40}=\pi_{4,3}^{-1}(\Pi_{30})$ is the open regular orbit.

The stabilizer $\mathbf{St}_{3,1}$ of point $(0,0,0,0,1)$ from the singular orbit Π_{31} is formed by matrices

$$A=\begin{Vmatrix}a_{11} & a_{12} & 0\\ 0 & a_{11}^2a_{33}^{-1} & 0\\ a_{31} & a_{32} & a_{33}\end{Vmatrix}$$

with the affine action $A^{(4)}:(0,0,0,0,1,u_4)\mapsto(0,0,0,0,1,a_{33}a_{11}^{-1}\ u_4+8a_{31}a_{11}^{-1})$. Therefore, $\Pi_{41}=\pi_{4,3}^{-1}(\Pi_{31})$ is an orbit.

Finally, the stabilizer $\mathbf{St}_{3,2}$ of point $(0,0,0,0,0)$ from orbit Π_{32} is formed by matrices

$$A=\begin{Vmatrix}a_{11} & a_{12} & 0\\ 0 & a_{22} & 0\\ a_{31} & a_{32} & a_{33}\end{Vmatrix}$$

with the action $A^{(4)}:(0,0,0,0,0,u_4)\mapsto(0,0,0,0,0,a_{33}^3a_{22}a_{11}^{-4}\ u_4)$. Therefore, the preimage $\pi_{4,3}^{-1}(\Pi_{32})$ of the singular orbit Π_{32} is a union of two orbits $\Pi_{43}=\{(x,u,u_1,0,0,0)\}$ and $\Pi_{42}=\pi_{4,3}^{-1}(\Pi_{31})\setminus\Pi_{43}$. Summarizing, we see that there is the only one open regular orbit Π_{40} and three singular orbits Π_{41}, Π_{42} and Π_{43}: $\mathbf{J}^4=\Pi_{40}\cup\Pi_{41}\cup\Pi_{42}\cup\Pi_{43}$. The points

$$p_{40}=(0,0,0,1,0,0)\in\Pi_{40},\quad p_{41}=(0,0,0,0,1,0)\in\Pi_{41},$$
$$p_{42}=(0,0,0,0,0,1)\in\Pi_{42},\quad p_{43}=(0,0,0,0,0,0)\in\Pi_{43}$$

can be taken as representatives of these orbits.

4.4. Orbits in 5-jet space

Let's begin with preimage of regular orbit Π_{40}.

The stabilizer $\mathbf{St}_{4,0}$ of point $(0,0,0,1,0,0)$ is formed by matrices

$$A=\begin{Vmatrix}a_{11} & a_{11}a_{31}a_{33}^{-1} & 0\\ 0 & a_{11}^2a_{33}^{-1} & 0\\ a_{31} & \frac{a_{31}^2a_{33}}{2} & a_{33}\end{Vmatrix}$$

and has the following action on the fibre:

$$A^{(5)}:(0,0,0,1,0,0,u_5)\longmapsto\left(0,0,0,1,0,0,a_{33}^3a_{11}^{-3}\ u_5\right).$$

Therefore, the preimage $\pi_{5,4}^{-1}(\Pi_{40})$ of the regular orbit is a union two orbits: the singular one Π_5 and the open regular orbit $\Pi_{50}=\pi_{5,2}^{-1}(\Pi_{20})\setminus\Pi_5$.

The stabilizer $\mathbf{St}_{4,1}$ of point $(0,0,0,0,1,0)$ from the singular orbit Π_{41} is formed by matrices

$$A=\begin{Vmatrix}a_{11} & a_{12} & 0\\ 0 & a_{11}^3a_{33}^{-2} & 0\\ 0 & a_{32} & a_{33}\end{Vmatrix}$$

and acts in the following way

$$A^{(5)} : (0,0,0,0,1,0,u_5) \longmapsto \left(0,0,0,0,1,0,a_{33}^2 a_{11}^{-2}\ u_5 - 10 a_{12} a_{33}^2 a_{11}^{-3}\right).$$

Therefore, $\Pi_{51} = \pi_{5,4}^{-1}(\Pi_{41})$ is an orbit.

The stabilizer $\mathbf{St}_{4,2}$ of point $(0,0,0,0,0,1)$ from the singular orbit Π_{42} is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^4 a_{33}^{-3} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

and acts as $A^{(5)} : (0,0,0,0,0,1,u_5) \mapsto (0,0,0,0,0,1,a_{33}a_{11}^{-1}\ u_5 + 15 a_{31} a_{11}^{-1})$. Therefore, $\Pi_{52} = \pi_{5,4}^{-1}(\Pi_{42})$ is an orbit too.

Finally, the stabilizer $\mathbf{St}_{4,3}$ of point $(0,0,0,0,0,0)$ from the singular orbit Π_{43} is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

and acts as $A^{(5)} : (0,0,0,0,0,0,u_5) \mapsto (0,0,0,0,0,1,a_{33}^3 a_{11}^{-6}\ u_5)$. Therefore, the preimage $\pi_{5,4}^{-1}(\Pi_{43})$ is a union of two orbits: $\Pi_{54} = \{(x,u,u_1,0,0,0,0)\}$ and $\Pi_{53} = \pi_{5,4}^{-1}(\Pi_{43}) \setminus \Pi_{54}$.

Summarizing, we conclude that $\mathbf{SL}_3(\mathbb{R})$-action in $\mathbf{J}^5$ has the orbit decomposition:

$$\mathbf{J}^5 = \Pi_{50} \cup \Pi_5 \cup \Pi_{51} \cup \Pi_{52} \cup \Pi_{53} \cup \Pi_{54},$$

where Π_{50} is the unique regular open orbit.

The points

$$p_{50} = (0,0,0,1,0,0,1) \in \Pi_{50}, \qquad p_5 = (0,0,0,1,0,0,0) \in \Pi_5,$$

$$p_{51} = (0,0,0,0,1,0,0) \in \Pi_{51}, \qquad p_{52} = (0,0,0,0,0,1,0) \in \Pi_{52},$$

$$p_{53} = (0,0,0,0,0,0,1) \in \Pi_{53}, \qquad p_{54} = (0,0,0,0,0,0,0) \in \Pi_{54}$$

can be taken as representatives of the orbits.

4.5. Orbits in 6-jet space

Let's begin with preimage of the regular orbit Π_{50}. Then the stabilizer $\mathbf{St}_{5,0}$ of the point p_{50} is formed by matrices

$$A = \begin{Vmatrix} a_{33} & a_{31} & 0 \\ 0 & a_{33} & 0 \\ a_{31} & \frac{1}{2} a_{31} a_{33}^{-1} & a_{33} \end{Vmatrix} \quad \text{with} \quad a_{33}^3 = 1.$$

Its action in the fibre has the following form

$$A^{(6)} : (0,0,0,1,0,0,1,u_6) \mapsto (0,0,0,1,0,0,1,u_6 + \frac{3a_{31}}{a_{33}}),$$

and therefore $\Pi_{60} = \pi_{6,5}^{-1}(\Pi_{50})$ is an open regular orbit.

The stabilizer $\mathbf{St}_5$ of the point p_5 is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{11}a_{31}a_{33}^{-1} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & \frac{1}{2}a_{31}^2 a_{33}^{-1} & a_{33} \end{Vmatrix}, \quad \text{where} \quad a_{11}^3 = 1,$$

which acts as: $A^{(6)} : (0,0,0,1,0,0,0,u_6) \mapsto (0,0,0,1,0,0,0,a_{33}^4 a_{11}^{-1} u_6)$. Therefore, preimage $\pi_{6,5}^{-1}(\Pi_5)$ is a union of three orbits $\pi_{6,5}(\Pi_5) = \Pi_{61} \cup \Pi_{62}$ with the following representatives

$$p_{61} = (0,0,0,1,0,0,0,1), p_{62} = (0,0,0,1,0,0,0,0).$$

The stabilizer $\mathbf{St}_{5,1}$ of the point $p_{51} = (0,0,0,0,1,0,0) \in \Pi_{51}$ generates by matrices

$$A = \begin{Vmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^3 a_{33}^{-2} & 0 \\ 0 & a_{32} & a_{33} \end{Vmatrix},$$

where $a_{11}^4 = a_{33}$. This groups acts in the fibre in the following way:

$$A^{(6)} : (0,0,0,0,1,0,0,u_6) \longmapsto \left(0,0,0,1,0,0,0,a_{11}^9 u_6 + 40 a_{11}^5 a_{32}\right).$$

Therefore, the preimage $\Pi_{63} = \pi_{6,5}^{-1}(\Pi_{51})$ is an orbit.

The stabilizer $\mathbf{St}_{5,2}$ of the point $p_{52} = (0,0,0,0,0,1,0) \in \Pi_{52}$ formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^4 a_{33}^{-3} & 0 \\ 0 & a_{32} & a_{33} \end{Vmatrix},$$

where $a_{11}^5 = a_{33}^2$, and acts in the following way

$$A^{(6)} : (0,0,0,0,0,1,0,u_6) \longmapsto \left(0,0,0,1,0,0,0,a_{11}^3 u_6\right).$$

Therefore, $\pi_{6,5}^{-1}(\Pi_{52})$ is a union of orbits $\pi_{6,5}^{-1}(\Pi_{52}) = \Pi_{64} \cup \Pi_{65}$ with representatives

$$p_{64} = (0,0,0,0,0,1,0,1), p_{65} = (0,0,0,0,0,1,0,0).$$

The stabilizer $\mathbf{St}_{5,3}$ of the point $p_{53} = (0,0,0,0,0,0,1) \in \Pi_{53}$ formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^5 a_{33}^{-4} & 0 \\ a_{13} & a_{32} & a_{33} \end{Vmatrix},$$

where $a_{11}^6 = a_{33}^3$, and acts in the following way:

$$A^{(6)} : (0,0,0,0,0,0,1,u_6) \longmapsto \left(0,0,0,0,0,0,1,a_{33}a_{11}^{-1}u_6 + 24\ a_{31}a_{11}^{-1}\right).$$

Therefore, $\Pi_{66} = \pi_{6,5}^{-1}(\Pi_{53})$ is an orbit with representative $p_{66} = (0,0,0,0,0,0,1,0)$. Finally, the stabilizer $\mathbf{St}_{5,4}$ of the point $p_{54} = (0,0,0,0,0,0,1) \in \Pi_{54}$ is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{13} & a_{32} & a_{33} \end{Vmatrix},$$

where $a_{11}a_{22}a_{33} = 1$, and acts in the following way:

$$A^{(6)} : (0,0,0,0,0,0,0,u_6) \longmapsto (0,0,0,1,0,0,0,a_{33}^4 a_{11}^7 u_6).$$

Therefore, the preimage $\pi_{6,5}^{-1}(\Pi_{54})$ is a union of two orbits Π_{67} and Π_{68} with representatives

$$p_{67} = (0,0,0,0,0,0,0,1) \text{ and } p_{68} = (0,0,0,0,0,0,0,0)$$

respectively.

Summarizing, we get the following result.

Theorem 1. $\mathbf{SL}_3(\mathbb{C})$*-action in* $\mathbf{J}^6$ *splits into the following orbit decomposition:*

$$\mathbf{J}^6 = \Pi_{60} \cup \Pi_{61} \cup \Pi_{62} \cup \Pi_{63} \cup \Pi_{64} \cup \Pi_{65} \cup \Pi_{66} \cup \Pi_{67} \cup \Pi_{68},$$

where Π_{60} *is the only open regular orbit.*

These orbits have the following representatives

$$\begin{array}{lll} p_{60} = (0,0,0,1,0,0,1,0), & p_{61} = (0,0,0,1,0,0,0,1), & p_{62} = (0,0,0,1,0,0,0,0), \\ p_{63} = (0,0,0,0,1,0,0,0), & p_{64} = (0,0,0,0,0,1,0,1), & p_{65} = (0,0,0,0,0,1,0,0), \\ p_{66} = (0,0,0,0,0,0,1,0), & p_{67} = (0,0,0,0,0,0,0,1), & p_{68} = (0,0,0,0,0,0,0,0). \end{array}$$

As a corollary of this theorem we get the following $\mathbf{SL}_3(\mathbb{C})$-classification of 6-jets of projective curves.

Theorem 2. *Let* $L \subset \mathbb{C}\mathbf{P}^2$ *be a projective curve. Then for any point* $a \in L$ *there are projective coordinates* (x, y) *such that* $x(a) = y(a) = 0$ *and the curve can be written in the form* $y = p(x) + \varepsilon(x)$*, where function* $\varepsilon(x)$ *has seventh order of smallness and polynomial* $p(x)$ *has one of the following form:*

$$\begin{array}{lll} p_{60}(x) = x^2 + x^5, & p_{61}(x) = x^2 + x^6, & p_{62}(x) = x^2, \\ p_{63}(x) = x^3, & p_{64} = x^4 + x^6, & p_{65} = x^4, \\ p_{66} = x^5, & p_{67} = x^6, & p_{68} = 0, \end{array}$$

where the polynomials p_{ij} *correspond to the orbits* Π_{ij}.

4.6. Stabilizers of regular orbit

The open orbit $\Pi_{60} = \mathbf{J}^6 \setminus \pi_{6,2}(\Pi_2) \setminus \pi_{6,5}(\Pi_5)$ as well as its elements will be called *regular*.

A point $a \in L$ on a projective curve we call *regular* if $[L]_a^6 \in \Pi_{60}$; in the opposite case it will be called *singular*.

It worth to note that our definitions differ from the standard ones: both regular and singular points belong to smooth complex curve, and their singularity has projective nature.

Remark also that the previous theorem states that the regular orbit is connected even though singular orbits Π_2 and Π_5 have codimension 1.

Before to consider differential invariants of projective curves we'll finish this section by description of stabilizers of regular point in $\mathbf{J}^k$, when $k = 2, 3, 4, 5, 6$.

Take 2-jet $p_{20} = (0,0,0,1)$. Then the stabilizer is a 4-dimensional group and consist of matrices

$$\mathbf{St}_2 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & \beta^{-1} & 0 \\ \gamma & \delta & \beta \end{Vmatrix} : \alpha, \gamma, \delta \in \mathbb{C},\ \beta \in \mathbb{C}^* \right\}.$$

For 3-jet $p_{30} = (0,0,0,1,0)$ the stabilizer is a 3 -dimensional group and consist of matrices

$$\mathbf{St}_3 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & \beta^{-1} & 0 \\ \alpha\beta & \gamma & \beta \end{Vmatrix} : \alpha, \gamma \in \mathbb{C},\ \beta \in \mathbb{C}^* \right\}.$$

For 4-jet $p_{40} = (0,0,0,1,0,0)$ the stabilizer is a 2 -dimensional group and consist of matrices

$$\mathbf{St}_4 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & \beta^{-1} & 0 \\ \alpha\beta & \frac{1}{2}\alpha^2\beta & \beta \end{Vmatrix} : \alpha \in \mathbb{C},\ \beta \in \mathbb{C}^* \right\}.$$

For 5-jet $p_{50} = (0,0,0,1,0,0,1)$ the stabilizer is a 1-dimensional group and consist of matrices

$$\mathbf{St}_5 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ \alpha & \frac{1}{2}\alpha^2 & 1 \end{Vmatrix} : \alpha \in \mathbb{C} \right\},$$

and for the 6-jet $p_{60} = (0,0,0,1,0,0,1,0)$ the stabilizer is trivial.

5. Projective Differential Invariants

5.1. Relative Invariants

Recall that a function f on the k-jet space $\mathbf{J}^k$ is said to be a *relative projective differential invariant of order* $\leq k$, if $f \circ g^{(k)} = C\left(g^{-1}\right) f$, for all element $g \in \mathbf{SL}_3(\mathbb{C})$, and a 1-cocycle C on the group.

An infinitesimal version of this states that $\mathbf{L}_{X^{(k)}}(f) = c(X)\ f$ for all vectors $X \in \mathfrak{sl}_3(\mathbb{C})$, and a 1-cocycle c on the Lie algebra.

Here we denote by $X^{(k)}$ the prolongation of the vector field X to the space of k-jets, and by $\mathbf{L}_{X^{(k)}}$ the correspondent Lie derivative.

To find relative invariants, we remark that, as we have seen, zeroes of functions

$$P_2 = u_2 \ \text{ and } \ P_5 = u_5 - \frac{5u_3u_4}{u_2} + \frac{40}{9}\frac{u_3^3}{u_2^2}$$

determine singular orbits Π_2 and Π_5. Therefore, these functions are relative invariants of the $\mathbf{SL}_3(\mathbb{C})$-action. Indeed, it can be easily check that $X^{(2)}(P_2) = \alpha_2(X) \cdot P_2$, where

$$X{=}(2a_{1,1}x{+}a_{2,2}x{+}a_{1,2}u{+}a_{1,3}{-}a_{3,1}x^2{-}a_{3,2}xu)\partial_x{+}(a_{1,1}u{+}2a_{2,2}u{+}a_{2,1}x{+}a_{2,3}{-}a_{3,1}xu{-}a_{3,2}u^2)\partial_u$$

is a general element of Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Here $\alpha_2(X) = -3\left(a_{1,2} - a_{3,2}x\right)u_1 - 3\,a_{1,1} + 3\,a_{3,1}x$ is the corresponding 1-cocycle.

In the similar way, $X^{(5)}(P_5) = \alpha_5(X) \cdot P_5$, where

$$\alpha_5(X) = -6\,(a_{1,2} - a_{3,2}x)\,u_1 + 3\,a_{3,2}u - 9\,a_{1,1} + 9\,a_{3,1}x - 3\,a_{2,2}.$$

For the same reasons zeros of the function P_7 define cubics, and therefore this function is a relative invariant. Indeed, $X^{(7)}(P_7) = \alpha_7(X) \cdot P_7$, where

$$\alpha_7(X) = -32\,(a_{1,2} - a_{3,2}x)\,u_1 - 40\,a_{1,1} + 40\,a_{3,1}x - 8\,a_{2,2} + 8\,a_{3,2}u.$$

Cocycles α_2, α_5 and α_7 are not independent, and obey the relation $16\alpha_2 + 8\alpha_5 - 3\alpha_7 = 0$.

Another relative invariant we can get from the volume form $\Omega = dx \wedge du$, because

$$X(\Omega) = \alpha_0(X)\Omega,$$

where $\alpha_0(X) = 3\,a_{1,1} + 3\,a_{2,2} - 3\,a_{3,1}x - 3\,a_{3,2}u$. This is not independent 1-cocycle, and we have the relation $\alpha_0 - 2\alpha_2 + \alpha_5 = 0$.

The last relative invariant can be obtained from the contact form $\omega = du - u_1dx$. In this case $X^{(1)}(\omega) = \alpha_1(X)\,\omega$, where 1-cocycle α_1 has the form

$$\alpha_1(X) = -(a_{1,2} - a_{3,2}x)u_1 + a_{1,1} + 2\,a_{2,2} - a_{3,1}x - 2\,a_{3,2}u.$$

This cocycle is linearly dependent with the previous ones: $2\alpha_0 - 3\alpha_1 + \alpha_2 = 0$.

These relations between 1-cocycles allow us to construct the following invariant tensors.

Theorem 3. *The following tensors on jet spaces are* $\mathbf{SL}_3(\mathbb{C})$*-invariants:*

Function $Q_7 = \dfrac{P_7^3}{P_5^8 P_2^{16}}$,

Differential 1**-form** $\omega_5 = \dfrac{P_7}{P_5^2\,P_2^7}\,\omega$,

Differential 2**-form** $\Omega_5 = \dfrac{P_5}{P_2^2}\,\Omega$.

5.2. Algebra of projective differential invariants

Let's denote by τ_k and ν_k the vector bundles on $\mathbf{J}^k$ induced by projection $\pi_{k,1}$ from the canonical bundles τ_1, ν_1 on $\mathbf{J}^1$, where

$$\tau_1\left([L]_a^1\right) = T_aL \quad \text{and} \quad \nu_1\left([L]_a^1\right) = T_a\mathbf{P}^2 \big/ T_aL.$$

As we have seen symmetric differential forms

$$\Theta_2 = u_2\frac{dx^2}{2!} \otimes \overline{\partial}_u \in S^2(\tau_2^*) \otimes \nu_2 \quad \text{and} \quad \Theta_5 = 60\,\sigma \cdot \Theta_2.$$

The form σ will be referred to a the *Study* 3*-form.* This form is obviously $\mathbf{SL}_3(\mathbb{C})$-invariant and in affine coordinates can be written by:

$$\sigma = \frac{P_5}{P_2}\,dx^3.$$

In addition to the Study form we introduce a Study derivation as a total derivation ∇ such that

$$\sigma(\nabla,\nabla,\nabla) = Q_7.$$

In affine coordinates this derivation has form

$$\nabla = \frac{P_7}{P_5^3 P_2^5}\frac{d}{dx}.$$

This is a $\mathbf{SL}_3(\mathbb{C})$-invariant derivation.

It is easy to check that the invariant Q_7 is an affine function in u_7 having the form

$$Q_7 = \frac{P_2^2}{P_5^5}u_7^3 + \cdots .$$

Applying the Study derivation we get an 8-th differential invariant

$$Q_8 = \nabla(Q_7) = \frac{Q_7\ P_2}{P_5^2}u_8 + \cdots , \tag{1}$$

and

$$Q_{k+1} = \nabla(Q_k),$$

for $k \geq 7$. All of these invariants are rational functions on the jet spaces which are defined on the preimages of regular orbit Π_{60}.

Let us specify now the notion of a differential invariant.

First of all remark that all bundles $\pi_{k,k-1} : \mathbf{J}^k \to \mathbf{J}^{k-1}$ are affine, when $k \geq 2$, and $\mathbf{J}^2$ is a total space of the bundle over $\mathbb{C}\mathbf{P}^2$ with fibres $\mathbb{C}\mathbf{P}^1$.

Therefore, all manifolds $\mathbf{J}^k$ are algebraic and we can talk about functions which are rational.

We say that a rational function f on manifold k-jets $\mathbf{J}^k$ is a $\mathbf{SL}_3(\mathbb{C})$-*differential invariant* (or simply *projective differential invariant*) of order k if $X^{(k)}(f) = 0$ for any vector field $X \in \mathfrak{sl}_3(\mathbb{C})$.

Therefore, due to the Rosenlicht theorem (see, [9]) differential invariants $Q_7, \ldots, Q_k$, separate regular $\mathbf{SL}_3(\mathbb{C})$-orbits in $\mathbf{J}^k$ and we arrive at the following result.

Theorem 4. *1. Any projective differential invariant of order k is a rational function of invariants $Q_7, \ldots, Q_k$.*

2. The field of differential invariants of order $\leq k$ separates regular orbits in $\mathbf{J}^k$.

6. Projective equivalence of algebraic plane curves

6.1. $\mathbf{SL}_3(\mathbb{C})$- action

Let L and $\widetilde{L}$ be an algebraic plane curves, and let $L^{(k)}, \widetilde{L}^{(k)} \subset \mathbf{J}^k$ be their prolongations. We say that L and $\widetilde{L}$ are *projectively equivalent* if $g(L) = \widetilde{L}$, for some element $g \in \mathbf{SL}_3(\mathbb{C})$.

All curves in this section are irreducible and not straight lines or quadrics. Then the values $Q_k(L) = Q_k|_{L^{(k)}}$ of invariants Q_k on the curve L are well defined.

The function $Q_7(L)$ we will call *projective curvature of the curve* (cf. [5]).

We will consider curves L such that the function $Q_8(L) \neq 0$, i.e., because $\nabla(Q_7) = Q_8$, L is not is a curve of constant projective curvature. Functions $Q_7(L)$ and $Q_8(L)$ are rational functions on L, and therefore they satisfy an algebraic relation

$$F(Q_7(L), Q_8(L)) = 0. \tag{2}$$

Denote by $\Sigma_L = F^{-1}(0) \subset \mathbb{C}^2$ the curve defining by (2).

We call this curve *defining curve,* and the minimal F (in 2) *defining function.*

It follows from the construction of the defining curve, that two projectively equivalent algebraic curves have the same defining curve. Moreover, the following result holds.

Theorem 5. *Two irreducible algebraic plane curves L and $\widetilde{L}$, which are not straight lines or quadrics, are projectively equivalent if and only if their defining curves coincide.*

Proof. Let's prove the sufficiency. First of all, function $Q_7(L)$ might be considered as local coordinate on L in an open domain. Then, in this domain, relation 2 can be viewed as a relation $Q_8(L) - \Phi(Q_7(L)) = 0$, for an analytical function Φ.

Let's consider now relation

$$Q_8 - \Phi(Q_7) = 0 \tag{3}$$

in jet space of the 8-th order as ordinary differential equation. Remark, that both curves L and $\widetilde{L}$ are local solutions of this equation. Moreover, relation (1) shows that solutions of the above differential equation are uniquely defined by their 8-jets.

Let us take points $a \in L$ and $\widetilde{a} \in \widetilde{L}$ from the corresponding domains, where the invariant Q_7 is a local coordinate such that $Q_7(L)(a) = Q_7(\widetilde{L})(\widetilde{a})$.

Then there is a projective transformation φ, which equalize 7-jets, $\varphi^{(7)}([L]^7_a) = [\widetilde{L}]^7_{\widetilde{a}}$. It follows from the fact that Q_7 is the only projective invariant of the order ≤ 7.

Relation (2) shows that $\varphi^{(8)}([L]^8_a) = [\widetilde{L}]^8_{\widetilde{a}}$. Remark that, projective transformations are symmetries of differential equation (3). Hence, $\varphi(L)$ is a solution (3) too. But 8-jets of $\widetilde{L}$ and $\varphi(L)$ at point $\widetilde{a}$ equal. Therefore, due to the uniqueness of solutions, $\widetilde{L} = \varphi(L)$. □

6.2. Cubics

As an example of application of the above theorem let's consider cubic curves. As we have seen these curves are solutions of equation

$$u_2 P_7 u_9 + P_8 = 0.$$

The left hand side of the equation is an obviously relative invariant.

This invariant can be rewritten in terms of invariants as follows:

$$\frac{P_2^5 P_5^5}{Q_7^2}\left(Q_9 Q_7 - \frac{11}{8}Q_8^2 - \frac{7}{72}Q_7 Q_8 - \frac{216}{35}Q_7^3 - \frac{49}{21600}Q_7^2\right).$$

Therefore, if the cubic curves, which satisfy the above Theorem, are solutions of the 9-th order differential equation

$$Q_9 Q_7 - \frac{11}{8}Q_8^2 - \frac{7}{72}Q_7 Q_8 - \frac{216}{35}Q_7^3 - \frac{49}{21600}Q_7^2 = 0. \tag{4}$$

Let Φ be the defining function of a cubic. Then, applying the Study derivative to the relation $Q_8 = \Phi(Q_7)$, we get $Q_9 = \Phi'(Q_7)\,\Phi(Q_7)$.

Relation (4) can be rewritten now as a differential equation for defining function $\Phi(\tau)$:

$$\frac{343}{36} - 259200\,\tau^3 - 12600\,\tau\Phi\Phi' + 14175\,\Phi^2 + 1225\,\Phi = 0.$$

Integrating this equation we get the following relation between invariants Q_7 and Q_8 which depends on arbitrary constant c and has the following form $F^3 + cG{Q_7}^9 = 0$, where

$$\begin{aligned} F =& \frac{49}{147456}\,Q_8^4 + \frac{343}{3317760}\,Q_8^3 + \left(\frac{2401}{199065600} + \frac{7}{192}\,Q_7^3\right) Q_8^2 + \\ &+ \left(-\frac{49}{25920}\,Q_7^3 + \frac{16807}{26873856000}\right) Q_8 + \left(Q_7^3 - \frac{343}{1036800}\right)\left(Q_7^3 - \frac{343}{9331200}\right) \end{aligned}$$

and

$$\begin{aligned} G =& 117649 - 6401203200\,Q_7^3 + 18151560\,Q_8 + 583443000\,Q_8^2 + 87071293440000\,Q_7^6 - \\ &- 493807104000\,Q_7^3 Q_8 + 3174474240000\,Q_7^3 Q_8^2 + 7001316000\,Q_8^3 + 28934010000\,Q_8^4. \end{aligned}$$

In other words, regular cubics are projectively defined by constant c.

REFERENCES

1. J.L. Coolidge, *A treatise on algebraic plane curves*, Dover Publ. NY, (1959), 513 p.
2. G.H. Halphen, *Sur les invariants différentiels*, Paris: Gauthier-Villars, (1878), 60 p.
3. F. Klein, W. Blaschke, *Vorlesungen über höhere Geometrie*, Berlin, J. Springer, (1926), 405 p.
4. F. Klein, S. Lie, *Uber diejenigen ebenen Curven, weiche aurch ein geschlossenes System von einfach unedlich veilen vertauschbaren linearen Transformationen in sich ubergehen*, Math. Annalen, **4** (1871), 50–85.
5. N. Konovenko, V. Lychagin, *On projective classification of plane curves*, Global and Stochastic Analysis, **1**:2 (2011), 241–264.
6. B. Kruglikov, V. Lychagin, *Global Lie-Tresse theorem*, preprint (2011), 38 p.; avilable at arXiv:1111.5480.
7. E.P. Lane, *Projective differential geometry of curves and surfaces*, The University of Chicago press, Chicago-Illinois, (1932).
8. A. Lascoux, *The differential equation satisfied by plane curve of degree n*, Bull. Sci. Math. **130**:4 (2006), 354–359.
9. M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1958), 401–443.
10. E.J. Wilczynski, *Interpretation of the symplest integral invariant of projective geometry*, Proc. Nat. Acad. Sci. USA, **2**:4 (1916), 248–252.

Received 15.11.2012

Revised 11.07.2013