TOPOLOGICAL PROPERTIES OF TAIMANOV SEMIGROUPS

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A semigroup $T$ is called Taimanov if $T$ contains two distinct points $0, \infty$ such that $xy = \infty$ for any distinct elements $x, y \in T \setminus \{0, \infty\}$ and $xy = 0$ in all other cases. We prove that any Taimanov semigroup $T$ has the following topological properties: (i) each $T_1$-topology with continuous shifts on $T$ is discrete; (ii) $T$ is closed in each $T_1$-topological semigroup containing $T$ as a subsemigroup; (iii) every non-isomorphic homomorphic image $Z$ of $T$ is a zero-semigroup and hence $Z$ is a topological semigroup in any topology on $Z$.

We shall follow the terminology of [5, 8, 10, 20].

The problem of non-discrete (Hausdorff) topologization of infinite groups was posed by Markov [17]. This problem was resolved by Ol’shanskiy [19] who constructed an infinite countable group $G$ admitting no non-discrete Hausdorff group topologies. On the other hand, Zelenyuk [23] proved that each group $G$ admits a non-discrete shift-continuous Hausdorff topology $\tau$ with continuous inversion $G \to G, x \mapsto x^{-1}$. In [1, 2.10] it was observed that Ol’shanskiy construction can be modified to produce for every non-zero $m \in \mathbb{Z} \setminus \{-2^n, 2^n : n \in \omega\}$ a countable infinite group $G_m$ admitting

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no non-discrete shift-continuous topology with continuous $m$-th power map $G_m \to G_m$, $x \mapsto x^m$.

Studying the topologizability problem in the class of inverse semigroups, Eberhart and Selden [9] proved that every Hausdorff semigroup topology on the bicyclic semigroup $\mathcal{B}(p, q)$ is discrete. This result was generalized by Bertman and West [4] who proved that every Hausdorff shift-continuous topology on $\mathcal{B}(p, q)$ is discrete. In [2, 3, 6, 7, 11, 12, 13, 14, 15, 16, 18] these topologizability results were extended to some generalizations of the bicyclic semigroup.

Studying the topologizability problem in the class of commutative semigroups [22], Taimanov in [21] constructed a commutative semigroup $\mathfrak{A}_\kappa$ of arbitrarily large cardinality $\kappa$, which admits no non-discrete Hausdorff semigroup topology, but any non-isomorphic homomorphic image $Z$ of $T$ is a zero-semigroup and hence is a topological semigroup in any topology on $Z$. We recall that a semigroup $Z$ is a zero-semigroup if the set $\{xy : x, y \in X\}$ is a singleton $\{z\}$. In this case the element $z$ is the zero-element of the semigroup $S$, i.e., a (unique) element $z \in S$ such that $xz = z = zx$ for all $x \in S$. In this paper we improve the mentioned Taimanov’s result proving that the Taimanov semigroup $\mathfrak{A}_\kappa$ admits no non-discrete shift-continuous $T_1$-topologies and is closed in any $T_1$-topological semigroup containing $\mathfrak{A}_\kappa$ as a subsemigroup. First we give an abstract definition of a Taimanov semigroup.

**Definition 1.** A semigroup $T$ is called Taimanov if it contains two distinct elements $0_T, \infty_T$ such that for any $x, y \in T$

$$x \cdot y = \begin{cases} \infty_T & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0_T, \infty_T\}; \\ 0_T & \text{if } x = y \text{ or } \{x, y\} \cap \{0_T, \infty_T\} \neq \emptyset. \end{cases}$$

The elements $0_T, \infty_T$ are uniquely determined by the algebraic structure of $T$: $0_T$ is a (unique) zero-element of $T$, and $\infty_T$ is the unique element of the set $TT \setminus \{0_T\}$.

It follows that each Taimanov semigroup $T$ is commutative. Concrete examples of Taimanov semigroups can be constructed as follows.

**Example 1.** For any non-zero cardinal $\kappa$ the set $\kappa \cup \{\kappa\}$ endowed with the commutative semigroup operation defined by

$$xy = \begin{cases} \kappa & \text{if } x \neq y \text{ and } x, y \in T \setminus \{0, \kappa\}; \\ 0 & \text{if } x = y \text{ or } \{x, y\} \cap \{0, \kappa\} \neq \emptyset. \end{cases}$$

is a Taimanov semigroup of cardinality $1 + \kappa$. Here we identify the cardinal $\kappa$ with the set $[0, \kappa)$ of ordinals, smaller than $\kappa$.

**Proposition 1.** Two Taimanov semigroups are isomorphic if and only if they have the same cardinality.

**Proof.** Given two Taimanov semigroups $T, S$ of the same cardinality, observe that any bijective map $f : T \to S$ with $f(0_T) = 0_S$ and $f(\infty_T) = \infty_S$ is an algebraic isomorphism of $T$ onto $S$. \qed
In this paper we show that any Taimanov semigroup $T$ has the following topological properties:

1. every shift-continuous $T_1$-topology on $T$ is discrete;
2. $T$ is closed in each $T_1$-topological semigroup containing $T$ as a subsemigroup;
3. every non-isomorphic homomorphic image $Z$ of $T$ is a zero-semigroup and hence any topology on $Z$ turns it into a topological semigroup.

The first statement generalizes the original result of Taimanov [21] and is proved in the following proposition.

**Proposition 2.** Every shift-continuous $T_1$-topology $\tau$ on any Taimanov semigroup $T$ is discrete.

**Proof.** The statement is trivial if the semigroup $T$ is finite. So, assume that $T$ is infinite. The topology $\tau$ satisfies the separation axiom $T_1$ and hence contains an open set $U \subset X$ such that $0_T \notin U$ and $\infty_T \notin U$.

First we prove that the points $0_T$ and $\infty_T$ are isolated in $T$. Chose any point $x \in T \setminus \{0_T, \infty_T\}$ and observe that $x \cdot 0_T = x \cdot \infty_T = 0_T \in U$. By the shift-continuity of the topology $\tau$, there exist neighborhoods $U_0 \in \tau$ of $0_T$ and $U_\infty \in \tau$ of $\infty_T$ such that $\{x \cdot U_0\} \cup \{x \cdot U_\infty\} \subset U$. We claim that $U_0 \setminus \{x, \infty_T\} = \{0_T\}$ and $U_\infty \setminus \{x, 0_T\} = \{\infty_T\}$.

In the opposite case we could find a point $y \in (U_0 \cup U_\infty) \setminus \{x, 0_T, \infty_T\}$ and conclude that $\infty_T = xy \in (U_0 \cup U_\infty) \subset U \subset T \setminus \{\infty_T\}$, which is a desired contradiction showing that the points $0_T$ and $\infty_T$ are isolated in $T$.

To show that each point $x \in T \setminus \{0_T, \infty_T\}$ is isolated in the topology $\tau$, observe that $xx = 0_T \in T \setminus \{\infty_T\} \in \tau$ and use the shift-continuity of the topology $\tau$ to find a neighborhood $U_x \in \tau$ of $x$ such that $xU_x \subset T \setminus \{\infty_T\}$. Assuming that $U_x \neq \{x\}$ we can choose any point $y \in U_x \setminus \{x\}$ and conclude that $\infty_T = xy \in T \setminus \{\infty_T\}$, which is a contradiction showing that $U_x = \{x\}$ and hence the point $x$ is isolated in the topology $\tau$.

The following example shows that any infinite Taimanov semigroup admits a non-discrete semigroup $T_0$-topology.

**Example 2.** For any infinite Taimanov semigroup $T$ the family of subsets

$$\tau := \{U \subset T : \text{if } 0_T \in U, \text{ then } \infty_T \in U \text{ and } |T \setminus U| < \omega\}$$

is a $T_0$-topology turning $T$ into a topological semigroup.

A semitopological semigroup $S$ will be called *square-topological* if the map $S \to S$, $x \mapsto x^2$, is continuous. It is clear that each topological semigroup is square-topological.

**Theorem 1.** A Taimanov semigroup $T$ is closed in any square-topological semigroup $S$ containing $T$ as a subsemigroup and satisfying the separation axiom $T_1$.

**Proof.** Assuming that $T$ is not closed in $S$, choose any point $s \in \overline{T} \setminus T$. We claim that $sx = \infty_T$ for any $x \in T \setminus \{0_T, \infty_T\}$. Assuming that $sx \neq \infty_T$ and using the shift-continuity of the $T_1$-topology of $S$, we can find a neighborhood $U_s \subset S$ of $s$ such that
\( U_\tau \cdot x \subset S \setminus \{ \infty \} \). Since \( s \) is an accumulation point of the set \( T \) in \( S \), there exists a point \( y \in U_s \setminus \{ x, 0_\tau, \infty_T \} \). For this point \( y \) we get \( \infty_T = yx \in U_\tau x \subset S \setminus \{ \infty_T \} \), which is a contradiction showing that \( ss = \infty_T \) for any \( x \in T \setminus \{ 0_T, \infty_T \} \). Next, we show that \( ss = \infty_T \). Assuming that \( ss \neq \infty_T \), we can use the shift-continuity of the \( T_1 \)-topology of \( S \) to find a neighborhood \( V_s \subset S \) of \( s \) such that \( sV_s \subset S \setminus \{ \infty_T \} \). Since \( s \) is an accumulation point of the set \( T \) in \( S \), there exists a point \( x \in V_s \cap T \setminus \{ 0_T, \infty_T \} \). For this point \( x \), we get \( \infty_T = sx \in sV_s \subset S \setminus \{ \infty_T \} \), which is a contradiction showing that \( ss = \infty_T \). By the separation axiom \( T_1 \), the set \( S \setminus \{ 0_T \} \) is an open neighborhood of \( \infty_T \) in \( S \). The continuity of the map \( S \to S, x \mapsto x^2 \), yields a neighborhood \( W_s \subset S \) such that \( x^2 \in S \setminus \{ 0_T \} \) for any \( x \in W_s \). Since \( s \) is an accumulation point of the set \( T \) in \( S \), there exists a point \( x \in W_s \cap T \setminus \{ 0_T, \infty_T \} \). For this point \( x \) we get \( 0_T = xx \in S \setminus \{ 0_T \} \), which is a desired contradiction showing that the set \( T \) is closed in \( S \). \( \square \)

The following example shows that any infinite Taimanov semigroup admits a (non-closed) embedding into a compact Hausdorff semitopological semigroup and also shows that the continuity of the map \( S \to S, x \mapsto x^2 \), in Theorem 1 is essential and cannot be replaced by the continuity of the map \( S \to S, x \mapsto x^m \), for some \( m \geq 3 \).

**Example 3.** Let \( T \) be a Taimanov semigroup and \( X \) be any \( T_1 \)-topological space containing \( T \) as a non-closed dense discrete subspace. Extend the semigroup operation of \( T \) to a binary operation of \( X \) defined by the formula:

\[
xy = \begin{cases} 
0_T & \text{if } x = y \in T \text{ or } \{ x, y \} \cap \{ 0_T, \infty_T \} \neq \emptyset; \\
\infty_T & \text{otherwise.}
\end{cases}
\]

Since \((xy)z = 0_T = x(0_T)\) for any \( x, y, z \in X \) the extended operation is associative and turns \( X \) into a commutative semigroup containing \( T \) as a subsemigroup. Observe that for \( a \in \{ 0_T, \infty_T \} \) the shift \( l_a = r_a : X \to X, x \mapsto xa = 0_T \), is constant and hence continuous. For any \( a \in T \setminus \{ 0_T, \infty_T \} \) the shift \( l_a = r_a : X \to X, x \mapsto xa = ax \), is almost constant in the sense that \( l_a^{-1}(\infty_T) = X \setminus \{ a, 0_T, \infty_T \} \) and hence is continuous (as the set \( \{ a, 0_T, \infty_T \} \) is closed and open in \( X \)). For any \( a \in X \setminus T \) the shift \( l_a = r_a : X \to X, x \mapsto xa = ax \), is almost constant in the sense that \( l_a^{-1}(\infty_T) = X \setminus \{ 0_T, \infty_T \} \) and hence is continuous. This shows that \( X \) is a semitopological commutative semigroup containing \( T \) as a non-closed dense subsemigroup. Observe also that for every \( m \geq 3 \) the map \( X^m \to X, (x_1, \ldots, x_m) \mapsto x_1 \cdots x_m = 0_T \), is constant and hence continuous. Then the map \( X \to X, x \mapsto x^m \), is continuous as well.

**Example 4.** For any topological zero-semigroup \( Z \) with zero \( 0_Z \) and any Taimanov semigroup \( T \) endowed with the discrete topology, any map \( h : T \to Z \) with \( h(0_T) = h(\infty_T) = 0_Z \) is a continuous semigroup homomorphism. Hence there exist many topological (zero-)semigroups containing continuous homomorphic images of Taimanov semigroups as non-closed subsemigroups.

**Proposition 3.** Any non-isomorphic homomorphic image \( S \) of a Taimanov semigroup \( T \) is a zero-semigroup.
Proof. Fix a non-injective surjective homomorphism $h : T \to S$. If $f(0_T) = f(\infty_T)$, then $SS = f(T) \cdot f(T) = f(TT) = f([0_T, \infty_T]) = \{f(0_T)\}$, which means that $S$ is a zero-semigroup. So, assume that $f(0_T) \neq f(\infty_T)$. Since $f$ is not injective, there exist two distinct points $a, b \in T$ with $f(a) = f(b)$. Since $f(0_T) \neq f(\infty_T)$, one of the points $a, b$, say $a$, belongs to $T \setminus \{0_T, \infty_T\}$. If $b \notin \{0_T, \infty_T\}$, then $ab = \infty_T$ and $aa = 0_T$ and hence $f(\infty_T) = f(ab) = f(a)f(b) = f(a)f(a) = f(aa) = f(0_T)$, which contradicts our assumption. This contradiction shows that $b \in \{0_T, \infty_T\}$ and hence $bc = 0_T$ for any $c \in T$.

If $|T| \geq 4$, then we can find a point $c \in T \setminus \{a, 0_T, \infty_T\}$ and conclude that $f(\infty_T) = f(ac) = f(a)f(c) = f(b)f(c) = f(bc) = f(0_T)$, which contradicts our assumption. So, $|T| \leq 3$ and hence $T = \{a, 0_T, \infty_T\}$ and $S = f(T) = \{f(a), f(0_T), f(\infty_T)\} = \{f(b), f(0_T), f(\infty_T)\} = \{f(0_T), f(\infty_T)\}$. Then $SS = f(\{xy : x, y \in \{0_T, \infty_T\}\}) = \{f(0_T)\}$, which means that $S$ is a zero-semigroup.

Since the semigroup operation $Z \times Z \to \{0_Z\} \subset Z$ of any zero-semigroup $Z$ is constant and hence is continuous with respect to any topology on $X$, Proposition 3 implies the following corollary.

**Corollary 1.** Every non-isomorphic homomorphic image $S$ of a Taimanov semigroup is a topological semigroup with respect to any topology on $S$.

We call that a semigroup $S$ is **algebraically complete** in a class $\mathcal{S}$ of semitopological semigroups if $S$ is a closed subsemigroup in each semitopological semigroup $T \in \mathcal{S}$ containing $S$ as a subsemigroup. Theorem 1 implies the following

**Corollary 2.** Each Taimanov semigroup $T$ is algebraically complete in the class of square-topological semigroups satisfying the separation axiom $T_1$. In particular, $T$ is algebraically complete in the class of $T_1$-topological semigroups.

**Remark 1.** Corollary 1 implies that for any Taimanov semigroup $T$ and any non-isomorphic surjective homomorphism $h : T \to S$ with the infinite image $S = h(T)$ the semigroup $S$ is a dense proper subsemigroup of some (compact) Hausdorff topological zero-semigroup.

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