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TOPOLOGICAL PROPERTIES OF TAIMANOV SEMIGROUPS

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A semigroup T is called *Taimanov* if T contains two distinct points $0, \infty$ such that $xy = \infty$ for any distinct elements $x, y \in T \setminus \{0, \infty\}$ and $xy = 0$ in all other cases. We prove that any Taimanov semigroup T has the following topological properties: (i) each T_1-topology with continuous shifts on T is discrete; (ii) T is closed in each T_1-topological semigroup containing T as a subsemigroup; (iii) every non-isomorphic homomorphic image Z of T is a zero-semigroup and hence Z is a topological semigroup in any topology on Z.

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Напівгрупа T називається *Таймановою*, якщо вона містить два різні елементи $0, \infty$ такі, що $xy = \infty$ для довільних різних точок $x, y \in T \setminus \{0, \infty\}$ і $xy = \infty$ у всіх інших випадках. Доведено, що довільна напівгрупа Тайманова T має такі топологічні властивості: (i) кожна T_1-топологія з неперервними зсувами на T є дискретною; (ii) T замкнена в довільній T_1-топологічній напівгрупі, що містить T як піднапівгрупу; (iii) кожен неізоморфний гомоморфний образ Z напівгрупи T є напівгрупою з нульовим множенням і, отже є топологічною напівгрупою в довільній топології на Z.

We shall follow the terminology of [5, 8, 10, 20].

The problem of non-discrete (Hausdorff) topologization of infinite groups was posed by Markov [17]. This problem was resolved by Ol'shanskiy [19] who constructed an infinite countable group G admitting no non-discrete Hausdorff group topologies. On the other hand, Zelenyuk [23] proved that each group G admits a non-discrete shift-continuous Hausdorff topology τ with continuous inversion $G \to G$, $x \mapsto x^{-1}$. In [1, 2.10] it was observed that Ol'shanskiy construction can be modified to produce for every non-zero $m \in \mathbb{Z} \setminus \{-2^n, 2^n : n \in \omega\}$ a countable infinite group G_m admitting

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no non-discrete shift-continuous topology with continuous m-th power map $G_m \to G_m$, $x \mapsto x^m$.

Studying the topologizability problem in the class of inverse semigroups, Eberhart and Selden [9] proved that every Hausdorff semigroup topology on the bicyclic semigroup $\mathscr{C}(p,q)$ is discrete. This result was generalized by Bertman and West [4] who proved that every Hausdorff shift-continuous topology on $\mathscr{C}(p,q)$ is discrete. In [2, 3, 6, 7, 11, 12, 13, 14, 15, 16, 18] these topologizability results were extended to some generalizations of the bicyclic semigroup.

Studying the topologizability problem in the class of commutative semigroups [22], Taimanov in [21] constructed a commutative semigroup $\mathfrak{A}_\kappa$ of arbitrarily large cardinality κ, which admits no non-discrete Hausdorff semigroup topology, but any non-isomorphic homomorphic image Z of T is a zero-semigroup and hence is a topological semigroup in any topology on Z. We recall that a semigroup Z is a *zero-semigroup* if the set $SS = \{xy : x, y \in X\}$ is a singleton $\{z\}$. In this case the element z is the *zero-element* of the semigroup S, i.e., a (unique) element $z \in S$ such that $xz = z = zx$ for al $x \in S$. In this paper we improve the mentioned Taimanov's result proving that the Taimanov semigroup $\mathfrak{A}_\kappa$ admits no non-discrete shift-continuous T_1-topologies and is closed in any T_1-topological semigroup containing $\mathfrak{A}_\kappa$ as a subsemigroup. First we give an abstract definition of a Taimanov semigroup.

Definition 1. A semigroup T is called *Taimanov* if it contains two distinct elements $0_T, \infty_T$ such that for any $x, y \in T$

$$x \cdot y = \begin{cases} \infty_T & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0_T, \infty_T\}; \\ 0_T & \text{if } x = y \text{ or } \{x, y\} \cap \{0_T, \infty_T\} \neq \varnothing. \end{cases}$$

The elements $0_T, \infty_T$ are uniquely determined by the algebraic structure of T: 0_T is a (unique) zero-element of T, and ∞_T is the unique element of the set $TT \setminus \{0_T\}$.

It follows that each Taimanov semigroup T is commutative. Concrete examples of Taimanov semigroups can be constructed as follows.

Example 1. For any non-zero cardinal κ the set $\kappa \cup \{\kappa\}$ endowed with the commutative semigroup operation defined by

$$xy = \begin{cases} \kappa & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0, \kappa\}; \\ 0 & \text{if } x = y \text{ or } \{x, y\} \cap \{0, \kappa\} \neq \varnothing. \end{cases}$$

is a Taimanov semigroup of cardinality $1 + \kappa$. Here we identify the cardinal κ with the set $[0, \kappa)$ of ordinals, smaller than κ.

Proposition 1. *Two Taimanov semigroups are isomorphic if and only if they have the same cardinality.*

Proof. Given two Taimanov semigroups T, S of the same cardinality, observe that any bijective map $f : T \to S$ with $f(0_T) = 0_S$ and $f(\infty_T) = \infty_S$ is an algebraic isomorphism of T onto S. $\square$

In this paper we show that any Taimanov semigroup T has the following topological properties:

(1) every shift-continuous T_1-topology on T is discrete;
(2) T is closed in each T_1-topological semigroup containing T as a subsemigroup;
(3) every non-isomorphic homomorphic image Z of T is a zero-semigroup and hence any topology on Z turns it into a topological semigroup.

The first statement generalizes the original result of Taimanov [21] and is proved in the following proposition.

Proposition 2. *Every shift-continuous T_1-topology τ on any Taimanov semigroup T is discrete.*

Proof. The statement is trivial if the semigroup T is finite. So, assume that T is infinite. The topology τ satisfies the separation axiom T_1 and hence contains an open set $U \subset X$ such that $0_T \in U$ and $\infty_T \notin U$.

First we prove that the points 0_T and ∞_T are isolated in T. Chose any point $x \in T \setminus \{0_T, \infty_T\}$ and observe that $x \cdot 0_T = x \cdot \infty_T = 0_T \in U$. By the shift-continuity of the topology τ, there exist neighborhoods $U_0 \in \tau$ of 0_T and $U_\infty \in \tau$ of ∞_T such that $(x \cdot U_0) \cup (x \cdot U_\infty) \subset U$. We claim that $U_0 \setminus \{x, \infty_T\} = \{0_T\}$ and $U_\infty \setminus \{x, 0_T\} = \{\infty_T\}$. In the opposite case we could find a point $y \in (U_0 \cup U_\infty) \setminus \{x, 0_T, \infty_T\}$ and conclude that $\infty_T = xy \in x \cdot (U_0 \cup U_\infty) \subset U \subset T \setminus \{\infty_T\}$, which is a desired contradiction showing that the points 0_T and ∞_T are isolated in T.

To show that each point $x \in T \setminus \{0_T, \infty_T\}$ is isolated in the topology τ, observe that $xx = 0_T \in T \setminus \{\infty_T\} \in \tau$ and use the shift-continuity of the topology τ to find a neighborhood $U_x \in \tau$ of x such that $xU_x \subset T \setminus \{\infty_T\}$. Assuming that $U_x \neq \{x\}$ we can choose any point $y \in U_x \setminus \{x\}$ and conclude that $\infty_T = xy \in T \setminus \{\infty_T\}$, which is a contradiction showing that $U_x = \{x\}$ and hence the point x is isolated in the topology τ. □

The following example shows that any infinite Taimanov semigroup admits a non-discrete semigroup T_0-topology.

Example 2. For any infinite Taimanov semigroup T the family of subsets

$$\tau := \{U \subset T : \text{if } 0_T \in U, \text{ then } \infty_T \in U \text{ and } |T \setminus U| < \omega\}$$

is a T_0-topology turning T into a topological semigroup.

A semitopological semigroup S will be called *square-topological* if the map $S \to S$, $x \mapsto x^2$, is continuous. It is clear that each topological semigroup is square-topological.

Theorem 1. *A Taimanov semigroup T is closed in any square-topological semigroup S containing T as a subsemigroup and satisfying the separation axiom T_1.*

Proof. Assuming that T is not closed in S, choose any point $s \in \bar{T} \setminus T$. We claim that $sx = \infty_T$ for any $x \in T \setminus \{0_T, \infty_T\}$. Assuming that $sx \neq \infty_T$ and using the shift-continuity of the T_1-topology of S, we can find a neighborhood $U_s \subset S$ of s such that

$U_s \cdot x \subset S \setminus \{\infty_T\}$. Since s is an accumulation point of the set T in S, there exists a point $y \in U_s \setminus \{x, 0_T, \infty_T\}$. For this point y we get $\infty_T = yx \in U_s x \subset S \setminus \{\infty_T\}$, which is a contradiction showing that $sx = \infty_T$ for any $x \in T \setminus \{0_T, \infty_T\}$. Next, we show that $ss = \infty_T$. Assuming that $ss \neq \infty_T$, we can use the shift-continuity of the T_1-topology of S to find a neighborhood $V_s \subset S$ of s such that $sV_s \subset S \setminus \{\infty_T\}$. Since s is an accumulation point of the set T in S, there exists a point $x \in V_s \cap T \setminus \{0_T, \infty_T\}$. For this point x, we get $\infty_T = sx \in sV_s \subset S \setminus \{\infty_T\}$, which is a contradiction showing that $ss = \infty_T$. By the separation axiom T_1, the set $S \setminus \{0_T\}$ is an open neighborhood of ∞_T in S. The continuity of the map $S \to S$, $x \mapsto x^2$, yields a neighborhood $W_s \subset S$ such that $x^2 \in S \setminus \{0_T\}$ for any $x \in W_s$. Since s is an accumulation point of the set T in S, there exists a point $x \in W_s \cap T \setminus \{0_T, \infty_T\}$. For this point x we get $0_T = xx \in S \setminus \{0_T\}$, which is a desired contradiction showing that the set T is closed in S. □

The following example shows that any infinite Taimanov semigroup admits a (non-closed) embedding into a compact Hausdorff semitopological semigroup and also shows that the continuity of the map $S \to S$, $x \mapsto x^2$, in Theorem 1 is essential and cannot be replaced by the continuity of the map $S \to S$, $x \mapsto x^m$, for some $m \geq 3$.

Example 3. Let T be a Taimanov semigroup and X be any T_1-topological space containing T as a non-closed dense discrete subspace. Extend the semigroup operation of T to a binary operation of X defined by the formula:

$$xy = \begin{cases} 0_T & \text{if } x = y \in T \text{ or } \{x, y\} \cap \{0_T, \infty_T\} \neq \varnothing; \\ \infty_T & \text{otherwise.} \end{cases}$$

Since $(xy)z = 0_T = x(yz)$ for any $x, y, z \in X$ the extended operation is associative and turns X into a commutative semigroup containing T as a subsemigroup. Observe that for $a \in \{0_T, \infty_T\}$ the shift $l_a = r_a : X \to X$, $x \mapsto ax = xa = 0_T$, is constant and hence continuous. For any $a \in T \setminus \{0_T, \infty_T\}$ the shift $l_a = r_a : X \to X$, $x \mapsto xa = ax$, is almost constant in the sense that $l_a^{-1}(\infty_T) = X \setminus \{a, 0_T, \infty_T\}$ and hence is continuous (as the set $\{a, 0_T, \infty_T\}$ is closed and open in X). For any $a \in X \setminus T$ the shift $l_a = r_a : X \to X$, $x \mapsto xa = ax$, is almost constant in the sense that $l_a^{-1}(\infty_T) = X \setminus \{0_T, \infty_T\}$ and hence is continuous. This shows that X is a semitopological commutative semigroup containing T as a non-closed dense subsemigroup. Observe also that for every $m \geq 3$ the map $X^m \to X$, $(x_1, \ldots, x_m) \mapsto x_1 \cdots x_m = 0_T$, is constant and hence continuous. Then the map $X \to X$, $x \mapsto x^m$, is continuous as well.

Example 4. For any topological zero-semigroup Z with zero 0_Z and any Taimanov semigroup T endowed with the discrete topology, any map $h : T \to Z$ with $h(0_T) = h(\infty_T) = 0_Z$ is a continuous semigroup homomorphism. Hence there exist many topological (zero-)semigroups containing continuous homomorphic images of Taimanov semigroups as non-closed subsemigroups.

Proposition 3. *Any non-isomorphic homomorphic image S of a Taimanov semigroup T is a zero-semigroup.*

Proof. Fix a non-injective surjective homomorphism $h : T \to S$. If $f(0_T) = f(\infty_T)$, then $SS = f(T) \cdot f(T) = f(TT) = f(\{0_T, \infty_T\}) = \{f(0_T)\}$, which means that S is a zero-semigroup. So, assume that $f(0_T) \neq f(\infty_T)$. Since f is not injective, there exist two distinct points $a, b \in T$ with $f(a) = f(b)$. Since $f(0_T) \neq f(\infty_T)$, one of the points a, b, say a, belongs to $T \setminus \{0_T, \infty_T\}$. If $b \notin \{0_T, \infty_T\}$, then $ab = \infty_T$ and $aa = 0_T$ and hence $f(\infty_T) = f(ab) = f(a)f(b) = f(a)f(a) = f(aa) = f(0_T)$, which contradicts our assumption. This contradiction shows that $b \in \{0_T, \infty_T\}$ and hence $bc = 0_T$ for any $c \in T$.

If $|T| \geq 4$, then we can find a point $c \in T \setminus \{a, 0_T, \infty_T\}$ and conclude that $f(\infty_T) = f(ac) = f(a)f(c) = f(b)f(c) = f(bc) = f(0_T)$, which contradicts our assumption. So, $|T| \leq 3$ and hence $T = \{a, 0_T, \infty_T\}$ and

$$S = f(T) = \{f(a), f(0_T), f(\infty_T)\} = \{f(b), \{f(0_T), f(\infty_T)\} = \{f(0_T), f(\infty_T)\}.$$

Then $SS = f(\{xy : x, y \in \{0_T, \infty_T\}\}) = \{f(0_T)\}$, which means that S is a zero-semigroup. □

Since the semigroup operation $Z \times Z \to \{0_Z\} \subset Z$ of any zero-semigroup Z is constant and hence is continuous with respect to any topology on X, Proposition 3 implies the following corollary.

Corollary 1. *Every non-isomorphic homomorphic image S of a Taimanov semigroup is a topological semigroup with respect to any topology on S.*

We call that a semigroup S is *algebraically complete* in a class $\mathscr{S}$ of semitopological semigroups if S is a closed subsemigroup in each semitopological semigroup $T \in \mathscr{S}$ containing S as a subsemigroup. Theorem 1 implies the following

Corollary 2. *Each Taimanov semigroup T is algebraically complete in the class of square-topological semigroups satisfying the separation axiom T_1. In particular, T is algebraically complete in the class of T_1-topological semigroups.*

Remark 1. *Corollary 1 implies that for any Taimanov semigroup T and any non-isomorphic surjective homomorphism $h : T \to S$ with the infinite image $S = h(T)$ the semigroup S is a dense proper subsemigroup of some (compact) Hausdorff topological zero-semigroup.*

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