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CONTINUITY POINTS OF SEPARATELY CONTINUOUS FUNCTIONS WITH VALUES IN $\aleph_0$-SPACES

Taras Banakh

Faculty of Mechanics and Mathematics, Ivan Franko National University of Lviv, Universytetska 1, Lviv

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It is proved that for any separately continuous function (more generally, KC-function) $f : X \times Y \to Z$ with values in an $\aleph_0$-space Z and any subset $S \subset Y$ having a countable base in Y the set $\{x \in X : \{x\} \times S \subset C(f)\}$ is comeager in X. On the other hand, there exists a separately continuous map $f : 2^\omega \times [0,\omega] \to Z$ to a Fréchet-Urysohn cosmic space Z with a unique non-isolated point such that $D(f) = 2^\omega \times \{\omega\}$.

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Доведено, що для нарізно неперервної функції (загальніше, KC-функції) $f : X \times Y \to Z$ зі значеннями у $\aleph_0$-просторі Z та підмножини $S \subset Y$ зі зліченною базою в Y, множина $\{x \in X : \{x\} \times S \subset C(f)\}$ залишкова в X. Також побудовано космічний простір Фреше-Урисона Z з єдиною неізольованою точкою та нарізно неперервну функцію $f : 2^\omega \times [0,\omega] \to Z$ з $D(f) = 2^\omega \times \{\omega\}$.

This short note is another step in numerous generalizations [7], [11]–[17], of the classical theorem of Baire [1] describing the largeness properties of the set $C(f)$ of continuity points of a separately continuous function $f : X \times Y \to Z$. For functions with values in metrizable spaces, one of the most general results was proved by Bouziad and Troallic [4]. To formulate their result we need to recall the notion of lower quasicontinuous multivalued function.

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E-mail: t.o.banakh@gmail.com

A multivalued function $F : X \multimap Y$ between topological spaces is called *lower quasicontinuous* at a point $x \in X$ if for any neighborhood $O_x \subset X$ of x and any open set $U \subset Y$ intersecting the set $F(x)$, there exists a non-empty open set $V \subset O_x$ such that $F(v) \cap U \neq \varnothing$ for all $v \in V$. We say that a multivalued function $F : X \multimap Y$ is *lower quasicontinuous* if it is lower quasicontinuous at each point $x \in X$.

It is clear that a function $f : X \to Y$ between topological spaces is *quasi-continuous* if and only if the multivalued function $F : X \multimap Y$, $F : x \mapsto \{f(x)\}$, is lower quasicontinuous.

A family $\mathcal{B}$ of open sets of a topological space Y is called a *base* at a subset $S \subset X$ if for any open set $U \subset Y$ and a point $y \in S \cap U$ there exists a set $B \in \mathcal{B}$ such that $y \in B \subset U$. We shall say that a subset S of a topological space Y has *countable base* in Y if Y has a countable base at S.

The following theorem was proved by Bouziad and Troallic in [4].

Theorem 1 (Bouziad-Troallic). *Let $f : X \times Y \to Z$ be a function defined on the product of two topological spaces with values in a metrizable space Z. Assume that the space Y has a countable base $\mathcal{B}$ at a subset $S \subset Y$ such that*

1) *for every $B \in \mathcal{B}$ the multivalued function $F_B : X \multimap Z$, $F_B : x \mapsto f(\{x\} \times B)$, is lower quasicontinuous;*
2) *for every $x \in X$ the function $f^x : Y \to Z$, $f^x : y \mapsto f(x, y)$, is continuous at each point $y \in S$.*

Then the set $\{x \in X : \{x\} \times S \subset C(f)\}$ is comeager in X.

We recall that a subset A of a topological space X is *comeager* if its complement $X \setminus A$ is *meager* in X, which means that $X \setminus A$ is a countable union of nowhere dense sets.

A function $f : X \times Y \to Z$ is called a *KC-function* if

- for every $y \in Y$ the function $f_y\colon X \to Z$, $f_y\colon x \mapsto f(x, y)$, is quasicontinuous;
- for every $x \in X$ the function $f^x\colon Y \to Z$, $f^x\colon y \mapsto f(x, y)$, is continuous.

It is clear that each separately continuous function is a KC-function.

Bouziad-Troallic Theorem 1 implies the following result of Maslyuchenko [12]. For separately continuous functions this result was proved by Calbrix and Troallic [5].

Corollary 1 (Maslyuchenko). *If $f : X \times Y \to Z$ is a KC-function defined on the product of two topological spaces with values in a metrizable space Z, then for any subset $S \subset Y$ with countable base in Y the set $\{x \in X : \{x\} \times S \subset C(f)\}$ is comeager in X.*

In this paper we show that Theorem 1 and Corollary 1 remain true for functions $f : X \times Y \to Z$ with values in an $\aleph_0$-space Z (i.e., regular spaces possessing a countable k-network).

A family $\mathcal{N}$ of subsets of a topological space X is called

- a *network* if for any open set $U \subset X$ and point $x \in U$ there exists a set $N \in \mathcal{N}$ such that $x \in N \subset U$;
- a *k-network* if for any open set $U \subset X$ and compact subset $K \subset U$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ with $K \subset \bigcup \mathcal{F} \subset U$.

It is clear that for a family $\mathcal{B}$ of subsets of a topological space we have the implications: (base of the topology $\Rightarrow$ k-network $\Rightarrow$ network).

A topological space X is called

- *cosmic* if X is regular and has a countable network;
- *$\aleph_0$-space* if X is regular and has a countable k-network.

For any topological space we have the implications:

$$\text{metrizable separable} \Rightarrow \text{an } \aleph_0\text{-space} \Rightarrow \text{cosmic}.$$

Cosmic spaces and $\aleph_0$-spaces form two important classes of generalized metric spaces, which are closed under many topological operations, see [6, §4 and §11]. These spaces have nice characterizations in terms of their function spaces $C_p(X)$ and $C_k(X)$. Here for a topological space X we denote by $C_p(X)$ and $C_k(X)$ the linear space of continuous real-valued functions on X, endowed with the topology of pointwise convergence and the compact-open topology, respectively.

The following characterization of cosmic spaces is well-known, see [8, 4.1.3], [6, 4.9].

Theorem 2. *For a Tychonoff space X the following conditions are equivalent:*

1) *X is cosmic;*
2) *the function space $C_p(X)$ is cosmic;*
3) *X is the image of a separable metrizable space M under a continuous surjective map $f : M \to X$.*

$\aleph_0$-Spaces have a similar (and even more interesting) characterization involving function spaces $C_k(X)$ and also subproper and compact-covering maps.

A map $f : X \to Y$ between topological spaces is called

- *compact-covering* if each compact subset $K_Y \subset Y$ coincides with the image $f(K_X)$ of some compact subset $K_X \subset X$;
- *subproper* if f admits a function $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$ and for every compact set $K \subset Y$ the closure $\overline{s(K)}$ of $s(K)$ in X is compact.

It is clear that each subproper map is compact-covering. Subproper maps were introduced and studied in [3].

Theorem 3. *For a Tychonoff space X the following conditions are equivalent:*

1) *X is an $\aleph_0$-space;*
2) *the function space $C_k(X)$ is an $\aleph_0$-space;*
3) *the function space $C_k(X)$ is cosmic;*
4) *X is the image of a separable metrizable space M under a continuous subproper map $f : M \to X$;*
5) *X is the image of a separable metrizable space M under a continuous compact-covering map $f : M \to X$.*

Proof. The implication (1) ⇒ (2) is a classical result of Michael [9] (see also [6, 11.5]), (2) ⇒ (3) is trivial and (3) ⇒ (1) can be found in [8, 4.1.3]. The implication (1) ⇒ (4) is proved in [3, 7.2] and (4) ⇒ (5) ⇒ (1) are trivial (for the last implication, see [6, p.494]). □

The following theorem is the main result of this note.

Theorem 4. *Let $f : X \times Y \to Z$ be a function defined on the product of two topological spaces with values in an $\aleph_0$-space Z. Assume that the space Y has a countable base $\mathcal{B}_Y$ at some subset $S \subset Y$ such that*

1) *for every $B \in \mathcal{B}_Y$ the multivalued function $F_B : X \multimap Z$, $F_B : x \mapsto f(\{x\} \times B)$, is lower quasicontinuous;*
2) *for every $x \in X$ the function $f^x : Y \to Z$, $f^x : y \mapsto f(x, y)$, is continuous at each point $y \in S$.*

Then the set $\{x \in X : \{x\} \times S \subset C(f)\}$ is comeager in X.

Proof. By Theorem 3, the function space $C_k(Z)$ is cosmic and by Theorem 2, $C_k(Z)$ is the image of a metrizable separable space T under a continuous map $\xi : T \to C_k(Z)$. It will be convenient to denote the image $\xi(t) \in C_k(Z)$ of an element $t \in T$ by ξ_t.

Now consider the function

$$g : X \times Y \times T \to \mathbb{R}, \quad g : (x, y, t) \mapsto \xi_t(f(x, y)).$$

Fix any countable base $\mathcal{B}_T$ of the topology of the metrizable separable space T. It is clear that the countable family $\mathcal{B} = \{U \times V : U \in \mathcal{B}_Y,\ V \in \mathcal{B}_T\}$ is a base of $Y \times T$ at the subset $S \times T \subset Y \times T$.

We claim that for any set $B = U \times V \in \mathcal{B}$ the multivalued function

$$G_B : X \multimap \mathbb{R}, \quad G_B : x \mapsto g(\{x\} \times U \times V),$$

is lower quasicontinuous at each point $x \in X$. Fix any neighborhood $O_x \subset X$ and any open set $W \subset \mathbb{R}$ containing some point $g(x, y, t) = \xi_t(f(x, y))$ with $(y, t) \in U \times V = B$. The continuity of the function $\xi_t \in C_k(Z)$ yields a neighborhood $O_{f(x,y)} \subset Z$ of $f(x, y)$ such that $\xi_t(O_{f(x,y)}) \subset W$.

By the lower quasicontinuity of the multivalued function $F_U : X \multimap Z$, $F_U : x' \mapsto f(\{x'\} \times U)$, there exists a non-empty open set $O'_x \subset O_x$ such that for every $x' \in O'_x$ there exists $y' \in U$ such that $f(x', y') \in O_{f(x,y)}$. Then $g(x', y', t) = \xi_t(f(x', y')) \in \xi_t(O_{f(x,y)}) \subset W$ and hence $G_B(x') \cap W = g(\{x'\} \times U \times V) \cap W \ni g(x', y', t)$ is not empty.

Next, we show that for every $x \in X$ the map $g^x : Y \times T \to \mathbb{R}$, $g^x : (y, t) \mapsto g(x, y, t)$, is continuous at each point (y, t) of the set $S \times T$. Since the space $Y \times T$ is first-countable at (y, t), it suffices to show that for any sequence $\{(y_n, t_n)\}_{n\in\omega} \subset Y \times T$ that converges to (y, t), the sequence $\big(g(x, y_n, t_n)\big)_{n\in\omega}$ converges to $g(x, y, t)$ in $\mathbb{R}$. The continuity of the function f^x at y ensures that the sequence $\big(f(x, y_n)\big)_{n\in\omega}$ converges to $f(x, y)$ in Z. The continuity of the map $\xi : T \to C_k(Z)$ guarantees that the function sequence $(\xi_{t_n})_{n\in\omega}$ converges to ξ_t in $C_k(Z)$. Now the definition of compact-open topology on $C_k(Z)$ implies that the sequence $\big(g(x, y_n, t_n)\big)_{n\in\omega} = \big(\xi_{t_n}(f(x, y_n))\big)_{n\in\omega}$ converges to $g(x, y, t) = \xi_t(f(x, y))$.

Now we can apply Theorem 1 and conclude that the set $R := \{x \in X : \{x\} \times S \times T \subset C(g)\}$ is comeager in X. We claim that $R \times S \subset C(f)$.

Assuming that f is discontinuous at some point $(x, y) \in R \times S$, we can find a neighborhood $O_{f(x,y)} \subset Z$ of $f(x, y)$ in Z such that $f(O_{(x,y)}) \not\subset O_{f(x,y)}$ for any neighborhood $O_{(x,y)} \subset X \times Y$ of (x, y). The $\aleph_0$-space Z is Tychonoff (being regular is Lindelöf). So, we can find a continuous function $\varphi : Z \to [0, 1]$ such that $\varphi(f(x, y)) = 0$ and $\varphi^{-1}([0, 1)) \subset O_{f(x,y)}$. Since $\varphi \in C_k(Z) = \xi(T)$, there exists $t \in T$ such that $\xi_t = \varphi$. It follows that $g(x, y, t) = \xi_t(f(x, y)) = 0$. By the continuity of the function g at (x, y, t), there exists a neighborhood $O_{(x,y)} \subset X \times Y$ such that $g(O_{(x,y)} \times \{t\}) \subset [0, 1)$. It follows that $\varphi(f(O_{(x,y)})) = \xi_t(f(O_{(x,y)})) = g(O_{(x,y)} \times \{t\}) \subset [0, 1)$ and hence $f(O_{(x,y)}) \subset \varphi^{-1}([0, 1)) \subset O_{f(x,y)}$, which contradicts the choice of the neighborhood $O_{f(x,y)}$. This contradiction shows that $R \times S \subset C(f)$ and hence the set $\{x \in X : \{x\} \times S \subset C(f)\} \supset R$ is comeager in X. □

Corollary 2. *If $f : X \times Y \to Z$ is a KC-function defined on the product of two topological spaces with values in an $\aleph_0$-space Z, then for any subset $S \subset Y$ with countable base in Y the set $\{x \in X : \{x\} \times S \subset C(f)\}$ is comeager in X.*

Corollary 2 implies the following result proved in [2, Corollary 3.10].

Corollary 3. *Let X be a topological space, Y be a second-countable topological space and Z be an $\aleph_0$-space. For any KC-function $f : X \times Y \to Z$ the set $D(f)$ of discontinuity points of f has meager projection on X.*

Theorem 4 and Corollary 2 cannot be generalized to cosmic spaces as shown by the following example, in which $2^\omega = \{0,1\}^\omega$ is the Cantor cube and $[0,\omega] = \omega \cup \{\omega\}$ is the closed segment of ordinals, endowed with the interval topology. So, $[0,\omega]$ is a compact countable space with a unique non-isolated point ω. Let us recall that a topological space X is *Fréchet-Urysohn* if for any subset $A \subset X$ and point $a \in \bar{A}$, there exists a sequence $\{a_n\}_{n\in\omega} \subset A$ that converges to a.

Example 1. There exist a Fréchet-Urysohn cosmic space Z with a unique non-isolated point, and a separately continuous function $f : 2^\omega \times [0,\omega] \to Z$ such that $C(f) = 2^\omega \times [0,\omega)$ and $D(f) = 2^\omega \times \{\omega\}$.

Proof. Let $2^{<\omega} = \bigcup_{n\in\omega} 2^n$, where $2 = \{0,1\}$. Observe that for any function $x \in 2^\omega$ and any $n \in \omega$ the restriction $x|n$ belongs to $2^n \subset 2^{<\omega}$.

Choose any point $\infty \notin 2^{<\omega}$ that does not belong to $2^{<\omega}$ and consider the space $Z = \{\infty\} \cup 2^{<\omega}$, endowed with the topology τ consisting of sets $U \subset Z$ having the following property: if $\infty \in U$, then for any $x \in 2^\omega$ there exists $n \in \omega$ such that $x|m \in U$ for all $m \ge n$ in ω. It is easy to see that the space Z is regular (being a T_1-space with a unique non-isolated point) and cosmic (being countable). The definition of the topology τ ensures that the space Z is Fréchet-Urysohn.

Now consider the function $f : 2^\omega \times [0,\omega] \to Z$ defined by

$$f(x,n) = \begin{cases} x|n & \text{if } n < \omega; \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to check (sf. [10, Theorem 1]) that f is separately continuoous and $C(f) = 2^\omega \times [0,\omega)$, so $D(f) = 2^\omega \times \{\omega\}$. □

Remark 1. The space Z from Example 1 is stratifiable (being a regular countable space with a unique non-isolated point).

A regular topological space X is called an $\aleph$-*space* if X has a σ-discrete k-network [6, §11]. The class of (paracompact) $\aleph$-spaces contains all metrizable spaces and all $\aleph_0$-spaces.

Problem 5. *Can Theorem 4 be generalized to functions with values in (paracompact) $\aleph$-spaces Z?*

The following proposition from [2, 4.5] yields a partial answer to Problem 5.

Proposition 1. *If $f : X \times Y \to Z$ is a separately continuous function defined on the product of a countably cellular space X and a second-countable space Y with values in an $\aleph$-space Z, then the set $\{x \in X : \{x\} \times Y \subset C(f)\}$ is comeager in X.*

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REFERENCES

1. R. Baire, *Sur les fonctions de variables reélles*, Ann. Math. Pura Appl. **3** (1899) 1–123.
2. T. Banakh, *Quasicontinuous and separately continuous functions with values in Maslyuchenko spaces*, Topology App. **230** (2017) 353–372.
3. T.Banakh, V.Bogachev, A.Kolesnikov, *k^*-Metrizable spaces and their applications*, J. Math. Sci. **155**:4 (2008) 475–522.
4. A. Bouziad, J.-P. Troallic, *Lower quasicontinuity, joint continuity and related concepts*, Topology Appl. **157**:18 (2010) 2889–2894.
5. J. Calbrix, J.-P. Troallic, *Applications séparément continues*, C. R. Acad. Sci. Paris Sér. A-B **288**:13 (1979), A647–A648.
6. G. Gruenhage, *Generalized metric spaces*, in: Handbook of set-theoretic topology, 423–501, North-Holland, Amsterdam, 1984.
7. V. Maslyuchenko, V. Nesterenko, *A new generalization of Calbrix-Troallic's theorem*, Topology Appl. **164** (2014), 162–169.
8. R. McCoy, I. Ntantu, *Topological properties of spaces of continuous functions*, Lecture Notes in Math. **1315**, Springer-Verlag (1988), 124 p.
9. E. Michael, *$\aleph_0$-spaces*, J. Math. Mech. **15** (1966) 983–1002.
10. Т.О. Банах, В.К. Маслюченко, О.І. Філіпчук, *Приклади нарізно неперервних відображень з суцільними розривами на горизонталях*, Мат. вісник НТШ, **14** (2017) 58–69.
11. В.К. Маслюченко, *Нарізно неперервні відображення зі значеннями в індуктивних границях*, Укр. мат. журн. **44**:3 (1992), 380–384.
12. В.К. Маслюченко, *Простори Гана і задача Діні*, Мат. методи і фіз.-мех. поля, **41**:4 (1998), 39–45.
13. В.К. Маслюченко, *Нарізно неперервні відображення від багатьох змінних зі значеннями в σ-метризовний просторах*, Неленійні коливання, **2**:3 (1999), 337–344.
14. В.К. Маслюченко, В.В. Михайлюк, О.І. Філіпчук, *Точки сукупної неперервності нарізно неперервних відображень зі значеннями в площині Немицького*, Мат. Студії, **26**:2 (2006), 217–221.
15. В.К. Маслюченко, В.В. Михайлюк, О.І. Філіпчук, *Сукупна неперервність K_hC-функцій зі значеннями в просторах Мура*, Укр. мат. журн., **60**:11 (2008) 1539–1549.
16. В.К. Маслюченко, В.В. Михайлюк, О.І. Шишина, *Сукупна неперервність горизонтально квазінеперервних відображень зі значеннями в σ-метризовних просторах*, Мат. методи і фіз-мех. поля. **45**:1 (2002) 42–46.
17. В.К. Маслюченко, В.В. Нестеренко, *Сукупна неперервність і квазінеперервність горизонтально квазінеперервних відображень*, Укр. мат. журн. **52**:12 (2000) 1711–1714.
18. В.К. Маслюченко, В.В. Нестеренко, *Точки сукупної неперервності та великі коливання*, Укр. мат. журн. **62**:6 (2010) 791–800.

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