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CONDITIONS OF SEMISCALAR EQUIVALENCE OF POLYNOMIAL 2-BY-2 MATRICES

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We consider the problem of determining whether two polynomial matrices are semiscalarly equivalent. Large difficulties in this problem arise already for matrices of second order. In this connection under certain restrictions the necessary and sufficient conditions of semiscalar equivalence of 2-by-2 polynomial matrices are found.

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Досліджується проблема визначення, коли дві поліноміальні матриці є напівскалярно еквівалентні. Значні труднощі у цій проблемі виникають вже для матриць другого порядку. У зв'язку з цим, при певних обмеженнях вказані необхідні та достатні умови напівскалярної еквівалентності 2×2 поліноміальних матриць.

Introduction

The notion of semiscalar equivalence of matrices was introduced to algebra by P.S. Kazimirskii and V.M. Petrychkovych in 1977. By definition [1], two polynomial matrices are called semiscalarly equivalent if they can be transformed one to the other by multiplying from the left by a nonsingular numerical matrice and from the right by

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an invertible polynomial matrice. The notion of *PS*-equivalence of matrices was introduced by J.A. Dias da Silva and T.J. Laffey in 1999. By definition [2], polynomial matrices $A(x)$ and $B(x)$ are called *PS*-equivalent provided $B(x) = P(x)A(x)Q$ for some invertible matrices Q – independent of x and $P(x)$ – dependent on x. We consider the problem of determining whether two matrices are semiscalarly equivalent. It contains the classical linear algebra problem of reducing a pair of numerical matrices to a canonical form by simultaneous similarity transformations. This problem has not been extensively studied in the literature. Large difficulties in this problem arise already for matrices of second order.

In this paper the semiscalar equivalence for one class of polynomial matrices of second order is investigated. In particular, necessary and sufficient conditions are found under which two 2-by-2 matrices over the ring $\mathbb{C}[x]$ are semiscalarly equivalent.

We consider the ring $M(2,\ \mathbb{C}[x])$ of order two polynomial matrices over the field of complex numbers $\mathbb{C}$. Let $A(x) \in M(2,\ \mathbb{C}[x])$. Suppose that the rank of matrix $A(x)$ is full. According to [1] (see also [3], Section IV, § 1 or [4]) the matrix $A(x)$ is semiscalarly equivalent to a matrix of lower triangular form with invariant multipliers on the main diagonal. Without loss of generality, we can assume that the first invariant multiplier of the considered matrix is identity. Therefore, this matrix can be considered in the form

$$A(x) = \left\| \begin{array}{cc} 1 & 0 \\ a(x) & \delta(x) \end{array} \right\|, \quad 0 < \deg a(x) < \deg \delta(x) \tag{1}$$

since the case with $a(x) \equiv 0$ is trivial.

In this paper we use the standard notations. In particular, $a^{(t)}(\alpha)$ is the value at $x = \alpha$ of the t-th derivative of the polynomial $a(x) \in \mathbb{C}[x]$ for $t \in \mathbb{N}$. Further, by the symbol $\bar{0}$ we denote the zero column of arbitrary height. The transpose operation is denoted by the symbol T. Let $M(2,\ \mathbb{C})$, $M(2,\ \mathbb{C}[x])$ denote the rings of 2×2 matrices over $\mathbb{C}$, $\mathbb{C}[x]$, respectively, and let $GL(2,\ \mathbb{C})$, $GL(2,\ \mathbb{C}[x])$ be their groups of units, respectively.

The determinant $|A(x)|$ is called the characteristic polynomial of $A(x)$ and its roots are called the characteristic roots of matrix $A(x)$. In this article the characteristic roots of a matrix can be of arbitrary multiplicity. The question of finding a complete set of invariants for *PS*-equivalence in the case 2-by-2 matrices without multiple characteristic roots is discussed in [2]. A different case of semiscalar equivalence of polynomial matrices of second order is considered in [5].

1. Main results

Let us denote by M the set of characteristic roots of a matrix $A(x)$ of the form (1). It is clear that the set M is an invariant of the matrix $A(x)$ with respect to semiscalar equivalence.

On the set M consider the equivalence relation

$$E = \{(\alpha, \beta) \in M \times M : a(\alpha) = a(\beta)\},$$

which determined a partition

$$M = \{E(\alpha) : \alpha \in M\} \tag{2}$$

of M into the equivalence classes $E(\alpha) := \{\beta \in M : (\alpha, \beta) \in E\}$.

The following two assertions yield some invariants of the matrix $A(x)$ with respect to semiscalar equivalence.

Proposition 1.1. *The partition* (2) *of the set* M *of characteristic roots of matrix* $A(x)$ *of the form* (1) *is invariant for the class* $\{CA(x)Q(x)\}$ *of semiscalarly equivalent matrices.*

Proof. Let the matrices $A(x)$ (1) and

$$B(x) = \begin{Vmatrix} 1 & 0 \\ b(x) & \Delta(x) \end{Vmatrix}, \quad \deg b(x) < \deg \Delta(x), \tag{3}$$

be semiscalarly equivalent, i.e.

$$SA(x)R(x) = B(x), \tag{4}$$

where $S \in GL(2, \mathbb{C})$ and $R(x) \in GL(2, \mathbb{C}[x])$. Introducing the notations $S = \|s_{ij}\|_1^2$, $R^{-1}(x) = \|r_{ij}(x)\|_1^2$, we deduce from (4) the relations

$$s_{11} + s_{12}a(x) = r_{11}(x), \tag{5}$$

$$s_{12}\delta(x) = r_{12}(x), \tag{6}$$

$$s_{21} + s_{22}a(x) = b(x)r_{11}(x) + \delta(x)r_{21}(x). \tag{7}$$

Setting $x = \alpha$ and $x = \beta$, $\alpha \neq \beta$, $\alpha,\ \beta \in M$, we obtain the equalities

$$s_{21} + s_{22}a(\alpha) = b(\alpha)r_{11}(\alpha) \text{ and } s_{21} + s_{22}a(\beta) = b(\beta)r_{11}(\beta).$$

If $a(\alpha) = a(\beta)$, then left sides of the resulting relations are equal. Therefore, from the equality of right sides, taking into account $r_{11}(\alpha) = r_{11}(\beta) \neq 0$ (see (5), (6)), we have $b(\alpha) = b(\beta)$. The notion of semiscalar equivalence is a symmetric relation. Then from $b(\alpha) = b(\beta)$ a similar argument yields $a(\alpha) = a(\beta)$. This completes the proof. □

Proposition 1.2. *Let α be a characteristic root of multiplicity n of a matrix $A(x)$ of form* (1). *Let also m be the lowest (non-zero) order of the non-zero derivative $a^{(m)}(\alpha) \neq 0$ of the entry $a(x)$ of this matrix at $x = \alpha$. Then the number m (and n) is invariant for the class $\{CA(x)Q(x)\}$ of semiscalarly equivalent matrices, if $m < n$.*

Proof. Let a matrix $A(x)$ (1) be semiscalarly equivalent to a matrix $B(x)$ (3) and m' be the lowest (non-zero) order of the non-zero derivative $b^{(m)}(\alpha) \neq 0$ of the entry $b(x)$ of the matrix $B(x)$ at $x = \alpha$. Suppose that $m < m'$. From relations (5) and (7) we obtain

$$s_{21} + s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x) = \delta(x)r_{21}(x). \tag{8}$$

Substituting $x = \alpha$ into (8), we find

$$s_{21} + s_{22}a(\alpha) - s_{11}b(\alpha) - s_{12}a(\alpha)b(\alpha) = 0. \tag{9}$$

Differentiating both sides of equality (8) m times at $x = \alpha$, we obtain

$$s_{22}a^{(m)}(\alpha) - s_{12}a^{(m)}(\alpha)b(\alpha) = 0.$$

The division of both sides of obtained equality by $a^{(m)}(\alpha) \neq 0$ and the substitution in (9) yields

$$\begin{cases} s_{22} - s_{12}b(\alpha) = 0, \\ s_{21} - s_{11}b(\alpha) = 0. \end{cases}$$

It is impossible, since the matrix $\|s_{ij}\|_1^2$ is nonsingular. Therefore, $m \geq m'$. Taking into account that the semiscalar equivalence is a symmetric relation, we conclude that $m = m'$. □

In this paper, we shall restrict our attention to considering the case in which the partition (2) of the set M has only two equivalence classes:

$$M = M_1 \cup M_2. \tag{10}$$

Recall that the invariance of the partition (2) was established in Proposition 1.

Since the matrix $A(x)$ has full rank, the entry $a(x)$ of the matrix $A(x)$ (1) satisfies the inequality $a(\alpha) \neq 0$ for some root $\alpha \in M$.

Theorem 1.3. *Let the partition* (2) *of the set M of characteristic roots of matrices $A(x)$* (1) *and $B(x)$* (3) *be of the form* (10). *Let also n_j be the multiplicity of an arbitrary root $\alpha_j \in M$; m_j be the lowest (non-zero) order of the non-zero derivatives*

$a^{(m_j)}(\alpha_j) \neq 0, b^{(m_j)}(\alpha_j) \neq 0$ *of the entries* $a(x), b(x)$ *of the matrices* $A(x), B(x)$ *at* $x = \alpha_j$*;*

$$a(x) = a_j + \sum_{t=0}^{s-m_j-1} a_{jt}(x-\alpha_j)^{m_j+t}, a_j = a(\alpha_j), a_{j0} \neq 0, \tag{11}$$

$$b(x) = b_j + \sum_{t=0}^{s-m_j-1} b_{jt}(x-\alpha_j)^{m_j+t}, b_j = b(\alpha_j), b_{j0} \neq 0, \tag{12}$$

are binomial decomposition of the entries $a(x), b(x)$ *into degrees of* $x - \alpha_j$ *if* $m_j < n_j$*;* $s = \deg \delta(x)$*;* $a_i = b_i = 0$ *for some* $\alpha_i \in M$*. Matrices* $A(x), B(x)$ *are semiscalarly equivalent if and only if there exists a number* $c \neq 0$ *satisfying the following conditions:*

1) $a_{i0} = cb_{i0}$ *and*

$$\begin{vmatrix} a_{i1} & a_{i2} & \dots & a_{i,s_i-1} & a_{is_i} \\ a_{i0} & a_{i1} & \dots & a_{i,s_i-2} & a_{i,s_i-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & a_{i1} & a_{i2} \\ 0 & & & a_{i0} & a_{i1} \end{vmatrix} = c^{s_i} \begin{vmatrix} b_{i1} & b_{i2} & \dots & b_{i,s_i-1} & b_{is_i} \\ b_{i0} & b_{i1} & \dots & b_{i,s_i-2} & b_{i,s_i-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & b_{i1} & b_{i2} \\ 0 & & & b_{i0} & b_{i1} \end{vmatrix}, \tag{13}$$

$s_i = 1, \dots, m_i - 1, m_i + 1, \dots, n_i - m_i - 1$, *for every root* $\alpha_i \in M$ *such that* $a_i = b_i = 0, m_i < n_i$*;*

2)

$$c\left(\frac{a_{i,m_i}}{a_{i0}^2} - \frac{a_{l,m_l}}{a_{l0}^2}\right) = \frac{b_{i,m_i}}{b_{i0}^2} - \frac{b_{l,m_l}}{b_{l0}^2} \tag{14}$$

for every pair of roots $\alpha_i, \alpha_l \in M$ *such that* $a_i = b_i = a_l = b_l = 0, 2m_i < n_i, 2m_l < n_l$*;*

3)

$$a_{i,m_i} + \frac{a_{i0}^2}{a_p} = c\left(b_{i,m_i} + \frac{b_{i0}^2}{b_p}\right) \tag{15}$$

for every root $\alpha_i \in M$ *such that* $a_i = b_i = 0, 2m_i < n_i$ *and for some root* $\alpha_p \in M$ *such that* $a_p, b_p \neq 0$*;*

4)

$$c\underbrace{\begin{vmatrix} 0 & \dots & 0 & a_{p0} & a_{p1} & \dots & a_{ps_p} \\ a_p & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & a_{p1} \\ & & \ddots & \ddots & \ddots & \ddots & a_{p0} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & a_p & 0 \end{vmatrix}}_{m_p+s_p} = \frac{a_p^{m_p+s_p+1}}{b_p^{m_p+s_p+1}} \underbrace{\begin{vmatrix} 0 & \dots & 0 & b_{p0} & b_{p1} & \dots & b_{ps_p} \\ b_p & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & b_{p1} \\ & & \ddots & \ddots & \ddots & \ddots & b_{p0} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & b_p & 0 \end{vmatrix}}_{m_p+s_p}, \tag{16}$$

$s_p = 0,\ 1,\ \dots,\ n_p - m_p - 1$, *for every root* $\alpha_p \in M$ *such that* $a_p,\ b_p \neq 0$, $m_p < n_p$.

2. Auxiliary statements

We first obtain two lemmas, which will be used in the proof of Theorem 1.3.

Lemma 2.1. *For the existence of the non-zero solution of the equation*

$$\left\| \begin{matrix} a_0 & b_0 & 0 \\ a_1 & b_1 & 0 \\ \vdots & \vdots & \vdots \\ a_{m-1} & b_{m-1} & 0 \\ a_m & b_m & a_0 b_0 \\ a_{m+1} & b_{m+1} & a_0 b_1 + a_1 b_0 \\ \vdots & \vdots & \vdots \\ a_k & b_k & \sum_{t=0}^{k-m} a_t b_{k-m-t} \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\| = \bar{0}, \tag{17}$$

over $\mathbb{C}$, *where* $a_0,\ b_0 \neq 0$, *it is necessary and sufficient that the following conditions hold:*

$$\begin{vmatrix} a_1 & a_2 & \dots & a_{v-1} & a_v \\ a_0 & a_1 & \dots & a_{v-2} & a_{v-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & a_1 & a_2 \\ 0 & & & a_0 & a_1 \end{vmatrix} = c^v \begin{vmatrix} b_1 & b_2 & \dots & b_{v-1} & b_v \\ b_0 & b_1 & \dots & b_{v-2} & b_{v-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & b_1 & b_2 \\ 0 & & & b_0 & b_1 \end{vmatrix}, \tag{18}$$

$v = 1,\ \dots,\ m-1,\ m+1,\ \dots,\ k, c = a_0/b_0$.

Under the conditions of Lemma every non-zero solution $\|x_{10}\ x_{20}\ x_{30}\|^T$ *of the equation* (17) *it is has* $x_{10},\ x_{20} \neq 0$.

Proof. **Necessity**. Let the equation (17) have a non-zero solution $\|x_{10}\ x_{20}\ x_{30}\|^T$. Then

$$\begin{cases} a_0x_{10} + b_0x_{20} = 0, \\ a_1x_{10} + b_1x_{20} = 0, \\ \dots\dots\dots\dots \\ a_{m-1}x_{10} + b_{m-1}x_{20} = 0, \end{cases} \tag{19}$$

and

$$\begin{cases} a_mx_{10} + b_mx_{20} + a_0b_0x_{30} = 0, \\ a_{m+1}x_{10} + b_{m+11}x_{20} + (a_0b_1 + a_1b_1)x_{30} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_kx_{10} + b_kx_{20} + x_{30}\sum_{t=0}^{k-m} a_tb_{k-m-t} = 0. \end{cases} \tag{20}$$

We assume that $x_{10} = 0$ or $x_{20} = 0$. Then, from first equalities (19) and (20) we obtain $x_{10} = x_{20} = x_{30} = 0$. This contradicts the assumption that the solution $\|x_{10}\ x_{20}\ x_{30}\|^T$ is non-zero. So, we have $x_{10},\ x_{20} \neq 0$. From equalities (19) we obtain the conditions (18), where $c = -x_{20}/x_{10} = a_0/b_0$, for $v = 1, \dots, m-1$.

By (20), if $a_m = cb_m$, then $x_{30} = 0$ and $a_h = cb_h$ for $h \in \{m+1, \dots, k\}$. For this reason in what follows $a_m \neq cb_m$, where $c = -x_{20}/x_{10} = a_0/b_0$. From the first and second equalities (20) by excluding x_{30} we obtain

$$a_0a_{m+1} - a_m(cb_1 + a_1) = c^2b_0b_{m+1} - cb_m(cb_1 + a_1). \tag{21}$$

If $m = 1$, then $a_0a_2 - a_1^2 = c^2(b_0b_2 - b_1^2)$. This means that conditions (20) are fulfilled for $v = m+1 = 2$. If $m > 1$, then $a_1 = cb_1$ and from (21) by multiplication of the both sides by $a_0^{m-1} = c^{m-1}b_0^{m-1}$ we find

$$a_0^m a_{m+1} - 2a_0^{m-1}a_1a_m = c(b_0^m b_{m+1} - 2b_0^{m-1}b_1b_m). \tag{22}$$

Denote by A_{uw}, B_{uw} the submatrices obtained, respectively, from matrices

$$\left\| \begin{matrix} a_1 & a_2 & \dots & a_m & a_{m+1} \\ a_0 & a_1 & \dots & a_{m-1} & a_m \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & a_1 & a_2 \\ 0 & & & a_0 & a_1 \end{matrix} \right\| \quad \text{and} \quad \left\| \begin{matrix} b_1 & b_2 & \dots & b_m & b_{m+1} \\ b_0 & b_1 & \dots & b_{m-1} & b_m \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & b_1 & b_2 \\ 0 & & & b_0 & b_1 \end{matrix} \right\| \tag{23}$$

by obliterating of two last columns and u-th and w-th rows. Denote also by $\Delta_{m+1}(A)$, $\Delta_{m+1}(B)$ the determinants of matrices (23), respectively. Decompose them for minors of order two that are contained in the last two columns. Because $|A_{uw}| =$

$|B_{uw}| = 0$ for $u \neq m+1$, we have

$$\Delta_{m+1}(A) = (-1)^{m+1}\Bigg(\begin{vmatrix} a_m & a_{m+1} \\ a_0 & a_1 \end{vmatrix} \cdot |A_{1,\,m+1}| -$$
$$- \begin{vmatrix} a_{m-1} & a_m \\ a_0 & a_1 \end{vmatrix} \cdot |A_{2,\,m+1}| + \ldots + \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \cdot |A_{m,\,m+1}|\Bigg),$$

and

$$\Delta_{m+1}(B) = (-1)^{m+1}\Bigg(\begin{vmatrix} b_m & b_{m+1} \\ b_0 & b_1 \end{vmatrix} \cdot |B_{1,\,m+1}| -$$
$$- \begin{vmatrix} b_{m-1} & b_m \\ b_0 & b_1 \end{vmatrix} \cdot |B_{2,\,m+1}| + \ldots + \begin{vmatrix} b_1 & b_2 \\ b_0 & b_1 \end{vmatrix} \cdot |B_{m,\,m+1}|\Bigg).$$

Since $\|a_0\ a_1 \ldots a_{m-1}\| = c\|b_0\ b_1 \ldots b_{m-1}\|$, each summand of expression in parenthesis for $\Delta_{m+1}(A)$, possibly except the first two, differs from the corresponding summand for $\Delta_{m+1}(B)$ by the multiplier c^{m+1}. From this fact and from the equality (22) the equality (18) follows for $v = m+1$.

Denote by $\Delta_v(A)$ and $\Delta_v(B)$ the determinants in the left and right sides of equality (18), respectively. Suppose by induction $\Delta_r(A) = c^r \Delta_r(B)$ for all r such that $m < r < k$. Accept for the sake of determinacy $r > 2m$. In the case where $r \leq 2m$ the proof is analogous. From the first r equalities (20) by excluding x_{30} and by sufficiently evident transformations we arrive at the system

$$\begin{cases} (a_{m+1} - (a_0b_0)^{-1}a_m \sum_{u=0}^{1} a_u b_{1-u})(-a_0)^m \Delta_{r-m}(A) = \\ \quad = c^{r+1}(b_{m+1} - (a_0b_0)^{-1}b_m \sum_{u=0}^{1} a_u b_{1-u})(-b_0)^m \Delta_{r-m}(B), \\ (a_{m+2} - (a_0b_0)^{-1}a_m \sum_{u=0}^{2} a_u b_{2-u})(-a_0)^{m+1} \Delta_{r-m-1}(A) = \\ \quad = c^{r+1}(b_{m+2} - (a_0b_0)^{-1}b_m \sum_{u=0}^{2} a_u b_{2-u})(-b_0)^{m+1} \Delta_{r-m-1}(B), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a_{r-m+1} - (a_0b_0)^{-1}a_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1} - a_{r-m+1} + \\ \quad +(a_0b_0)^{-1}a_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1})(-a_0)^{r-m} \Delta_m(A) = \\ \quad = c^{r+1}\big(b_{r-m+1} - (a_0b_0)^{-1}b_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1} - b_{r-m+1} + \\ \quad +(a_0b_0)^{-1}b_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1}\big)(-b_0)^{r-m} \Delta_m(B), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a_r - (a_0b_0)^{-1}a_m \sum_{u=0}^{r-m} a_u b_{r-m-u})(-a_0)^{r-1} \Delta_1(A) = \\ \quad = c^{r+1}(b_r - (a_0b_0)^{-1}b_m \sum_{u=0}^{r-m} a_u b_{r-m-u})(-b_0)^{r-1} \Delta_1(B), \\ (a_{r+1} - (a_0b_0)^{-1}a_m \sum_{u=0}^{r-m+1} a_u b_{r-m-u+1})(-a_0)^r = \\ \quad = c^{r+1}(b_{r+1} - (a_0b_0)^{-1}b_m \sum_{u=0}^{r-m+1} a_u b_{r-m-u+1})(-b_0)^r, \end{cases} \tag{24}$$

where $c = a_0/b_0$.

Adding the left sides of equality (24) and separately the right sides, we obtain the equality:

$$\begin{aligned}
&(-a_0)^r a_{r+1} + (-a_0)^{r-1} a_r \Delta_1(A) + \dots + (-a_0)^{r-m} a_{r-m+1} \Delta_m(A) + \dots \\
&+ (-a_0)^m a_{m+1} \Delta_{r-m}(A) + (-a_0)^{m-1} a_m \Delta_{r-m+1}(A) - \\
&- (a_0 b_0)^{-1} a_m (b_1 \Delta_{r-m}(A)(-a_0)^{m+1} + b_2 \Delta_{r-m-1}(A)(-a_0)^{m+2} + \dots \\
&+ b_{j,\, r-m_j} \delta_{j1}(A)(-a_{j0})^r) + (a_0 b_0)^{-1} a_m (b_1 \Delta_{r-m}(A)(-a_0)^{m+1} + \\
&+ b_2 \Delta_{r-m-1}(A)(-a_0)^{m+2} + \dots + b_{r-m} \Delta_1(A)(-a_0)^r + b_{r-m+1}(-a_0)^{r+1}) - \\
&- (a_{r-m+1} - (a_0 b_0)^{-1} a_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1})(-a_0)^{r-m} \Delta_m(A) = \\
&= c^{r+1}((-b_0)^r b_{r+1} + (-b_0)^{r-1} b_r \Delta_1(B) + \dots + (-b_0)^{r-m} b_{r-m+1} \Delta_m(B) + \dots \\
&+ (-b_0)^m b_{m+1} \Delta_{r-m}(B) + (-b_0)^{m-1} b_m \Delta_{r-m+1}(B) - \\
&- (a_0 b_0)^{-1} b_m (a_1 \Delta_{r-m}(B)(-b_0)^{m+1} + a_2 \Delta_{r-m-1}(B)(-b_0)^{m+2} + \dots \\
&+ a_{r-m} \Delta_1(B)(-b_0)^r) + (a_0 b_0)^{-1} b_m (a_1 \Delta_{\, r-m}(B)(-b_0)^{m+1} + \\
&+ a_2 \Delta_{r-m-1}(B)(-b_0)^{m+2} + \dots + a_{r-m} \Delta_1(B)(-b_0)^r + a_{r-m+1}(-b_0)^{r+1}) - \\
&- (b_{r-m+1} - (a_0 b_0)^{-1} b_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1})(-b_{j0})^{r-m_j} \Delta_m(B)).
\end{aligned} \tag{25}$$

Grouping similar terms in both sides of obtained equality we arrive to

$$\begin{aligned}
&(-a_0)^r a_{r+1} + (-a_0)^{r-1} a_r \Delta_1(A) + \dots + (-a_0)^{r-m} a_{r-m+1} \Delta_m(A) + \dots + \\
&+ (-a_0)^m a_{m+1} \Delta_{r-m}(A) + (-a_0)^{m-1} a_m \Delta_{r-m+1}(A) + (a_0 b_0)^{-1} a_m b_{r-m+1} (-a_0)^{r+1} - \\
&- \big(a_{r-m+1} - (a_0 b_0)^{-1} a_m \sum_{u=1}^{r-2m+1} a_u b_{r-2m-u+1}\big)(-a_0)^{r-m} \Delta_m(A) = \\
&= c^{r+1}((-b_0)^r b_{r+1} + (-b_0)^{r-1} b_r \Delta_1(B) + \dots + (-b_0)^{r-m} b_{r-m+1} \Delta_m(B) + \dots \\
&+ (-b_0)^m b_{m+1} \Delta_{r-m}(B) + (-b_0)^{m-1} b_m \Delta_{r-m+1}(B) + (a_0 b_0)^{-1} b_m a_{r-m+1} (-b_0)^{r+1} - \\
&- \big(b_{r-m+1} - (a_0 b_0)^{-1} b_m \sum_{u=1}^{r-2m+1} a_u b_{r-2m-u+1}\big)(-b_0)^{r-m} \Delta_m(B)).
\end{aligned} \tag{26}$$

It follows from (20), that

$$\begin{aligned}
a_{r-m+1} + (a_0 b_0)^{-1} a_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1} &= \\
&= c(b_{r-m+1} + (a_0 b_0)^{-1} b_m \sum_{u=0}^{r-2m+1} a_u b_{r-2m-u+1}).
\end{aligned}$$

From this relation it is easy to be sure that the following equality is true

$$\begin{aligned}
&(a_0b_0)^{-1}a_m b_{r-m+1}(-a_0)^{r+1}- \\
&\quad -\big(a_{r-m+1}-(a_0b_0)^{-1}a_m\sum_{u=0}^{r-2m+1}a_u b_{r-2m-u+1}\big)(-a_0)^{r-m}\Delta_m(A)= \\
&= c^{r+1}\big((a_0b_0)^{-1}b_m a_{r-m+1}(-b_0)^{r+1}- \\
&\quad -\big(b_{r-m+1}(a_0b_0)^{-1}b_m\sum_{u=0}^{r-2m+1}a_u b_{r-2m-u+1}\big)(-b_0)^{r-m}\Delta_m(B)\big).
\end{aligned} \tag{27}$$

From (19) and the induction hypothesis we can write

$$\begin{aligned}
&(-a_0)^{m-2}a_{m-1}\Delta_{r-m+2}(A)+\ \dots\ +(-a_0)a_2\Delta_{r-1}(A)+a_1\Delta_r(A)= \\
&c^{r+1}((-b_0)^{m-2}b_{m-1}\Delta_{r-m+2}(B)+\dots+(-b_{j0})b_{j2}\delta_{j,r-1}(B)+b_1\Delta_r(B)).
\end{aligned} \tag{28}$$

Comparing (26), (27) and (28), we obtain equality

$$\begin{aligned}
&(-a_0)^r a_{r+1}+(-a_0)^{r-1}a_r\Delta_1(A)+\dots+(-a_0)a_2\Delta_{r-1}(A)+a_1\Delta_r(A)= \\
&c^{r+1}\big((-b_0)^r b_{r+1}+(-b_0)^{r-1}b_r\Delta_1(B)+\dots+(-b_0)b_2\Delta_{r-1}(B)+b_1\Delta_r(B)\big),
\end{aligned} \tag{29}$$

i.e. $\Delta_{r+1}(A)=c^{r+1}\Delta_{r+1}(B)$, where $c=a_0/b_0$. The necessity of conditions of the Lemma is proved.

Sufficiency. Consider equalities (19) and (20) as one system of equations in three unknowns $x_{10},\ x_{20},\ x_{30}$. In conditions (18) $c=a_0/b_0$. This means that $x_{10}=1$, $x_{20}=-c$ satisfies the first equation of the system (19). From condition with $v=1$ it follows that $x_{10}=1$, $x_{20}=-c$ satisfies the second equation of (19). Next, third and all following equalities in system (19) for $x_{10}=1$, $x_{20}=-c$ can be recurrently obtained from conditions (18) with $v=2,\ \dots,\ m-1$. Evidently, the values

$$x_{10}=1, x_{20}=-c, x_{30}=(cb_m-a_m)(a_0b_0)^{-1}, \tag{30}$$

satisfy the first equation (20). We compute both determinants in equality (20) with $v=m+1$. After annihilation of equal summands on both sides of the obtained equality and after division by $a_0^m=c^m b_0^m$ with the help of simple transformations we can obtain the following relation

$$a_{m+1}-cb_{m+1}+(cb_m-a_m)(a_0b_0)^{-1}(a_0b_1+a_1b_1)=0.$$

This means that (30) satisfies the second equation of system (20).

Assume by induction that (30) satisfies the first $r-m+1$ equations of system (20), i.e.

$$\begin{cases} a_m - cb_m + (cb_m - a_m)(a_0b_0)^{-1}a_0b_0 = 0, \\ a_{m+1} - cb_{m+1} + (cb_m - a_m)(a_0b_0)^{-1}\sum_{u=0}^{1} a_u b_{1-u} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_r - cb_r + (cb_m - a_m)(a_0b_0)^{-1}\sum_{u=0}^{r-m} a_u b_{r-m-u} = 0. \end{cases} \tag{31}$$

In so doing, we may think for the sake of determinacy $r > 2m$. In opposite case the proof is analogous. Taking into account the conditions (18) and inductive assumption, we can write equalities (27), (28) and (29). From these equalities we obtain relation (26). This relation implies the equality (25). It is evident that from the second and all following equalities of (31) we find that first $r-m$ equalities of (24) are valid. The first $r-m$ equalities of (24) along with relation (25) yield the last equality of (24). This equality after shortening in $(-a_0)^r = c^r(-b_0)^r$ and after some simplifications can be written in the form

$$a_{r+1} - cb_{r+1} + (cb_m - a_m)(a_0b_0)^{-1}\sum_{u=0}^{r-m+1} a_u b_{r-m-u+1} = 0.$$

This means that (30) is the solution of $(r-m+2)$-th equation of system (20).

We inductively proved the existence of the non-zero solution (30) of systems (19) and (20). Thus, matrix equation (17) has non-zero solution. □

Lemma 2.2. *For the existence of the non-zero solution of the equation*

$$\left\| \begin{matrix} a & b & ab \\ a_0 & b_0 & ab_0 + a_0b \\ a_1 & b_{-1} & ab_1 + a_1b \\ \vdots & \vdots & \vdots \\ a_{m-1} & b_{m-1} & ab_{m-1} + a_{m-1}b \\ a_m & b_m & ab_m + a_0b_0 + a_mb \\ a_{m+1} & b_{m+1} & ab_{m+1} + a_0b_1 + a_1b_0 + a_{m+1}b \\ \vdots & \vdots & \vdots \\ a_k & b_k & ab_k + \sum_{t+0}^{k-m} a_{\_t} b_{k-m-t} + a_kb \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\| = \bar{0}, \tag{32}$$

over $\mathbb{C}$, *where* $a,\ b,\ a_0,\ b_0 \neq 0$, $k \geq m$, $m \geq 1$, *it is necessary and sufficient that* *the following conditions hold:*

$$\underbrace{\begin{vmatrix} 0 & \dots & 0 & a_0 & a_1 & \dots & a_v \\ a & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & a_1 \\ & & \ddots & \ddots & \ddots & \ddots & a_0 \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & a & 0 \end{vmatrix}}_{m+v} = \frac{a^{m+v-1}a_0}{b^{m+v-1}b_0} \underbrace{\begin{vmatrix} 0 & \dots & 0 & b_0 & b_1 & \dots & b_v \\ b & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & b_1 \\ & & \ddots & \ddots & \ddots & \ddots & b_0 \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & b & 0 \end{vmatrix}}_{m+v} \tag{33}$$

for $v \in \{0, 1, \dots, k\}$.

Under the conditions of Lemma in every non-zero solution $\|x_{10}\ \ x_{20}\ \ x_{30}\|^T$ *of the equation* (17) *it is necessary that* $x_{10},\ x_{20} \neq 0$.

Proof. **Necessity.** The equality (33) for $v = 0$ holds true trivially. Assume that the equation (32) has a non-zero solution. Then the matrix of this equation has rank less than 3. The two first rows of the matrix are linearly independent. Then, each other row of the matrix linearly depends on these first two rows. Because

$$\begin{vmatrix} a & b & ab \\ a_0 & b_0 & ab_0 + a_0 b \\ a_v & b_v & ab_v + a_v b \end{vmatrix} = 0, \quad v = 1, \dots, m-1,$$

we have $\frac{a_v}{a_0} = \frac{b_v}{b_0}$. This implies the equality (33) for $v = 1,\ \dots,\ m-1$. From the equality

$$\begin{vmatrix} a & b & ab \\ a_0 & b_0 & ab_0 + a_0 b \\ a_m & b_m & ab_m + a_0 b_0 + a_m b \end{vmatrix} = 0$$

we have

$$ab_0^2 a_0 - a_0 b_m ab + a_m b_0 ab - a_0^2 b_0 b = 0,$$

whence, after division of both sides by aba_0b_0, we obtain equality (33) for $v = m$. Assume by induction that equality (33) holds true for all v with $m \le v \le r < k$. Consider the equality

$$\begin{vmatrix} a & b & ab \\ a_0 & b_0 & ab_0 + a_0 b \\ a_{r+1} & b_{r+1} & ab_{r+1} + \sum_{t=0}^{r+1-m} a_t b_{r+1-m-t} + a_{r+1} b \end{vmatrix} = 0.$$

After calculation of the determinant, we obtain the equation

$$ab_0 \sum_{t=0}^{r+1-m} a_t b_{r+1-m-t} - a_0 b_{r+1} ab + a_{r+1} b_0 ab - a_0 b \sum_{t=0}^{r+1-m} a_t b_{r+1-m-t} = 0.$$

We can divide by aba_0b_0 its both sides to obtain the equality

$$\frac{a_{r+1}}{a_0} - \frac{b_{r+1}}{b_0} + \Big(\frac{1}{a_0 b} - \frac{1}{ab_0}\Big) \sum_{t=0}^{r+1-m} a_t b_{r+1-m-t} = 0. \tag{34}$$

Let us denote by $\Delta_v(a)$ and $\Delta_v(b)$ the determinants in the left and right sides of the equality (33), respectively. Considering the inductive assumption, we can write the equalities

$$\begin{cases} (-1)^{m-1} a_0 a^{-1}((a^r a_0)^{-1}\Delta_{r+1-m}(a) - (b^r b_0)^{-1}\Delta_{r+1-m}(b)) = 0, \\ (-1)^{m} a_1 a^{-1}((a^{r-1} a_0)^{-1}\Delta_{r-m}(a) - (b^{r-1} b_0)^{-1}\Delta_{r-m}(b)) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (-1)^{r-1} a_{r-m} a^{-1}((a^m a_0)^{-1}\Delta_1(a) - (b^m b_0)^{-1}\Delta_1(b)) = 0, \\ (-1)^{r} a_{r+1-m} a^{-1}((a^{m-1} a_0)^{-1}\Delta_0(a) - (b^{m-1} b_0)^{-1}\Delta_0(b)) = 0, \end{cases} \tag{35}$$

$$\begin{cases} (-1)^{m-1} b_0 b^{-1}((a^r a_0)^{-1}\Delta_{r+1-m}(a) - (b^r b_0)^{-1}\Delta_{r+1-m}(b)) = 0, \\ (-1)^{m} b_1 b^{-1}((a^{r-1} a_0)^{-1}\Delta_{r-m}(a) - (b^{r-1} b_0)^{-1}\Delta_{r-m}(b)) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (-1)^{r-1} b_{r-m} b^{-1}((a^m a_0)^{-1}\Delta_1(a) - (b^m b_0)^{-1}\Delta_1(b)) = 0, \\ (-1)^{r} b_{r+1-m} b^{-1}((a^{m-1} a_0)^{-1}\Delta_0(a) - (b^{m-1} b_0)^{-1}\Delta_0(b)) = 0. \end{cases} \tag{36}$$

Decompose the determinants $\Delta_{r+1}(a)$ and $\Delta_{r+1}(b)$ for entries of their last columns. Then the left and right sides of equality (33) for $v = r+1$ can be written in the form

$$\begin{aligned} (a^{m+r} a_0)^{-1}\Delta_{r+1}(a) &= (a^{m+r} a_0)^{-1}((-1)^{m+r} a_{r+1} a^{m+r} + (-1)^r a_{r+1-m} a^r \Delta_0(a) + \\ &+ (-1)^{r-1} a_{r-m} a^{r-1}\Delta_1(a) + \ \dots\ + (-1)^{m+1} a_2 a^{m+1}\Delta_{r-m-1}(a) + \\ &+ (-1)^m a_1 a^m \Delta_{r-m}(a) + (-1)^{m-1} a_0 a^{m-1}\Delta_{r-m+1}(a)), \end{aligned} \tag{37}$$

$$\begin{aligned} (b^{m+r} b_0)^{-1}\Delta_{r+1}(b) &= (b^{m+r} b_0)^{-1}((-1)^{m+r} b_{r+1} b^{m+r} + (-1)^r b_{r+1-m} b^r \Delta_0(b) + \\ &+ (-1)^{r-1} b_{r-m} a^{r-1}\Delta_1(b) + \ \dots\ + (-1)^{m+1} b_2 b^{m+1}\Delta_{r-m-1}(b) + \\ &+ (-1)^m b_1 b^m \Delta_{r-m}(b) + (-1)^{m-1} b_0 b^{m-1}\Delta_{r-m+1}(b)). \end{aligned} \tag{38}$$

It is easy to make sure that the following identity is true

$$
\begin{aligned}
& b^{-1}(b_0(a^r a_0)^{-1}\Delta_{r-m+1}(a)(-1)^{m-1} + b_1(a^{r-1}a_0)^{-1}\Delta_{r-m}(a)(-1)^m + \ldots + \\
& + b_{r-m}(a^m a_0)^{-1}\Delta_1(a)(-1)^{r-1} + b_{r-m+1}(a^{m-1}a_0)^{-1}\Delta_0(a)(-1)^r) + \\
& + a^{-1}(-a_0(b^r b_0)^{-1}\Delta_{r-m+1}(b)(-1)^{m-1} - a_1(b^{r-1}b_0)^{-1}\Delta_{r-m}(b)(-1)^m - \ldots - \\
& - a_{r-m}(b^m b_0)^{-1}\Delta_1(b)(-1)^{r-1} - a_{r-m+1}(b^{m-1}b_0)^{-1}\Delta_0(b)(-1)^r) + \qquad (39) \\
& + (-1)^{m+r}(a_0 b)^{-1}\sum_{t=0}^{r+1-m} a_t b_{r+1-m-t} - (-1)^{m+r}(ab_0)^{-1}\sum_{t=0}^{r+1-m} a_t b_{r+1-m-t} = 0.
\end{aligned}
$$

We add left sides of equalities (35), (36) and multiplied by $(-1)^{m+r}$ left side of the equality (34) and separately we add right sides of the these equalities. Taking into account the expression (37), (38) for $(a^{m+r}a_0)^{-1}\Delta_{r+1}(a)$, $(b^{m+r}b_0)^{-1}\Delta_{r+1}(b)$ and the identity (39), we obtain $(a^{m+r}a_0)^{-1}\Delta_{r+1}(a) - (b^{m+r}b_0)^{-1}\Delta_{r+1}(b) = 0$. This means that the equality (33) for $v = r+1$ is true. The necessity of the conditions of Lemma is proved.

Sufficiency. Assume that the conditions (33) is satisfied for $v = 1, \ldots, m-1$. Then, the submatrix formed by first $m+1$ rows of the matrix from the equation (32) has rank less than 3. In fact, this rank is equal to 2, because, as has been stated above, the first two rows of this matrix are linearly independent. The equality (33) for $v = m$ implies that $(m+2)$-th row of the matrix of equation (32) linearly depends on its first two rows. Our inductive assumption is the following. Let each of the first $r+2$, $m \le r < k$, rows of the matrix of equation (32) linearly depend on its first two rows. We now reverse the order of arguments, as compared to the proof of the necessity, passing from relation (33) for $v = r+1$ to relation (34). This relation implies that the minor of order 3 of the matrix of equation (32) that is contained in 1-th, 2-nd and $(r+3)$-th rows is equal to zero. This means that indicated rows are linearly dependent. The above argument inductively proves that the matrix of equation (32) has the 2. This implies that the equation (32) has a non-zero solution.

The remaining part of the lemma will be proved by contradiction. Let $\|x_{10}\ x_{20}\ x_{30}\|^T$ be a non-zero solution of the equation (32), where $x_{10} = 0$ or $x_{20} = 0$. For this reason we have the equality

$$
\left\| \begin{matrix} b & ab \\ b_0 & ab_0 + a_0 b \end{matrix} \right\| \cdot \left\| \begin{matrix} x_{20} \\ x_{30} \end{matrix} \right\| = \bar{0} \quad \text{or} \quad \left\| \begin{matrix} a & ab \\ a_0 & ab_0 + a_0 b \end{matrix} \right\| \cdot \left\| \begin{matrix} x_{10} \\ x_{30} \end{matrix} \right\| = \bar{0}.
$$

Since $a,\ b,\ a_0,\ b_0 \neq 0$, the determinants of the 2×2 matrices of these equations are different from 0. Hence, $\|x_{20}\ x_{30}\|^T = \bar{0}$ or $\|x_{10}\ x_{30}\|^T = \bar{0}$, i.e., actually, $\|x_{10}\ x_{20}\ x_{30}\|^T = \bar{0}$, contrary to the our assumption. Therefore, in the non-zero solution $\|x_{10}\ x_{20}\ x_{30}\|^T$ of the equation (32) necessarily $x_{10},\ x_{20} \neq 0$. Lemma is proved. □

3. Proof of the Theorem

Necessity. Assume that matrices $A(x)$ and $B(x)$ of the forms (1) and (3) are semiscalarly equivalent. Then, the entries $a(x)$ and $b(x)$ of these matrices satisfy the congruence

$$s_{22}a(x) - s_{11}b(x) - s_{12}a(x)b(x) \equiv 0 \mod \delta(x), \tag{40}$$

where $s_{11},\ s_{22} \neq 0$. Construct the decompositions of the entries $a(x)$, $b(x)$ into degrees of binomial $x - \alpha_i$ (see (11), (12)) for $\alpha_i \in M$ such that $a_i = b_i = 0$, $m_i < n_i$, and compare the coefficients of equal degrees of binomial $x - \alpha_i$ on both sides of the congruence (40). Then, we obtain

$$s_{22}a_{it} - s_{11}b_{it} = 0 \text{ for } t \in \{0, 1, \dots, l_i - m_i - 1\} \text{ where } l_i := \min(2m_i, n_i).$$

This implies $s_{11}/s_{22} = a_{i0}/b_{i0}$ and equality (13), where $c = s_{11}/s_{22} = a_{i0}/b_{i0}$, for all $\alpha_i \in M$ such that $2m_i \geq n_i$.

Taking into account the roots $\alpha_i \in M$ such that $2m_i < n_i$ and comparing the coefficients of equal degrees of binomial $x - \alpha_i$ on both sides of the congruence (40), we obtain the system of the equalities, written in matrix form:

$$\begin{Vmatrix} a_{i0} & b_{i0} & 0 \\ a_{i1} & b_{i1} & 0 \\ \vdots & \vdots & \vdots \\ a_{i,\,m_i-1} & b_{i,\,m_i-1} & 0 \\ a_{im_i} & b_{im_i} & a_{i0}b_{i0} \\ a_{i,\,m_i+1} & b_{i,\,m_i+1} & a_{i0}b_{i1} + a_{i1}b_{i0} \\ \vdots & \vdots & \vdots \\ a_{i,\,n_i-m_i-1} & b_{i,\,n_i-m_i-1} & \sum_{t=0}^{n_i-2m_i-1} a_{it}b_{i,\,n_i-2m_i-1-t} \end{Vmatrix} \cdot \begin{Vmatrix} s_{22} \\ -s_{11} \\ -s_{12} \end{Vmatrix} = \bar{0}.$$

Since $s_{11}, s_{22} \neq 0$, this equality implies the first condition of Theorem 1.3 (according to Lemma 2.1).

Let $\alpha_i,\ \alpha_l \in M$ be the arbitrary pair of the entries such that $a_i = b_i = 0$, $2m_i < n_i$, $a_l = b_l = 0$, $2m_l < n_l$. For the coefficients $a_{im_i},\ a_{lm_l},\ b_{im_i},\ b_{lm_l}$, $a_{i0} \neq 0,\ \ a_{l0} \neq 0$, $b_{i0} \neq 0,\ \ b_{l0} \neq 0$ of the decomposition of the forms (11), (12) into degrees of binomials $x - \alpha_i$, $x - \alpha_l$, the congruence (40) implies the relations

$$s_{22}a_{im_i} - s_{11}b_{im_i} - s_{12}a_{i0}b_{i0} = 0,\ \ s_{22}a_{lm_l} - s_{11}b_{lm_l} - s_{12}a_{l0}b_{l0} = 0.$$

Excluding s_{12} from these equalities and considering that $a_{i0} = cb_{i0}$, $a_{l0} = cb_{l0}$, $c = s_{11}/s_{22}$, we obtain the equality (14) from the second condition of the Theorem.

For the coefficients $a_{i0},\ a_{im_i},\ b_{i0},\ b_{im_i}$ and for the free terms of the decompositions of the form (11), (12) into degrees of $x-\alpha_i$, $x-\alpha_p$ for roots $\alpha_i,\ \alpha_p \in M$ such that $a_i = b_i = 0$, $2m_i < n_i$, $a_p,\ b_p \neq 0$, the congruence (40) yields the system

$$\begin{cases} s_{22}a_{i0} - s_{11}b_{i0} = 0, \\ s_{22}a_{im_i} - s_{11}b_{im_i} - s_{12}a_{i0}b_{i0} = 0, \\ s_{22}a_p - s_{11}b_p - s_{12}a_pb_p = 0. \end{cases}$$

Excluding s_{12} from two last equalities of these system and putting $a_{i0} = cb_{i0}$ for $c = s_{11}/s_{22}$, we obtain have equality (15) from the third condition of Theorem 1.3.

For the coefficients of the decompositions (11), (12) into degrees of binomial $x-\alpha_p$ for roots $\alpha_p \in M$ such that $a_p,\ b_p \neq 0$, $m_p < n_p$, the congruence (40) yields the system

$$\begin{cases} s_{22}a_p - s_{11}b_p - s_{12}a_pb_p = 0, \\ s_{22}a_{pt} - s_{11}b_{pt} - s_{12}(a_{pt}b_p + b_{pt}a_p) = 0, \end{cases}$$

for $t \in \{0, 1, \ldots, l_p - m_p - 1\}$ where $l_p = \min(2m_p, n_p)$. After excluding s_{12} from this system, we obtain

$$c\frac{a_{pt}}{a_p^2} = \frac{b_{pt}}{b_p^2} \tag{41}$$

for $t \in \{0, 1, \ldots, l_p - m_p - 1\}$, where $c = s_{11}/s_{22}$. This implies the equality (16) for the roots $\alpha_p \in M$ such that $a_p,\ b_p \neq 0$, $m_p < n_p$, $2m_p \geq n_p$. If $2m_p < n_p$, then the congruence (40) implies the system of the equalities, written in the matrix form as

$$\|F\ G\ H\| \cdot \|s_{22}\ \ -s_{11}\ \ -s_{12}\|^T = \bar{0},$$

where

$$F = \left\| \begin{matrix} a_p \\ a_{p0} \\ a_{p1} \\ \vdots \\ a_{p,\,m_p-1} \\ a_{pm_p} \\ a_{p,\,m_p+1} \\ \vdots \\ a_{p,\,n_p-m_p-1} \end{matrix} \right\|, \quad G = \left\| \begin{matrix} b_p \\ b_{p0} \\ b_{p1} \\ \vdots \\ b_{p,\,m_p-1} \\ b_{pm_p} \\ b_{p,\,m_p+1} \\ \vdots \\ b_{p,\,n_p-m_p-1} \end{matrix} \right\|, \tag{42}$$

$$H=\left\|\begin{array}{c} a_pb_p \\ a_{p0}b_p+a_pb_{p0} \\ a_{p1}b_p+a_pb_{p1} \\ \vdots \\ a_{p,\,m_p-1}b_p+a_pb_{p,\,m_p-1} \\ a_{pm_p}b_p+a_{p0}b_{p0}+a_pb_{pm_p} \\ a_{p,\,m_p+1}b_p+a_{p0}b_{p1}+a_{p1}b_{p0}+a_pb_{p,\,m_p+1} \\ \vdots \\ a_{p,n_p-m_p-1}b_p+\sum_{t=0}^{n_p-2m_p-1}a_{pt}b_{p,n_p-2m_p-1-t}+a_pb_{p,n_p-m_p-1} \end{array}\right\|. \tag{43}$$

Since $s_{11},\ s_{22}\neq 0$, by the Lemma 2 from the obtained equality we have

$$\underbrace{\left|\begin{array}{ccccccc} 0 & \dots & 0 & a_{p0} & a_{p1} & \dots & a_{ps_p} \\ a_p & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & a_{p1} \\ & & \ddots & \ddots & \ddots & \ddots & a_{p0} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & a_p & 0 \end{array}\right|}_{m_p+s_p}=\frac{a_p^{m_p+s_p-1}a_{p0}}{b_p^{m_p+s_p-1}b_{p0}}\underbrace{\left|\begin{array}{ccccccc} 0 & \dots & 0 & b_{p0} & b_{p1} & \dots & b_{ps_p} \\ b_p & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & b_{p1} \\ & & \ddots & \ddots & \ddots & \ddots & b_{p0} \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \vdots \\ 0 & & & & & b_p & 0 \end{array}\right|}_{m_p+s_p}, \tag{44}$$

for $s_p\in\{0,1,\dots,n_p-m_p-1\}$. Here for $t\in\{0,1,\dots,m_p-1\}$ the equalities (41) hold. Now multiply the left side of the equality (44) by the left side of the equality (41) at $t=0$ and carry out analogous operation with right sides. As a result, we obtain the equality (16). This proves the last condition of Theorem 1.3 and necessity entirely.

Sufficiency. Assume that the conditions 1) – 4) of Theorem 1.3 are fulfilled. Consider the matrix equations

$$\left\|\begin{array}{ccc} a_{i0} & b_{i0} & 0 \\ a_{i1} & b_{i1} & 0 \\ \vdots & \vdots & \vdots \\ a_{i,\,m_i-1} & b_{i,\,m_i-1} & 0 \end{array}\right\|\cdot\left\|\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\|=\bar{0}, \tag{45}$$

$$\left\|\begin{array}{ccc} a_{im_i} & b_{im_i} & a_{i0}b_{i0} \\ a_{i,\,m_i+1} & s_{i,\,m_i+1} & a_{i0}b_{i1}+a_{i1}b_{i0} \\ \vdots & \vdots & \vdots \\ a_{i,\,n_i-m_i-1} & b_{i,\,n_i-m_i-1} & \sum_{t=0}^{n_i-2m_i-1}a_{it}b_{i,\,n_i-2m_i-1-t} \end{array}\right\|\cdot\left\|\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\|=\bar{0}, \tag{46}$$

in the unknown vector-column $\|x_1\ x_3\ x_3\|^T$, were written by the coefficients of the decompositions (11), (12) for some root $\alpha_i \in M$. For an arbitrary fixed root $\alpha_i \in M$ such that $a_i = b_i = 0$, $m_i < n_i$, $2m_i \geq n_i$ by Lemma 2.1 the equation (45) has a non-zero solution. If $2m_i < n_i$, by Lemma 2.1 both these equations (45), (46) have common non-zero solution. Evidently, that first two rows of the matrix of each of these equations are linearly independent and each other row of this matrix is a linear combination of the first two rows. Taking into account that

$$\left\| \begin{matrix} a_{i0} & b_{i0} & 0 \\ a_{im_i} & b_{im_i} & a_{i0}b_{i0} \end{matrix} \right\| \cdot \left\| \begin{matrix} 1 \\ -c \\ (cb_{im_i} - a_{im_i})(a_{i0}b_{i0})^{-1} \end{matrix} \right\| = \bar{0},$$

we conclude that

$$\|x_1\ x_2\ x_3\|^T = \|1\ \ -c\ \ (cb_{im_i} - a_{im_i})(a_{i0}b_{i0})^{-1}\|^T \tag{47}$$

is the common non-zero solution of the equations (45), (46).

From the condition 2) we deduce the relation

$$(cb_{im_i} - a_{im_i})(a_{i0}b_{i0})^{-1} = (cb_{lm_l} - a_{lm_l})(a_{l0}b_{l0})^{-1}.$$

for other arbitrary root $\alpha_l \in M$ such that $a_l = b_l = 0$, $2m_l < n_l$, different from $\alpha_i \in M$. This means that the solution (47) does not depend on the choice of the root $\alpha_i \in M$ satisfying the condition 1). So, we can write the congruence

$$a(x) - cb(x) - c_1 a(x)b(x) \equiv 0 \mod (x - \alpha_i)^{n_i}, \tag{48}$$

where

$$c_1 = (a_{im_i} - cb_{im_i})(a_{i0}b_{i0})^{-1}, \tag{49}$$

for arbitrary root $\alpha_i \in M$ such that $a_i = b_i = 0$ (including the roots α_i with $m_i \geq n_i$, if such roots exist). Thus, one subset of the partition (10) of the set M of roots α_i is exhausted.

The fulfillment of the equality (16) for $s_p = 0$ means that $\frac{ca_{p0}}{a_p^2} = \frac{b_{p0}}{b_p^2}$. Taking into account this fact, we see that the relation (44) follows from the the equality (16) for every $s_p \in \{1, \ldots, n_p - m_p - 1\}$. Moreover, by Lemma 2.2 the equation

$$\|F\ G\ H\| \cdot \|x_1\ x_2\ x_3\|^T = \bar{0}, \tag{50}$$

(with F, G and H defined as in (42), (43)) has a non-zero solution. Since the first two rows of the matrix of the equation (50) are linearly independent and

$$\left\| \begin{matrix} a_p & b_p & a_p b_p \\ a_{p0} & b_{p0} & a_{p0}b_p + a_p b_{p0} \end{matrix} \right\| \left\| \begin{matrix} 1 \\ -c \\ (cb_p - a_p)(a_p b_p)^{-1} \end{matrix} \right\| = \bar{0},$$

the column $\|x_1 \ x_3 \ x_3\|^T = \|1 \ \ -c \ \ (cb_p - a_p)(a_p b_p)^{-1}\|^T$ is the solution of the equation (50) for arbitrary root $\alpha_p \in M$ such that $a_p, b_p \neq 0$. From the condition 3) we deduce the relation

$$(cb_p - a_p)(a_p b_p)^{-1} = (cb_{im_i} - a_{im_i})(a_{i0} b_{i0})^{-1}.$$

Therefore, a solution of the equation (50) can be written in the form (47). This means that the following congruence is satisfied

$$a(x) - cb(x) - c_1 a(x) b(x) \equiv 0 \mod (x - \alpha_p)^{n_p}, \tag{51}$$

where c_1 is defined as in congruence (48) (see (49)). In addition, the congruence (51) is fulfilled for arbitrary root $\alpha_p \in M$ such that $a_p, b_p \neq 0$. Now comparing (48) and (51), we have the congruence

$$a(x) - cb(x) - c_1 a(x) b(x) \equiv 0 \mod \delta(x).$$

Let $r_{21}(x)$ be determined as the result of dividing the left side of the congruence by $\delta(x)$. Then, we write

$$a(x) - cb(x) - c_1 a(x) b(x) = \delta(x) r_{21}(x). \tag{52}$$

Write (52) in the form $a(x) = b(x) r_{11}(x) + \delta(x) r_{21}(x)$, where $r_{11}(x) = c + c_1 a(x)$, and denote $r_{12}(x) = c_1 \delta(x)$, $r_{22}(x) = 1 - b(x) r_{12}(x)$. Now it can be written

$$\left\| \begin{matrix} c & c_1 \\ 0 & 1 \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ a(x) & \delta(x) \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 \\ b(x) & \delta(x) \end{matrix} \right\| \left\| \begin{matrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{matrix} \right\|.$$

It is evident that the left factor on the left side of determined equation and the right factor on the right side are invertible matrices. Thus, the matrices $A(x)$ and $B(x)$ are semiscalarly equivalent. Theorem 1.3 is proved.

4. Some corollaries of Theorem 1.3

Corollary 4.1. *Let the partition* (2) *of the set M of characteristic roots of matrices $A(x)$ be of the form* (10)*. Let also the entry $a(x)$ of the matrix $A(x)$ in the notations of Theorem 1.3 satisfy the following conditions:*

1) $m_i < n_i \leq 2m_i$, $a_i = 0$, for the roots $\alpha_i \in M$, $i \in \{1, \dots, r\}$ where $r \geq 1$;

2) $m_p < n_p \leq 2m_p$, $a_p \neq 0$, for roots $\alpha_p \in M$, $p \in \{r+1, \dots, q\}$ where $q > r$;

3) $m_h \geq n_h$ for all $\alpha_h \in M$ which are different from α_i, α_p.

Such a matrix $A(x)$ in the class $\{CA(x)Q(x)\}$ of semiscalarly equivalent matrices is determined up to constant factors $c \neq 0$ and $1/c$ of the rows

$$\|a_{10} \;\; a_{11} \;\; \dots \;\; a_{1,n_1-m_1-1} \;\; \dots \;\; a_{r0} \;\; a_{r1} \;\; \dots \;\; a_{r,n_r-m_r-1}\|$$

and

$$\frac{1}{a_{r+1}^2}\left\|a_{r+1,0} \;\; a_{r+1,1} \;\; \dots \;\; a_{r+1,n_{r+1}-m_{r+1}-1} \;\; \dots \;\; a_{q0} \;\; a_{q1} \;\; \dots \;\; a_{q,n_q-m_q-1}\right\|,$$

respectively.

Proof. Since $n_i \le 2m_i$ and $n_p \le 2m_p$, it follows that $n_i - m_i - 1 \le m_i - 1$ and $n_p - m_p - 1 \le m_p - 1$. By Theorem 1.3, in the case under consideration matrices $A(x)$, $B(x)$ are semiscalarly equivalent if and only if the ratio $c = a_{i0}/b_{i0}$ does not depend on the choice of $i \in \{1, \dots, r\}$ and the equalities (13), (16) are valid for every $s_i \in \{1, \dots, n_i - m_i - 1\}$ and $s_p \in \{0, 1, \dots, n_p - m_p - 1\}$, respectively. The fulfillment of the equality (13), where $c = a_{i0}/b_{i0}$, for $s_i \in \{1, \dots, n_i - m_i - 1\}$, is equivalent to the fulfillment of the equality

$$\|a_{i0} \; a_{i1} \; \dots \; a_{i,n_i-m_i-1}\| = c\,\|b_{i0} \; b_{i1} \; \dots \; b_{i,n_i-m_i-1}\|$$

for arbitrary $i \in \{1, \dots, r\}$. The equality (16) for $s_p \in \{0, 1, \dots, n_p - m_p - 1\}$ is valid if and only if when

$$\frac{1}{a_p^2}\left\|a_{p0} \; a_{p1} \; \dots \; a_{p,n_p-m_p-1}\right\| = \frac{1}{cb_p^2}\left\|b_{p0} \; b_{p1} \; \dots \; b_{p,n_p-m_p-1}\right\|$$

for arbitrary $p \in \{r+1, \dots, q\}$. □

Corollary 4.2. *Let the partition* (2) *of the set M of characteristic roots of matrices $A(x)$ be of the form* (10). *Let also in the notations of Theorem 1.3 $a_i = 0$ for some root $\alpha_i \in M$ and $m_j \ge n_j$ for every root $\alpha_j \in M$. Such matrix $A(x)$ in the class $\{CA(x)Q(x)\}$ of semiscalarly equivalent matrices is determined up to constant factor of its entry $a(x)$.*

Proof. Let the entry $b(x)$ of the matrix $B(x)$ of the form (3) satisfy the condition $b(\alpha_i) = b_i = 0$ for the same $\alpha_i \in M$ for which $a_i = 0$. By Proposition 1.1, if matrices $A(x)$ and $B(x)$ are semiscalarly equivalent, then the equivalence $a_l = 0 \Leftrightarrow b_l = 0$ is valid for arbitrary $\alpha_l \in M$. Moreover, by (the same) Proposition 1.1, all non-zero values of the polynomial $b(x)$ (as $a(x)$) at the points of the set M are equal, i.e. for arbitrary pair $\alpha_p,\ \alpha_q \in M$ such that $a_p,\ a_q \neq 0$ we have $a_p = a_q \neq 0$ and $b_p = b_q \neq 0$. We take into consideration that by Proposition 1.2,

$a^{(t)}(\alpha_j) = b^{(t)}(\alpha_j) = 0$, $t \in \{1, \ldots, n_j - 1\}$, for every $\alpha_j \in M$. From what has been said it follows that there exists a number $c \in \mathbb{C}$ such that every characteristic root $\alpha_j \in M$ of multiplicity n_j is the root of the polynomial $\varphi(x) = a(x) - cb(x)$ of the same multiplicity n_j. Taking into account that $\deg a(x), \deg b(x) < \deg \delta(x)$, we conclude that $\varphi(x) \equiv 0$ and thus $a(x) = c\, b(x)$. □

Remark. The conditions of Corollary 4.2 hold for any matrix $A(x)$ of the form (1) without multiple characteristic roots, if its entry $a(x)$ takes on the set of characteristic roots exactly two different values, one of which is zero.

5. Examples

Example 5.1. Consider the matrices

$$A(x) = \left\| \begin{matrix} 1 & 0 \\ x^2 - x^3 + 4x^4 - 3x^5 & x^4 - 4x^5 + 6x^6 - 4x^7 + x^8 \end{matrix} \right\|,$$

$$B(x) = \left\| \begin{matrix} 1 & 0 \\ x^2 - x^3 + \frac{21}{4}x^4 - \frac{45}{4}x^5 + \frac{37}{4}x^6 - \frac{11}{4}x^7 & x^4 - 4x^5 + 6x^6 - 4x^7 + x^8 \end{matrix} \right\|.$$

Each of the characteristic roots $\alpha_1 = 0$, $\alpha_2 = 1$ of these matrices are of multiplicity 4. The decompositions of entries

$$a(x) = x^2 - x^3 + 4x^4 - 3x^5, \quad b(x) = x^2 - x^3 + \tfrac{21}{4}x^4 - \tfrac{45}{4}x^5 + \tfrac{37}{4}x^6 - \tfrac{11}{4}x^7$$

into degrees of binomial $x - 1$ are of the form:

$$a(x) = 1 - 8(x-1)^2 - 15(x-1)^3 - 11(x-1)^4 - 3(x-1)^5,$$
$$b(x) = \frac{1}{2} - 2(x-1)^2 - \tfrac{15}{4}(x-1)^3 - \tfrac{17}{2}(x-1)^4 - \tfrac{27}{2}(x-1)^5 - 10(x-1)^6 - \tfrac{35}{2}(x-1)^7.$$

By Corollary 4.1, the matrices $A(x)$, $B(x)$ are semiscalarly equivalent. In fact,

$$A(x) = \left\| \begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} \right\| \cdot B(x) \cdot$$
$$\cdot \left\| \begin{matrix} 1 + x^2 - x^3 + 4x^4 - 3x^5 & x^4 - 4x^5 + 6x^6 - 4x^7 + x^8 \\ -\frac{9}{4} - \frac{5}{4}x - x^2 + \frac{23}{4}x^3 - \frac{33}{4}x^4 & 1 - x^2 + x^3 - \frac{21}{4}x^4 + \frac{45}{4}x^5 - \frac{37}{4}x^6 + \frac{11}{4}x^7 \end{matrix} \right\|.$$

Example 5.2. Consider the matrix

$$A(x) = \left\| \begin{matrix} 1 & 0 \\ 2 - 3x + x^3 & 1 - 2x^2 + x^4 \end{matrix} \right\|$$

with characteristic roots $\alpha_1 = 1$, $\alpha_2 = -1$, each of multiplicity 2. The decompositions of entry $a(x) = 2 - 3x + x^3$ into degrees of binomials $x - 1$, $x + 1$ are of the form:

$$a(x) = 3(x-1)^2 + (x-1)^3 = 4 - 3(x+1)^2 + (x+1)^3.$$

Matrix $A(x)$ satisfies the conditions of Corollary 4.2. Therefore, for matrices $A(x)$ and $B(x) = \begin{Vmatrix} 1 & 0 \\ b(x) & 1 - 2x^2 + x^4 \end{Vmatrix}$, where $b(1) = 0$, to be semiscalarly equivalent, it is necessary and sufficient that $a(x) = cb(x)$, where $c \in \mathbb{C}$, $c \neq 0$.

Remark. Assume that the matrices $A(x)$ and $B(x)$ from Theorem 1.3 are semiscalarly equivalent, i.e. $SA(x)R(x) = B(x)$. In this case Theorem 1.3 provides a method for constructing the transforming matrices S, $R(x)$. Such method was used in Example 5.1. However, this is the subject of another study.

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