Математичний Вісник
Наукового Товариства
ім. Тараса Шевченка
2016. — Т.13

Mathematical Bulletin
of Taras Shevchenko
Scientific Society
2016. — V.13

CLASSIFYING LOCALLY COMPACT SEMITOPOLOGICAL POLYCYCLIC MONOIDS

SERHII BARDYLA

Faculty of Mechanics and Mathematics, Ivan Franko National University, Lviv

S. Bardyla, *Classifying locally compact semitopological polycyclic monoids*, Math. Bull. Shevchenko Sci. Soc. **13** (2016) 21–28.

We present a complete classification of Hausdorff locally compact polycyclic monoids up to a topological isomorphism. A *polycyclic monoid* is an inverse monoid with zero, generated by a subset Λ such that $xx^{-1} = 1$ for any $x \in \Lambda$ and $xy^{-1} = 0$ for any distinct $x, y \in \Lambda$. We prove that any non-discrete Hausdorff locally compact topology with continuous shifts on a polycyclic monoid S coincides with the topology of one-point compactification of the discrete space $S \setminus \{0\}$.

С. Бардила. *Класифікація локально компактних напівтопологічних поліциклічних моноїдів* // Мат. вісн. Наук. тов. ім. Шевченка. — 2016. — Т.13. — С. 21–28.

Отримано повну класифікацію гаусдорфових локально компактних напівтопологічних поліциклічних моноїдів. *Поліциклічним моноїдом* називається інверсний моноїд з нулем і множиною генераторів Λ такою, що $xx^{-1} = 1$ для довільного $x \in \Lambda$ і $xy^{-1} = 0$ для довільних різних $x, y \in \Lambda$. Доведено, що кожна недискретна гаусдорфова локально компактна топологія з неперервними зсувами на поліциклічному моноїді S збігається з топологією одноточкової компактифікації дискретного простору $S \setminus \{0\}$.

Introduction

In this paper we present a complete classification of locally compact semitopological polycyclic monoids up to a topological isomorphism.

We shall follow the terminology of [6, 8, 14, 17]. First we recall some information on inverse semigroups and monoids. We identify cardinals with the sets of ordinals of smaller cardinality.

A *semigroup* is a set S endowed with an associative binary operation $\cdot : S \times S \to S$, $\cdot : (x, y) \mapsto xy$. An element $e \in S$ is called the *unit* (resp. *zero*) of S if $xe = x = ex$ (resp. $xe = e = ex$) for all $x \in S$. A semigroup can contains at most one unit (which

2010 *Mathematics Subject Classification:* 20M18, 22A15 *УДК:* 512.536

Key words and phrases: locally compact semitopological semigroup, bicyclic semigroup, polycyclic monoid, λ-polycyclic monoid.

E-mail: sbardyla@yahoo.com

will be denoted by 1) and at most one zero (denoted by 0). A *monoid* if a semigroup with a unit.

A semigroup S is called *inverse* if for every element $a \in S$ there exists a unique element a^{-1} (called the *inverse* of a) such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. An *inverse monoid* is an inverse semigroup with unit. We say that an inverse monoid S *is generated* by a subset $\Lambda \subset S$ if S coincides with the smallest subsemigroup of S containing the set $\Lambda \cup \Lambda^{-1}$.

A *polycyclic monoid* is an inverse monoid S with zero $0 \neq 1$, which is generated by a subset $\Lambda \subset S$ such that $xx^{-1} = 1$ for all $x \in \Lambda$ and $xy^{-1} = 0$ for any distinct $x, y \in \Lambda$. If the generating set Λ has cardinality λ, then S is called a *λ-polycyclic monoid*. We claim that $|\Lambda| \geq 2$. In the opposite case, $\Lambda = \{x\}$ is a singleton and $0 \in S = \{x^{-n}x^m : n, m \in \omega\}$, which implies that $0 = x^{-n}x^m$ for some non-negative numbers n, m. Then $0 = x^{n+1} \cdot 0 \cdot x^{-m} = x^{n+1}(x^{-n}x^m)x^{-m} = x$ and hence $1 = xx^{-1} = 0x^{-1} = 0$, but this contradicts the definition of a polycyclic monoid.

A canonical example of a λ-polycyclic monoid can be constructed as follows. Let $M_{\lambda\pm}$ be the monoid of all words in the alphabet $\{x, x^{-1} : x \in \lambda\}$, endowed with the semigroup operation of concatenation of words. The empty word is the unit 1 of the monoid $M_{\lambda\pm}$. Let $M^0_{\lambda\pm} := M_{\lambda\pm} \cup \{0\}$ be the monoid $M_{\lambda\pm}$ with the attached external zero, i.e., an element $0 \notin M_{\lambda\pm}$ such that $0 \cdot x = 0 = x \cdot 0$ for all $x \in M^0_{\lambda\pm}$. On the monoid $M^0_{\lambda\pm}$ consider the smallest congruence $\sim$ containing the pairs $(xx^{-1}, 1)$ and $(xy^{-1}, 0)$ for all distinct elements $x, y \in \lambda$. Then the quotient semigroup $M^0_{\lambda\pm}/_\sim$ is the required canonical example of a λ-polycyclic monoid, which will be denoted by P_λ and called *the λ-polycyclic monoid*.

Algebraic properties of the λ-polycyclic monoid were deeply investigated in the papers [16, 11, 12, 4] and the monograph [14, §6.3]. According to Theorem 5 in [14, §6.3] and Theorem 2.5 in [4], the semigroup P_λ is congruence-free, which implies that each λ-polycyclic monoid is algebraically isomorphic to P_λ.

The aim of this paper is to describe Hausdorff locally compact topologies on P_λ, compatible with the algebraic structure of the semigroup P_λ. A suitable compatibility condition is given by the notion of a semitopological semigroup.

A *semitopological semigroup* is a semigroup S endowed with a Hausdorff topology making the binary operation $S \times S \to S$, $(x, y) \mapsto xy$, separately continuous. If this operation is jointly continuous, then S is called a *topological semigroup*.

For a cardinal $\lambda \geq 2$ by P^d_λ we shall denote the λ-polycyclic monoid P_λ endowed with the discrete topology, and by P^c_λ the monoid P_λ endowed with the compact topology $\tau = \big\{U \subset P_\lambda : 0 \in U \Rightarrow (P_\lambda \setminus U \text{ is finite})\big\}$ of one-point compactification of the discrete space $P_\lambda \setminus \{0\}$. It is clear that P^d_λ is a topological monoid. On the other hand, P^c_λ is a compact semitopological monoid, which is not a topological semigroup.

By [4], each locally compact topological λ-polycyclic monoid is discrete and hence is topologically isomorphic to P^d_λ. In the semitopological case we have the following dichotomy, which is the main result of this paper.

Main Theorem. *Any locally compact semitopological polycyclic monoid S is either discrete or compact. More precisely, S is topologically isomorphic either to P^d_λ or to*

P_λ^c *for a unique cardinal* $\lambda \geq 2$.

Since the compact semitopological λ-polycyclic monoid P_λ^c fails to be a topological semigroup, Main Theorem implies the mentioned result of [4]:

Corollary. *Any locally compact topological polycyclic monoid S is discrete. More precisely, S is topologically isomorphic to the topological λ-polycyclic monoid P_λ^d for a unique cardinal $\lambda \geq 2$.*

Some other topologizability results of the same flavor can be found in [7, 19, 18, 13, 1, 9, 15, 2, 10, 3, 4, 5].

Proof of Main Theorem

The proof of Main Theorem is divided into a series of 12 lemmas.

Let S be a non-discrete locally compact semitopological polycyclic monoid and let Λ be its generating set. By Theorem 5 in [14, §6.3] and Theorem 2.5 in [4], the polycyclic monoids are conguence-free, which implies that S is algebraically isomorphic to the λ-polycyclic monoid P_λ for some $\lambda \geq 2$. Theorem 2.2 in [4] implies that the cardinal λ is unique. So, we can identify S with P_λ and the cardinal λ with the generating set Λ of the inverse monoid S.

Let S^+ be the submonoid of S, generated by the set Λ (i.e., S^+ is the smallest submonoid of S containing the generating set Λ). Elements of S^+ can be identified with words in the alphabet Λ. Such words will be called *positive*. The relations between the generators of S guarantee that each non-zero element a of S can be uniquely written as $u^{-1}v$ for some positive words $u, v \in S^+$. Then by $\downarrow a$ we denote the set of all prefixes of the word $u^{-1}v$. For a subset $C \subset S$ we put $\downarrow C = \bigcup_{a \in C} \downarrow a$.

The following algebraic property of a polycyclic monoid is proved in [4, Proposition 2.7].

Lemma 1. *For any non-zero elements $a, b, c \in S$, the set $\{x \in S : axb = c\}$ is finite.*

This lemma will be applied in the proof of the following useful fact that can be found in [4, Proposition 3.1].

Lemma 2. *All non-zero elements of S are isolated points in the space S.*

Proof. For convenience of the reader we present a short proof of this important lemma. First we show that the unit 1 is an isolated point of the semitopological monoid S. Take any generator $g \in \Lambda$ and consider the idempotent $e = g^{-1}g$ of S. Since the map $S \to eS$, $x \mapsto ex$, is a retraction of the Hausdorff space S onto eS, the principal right ideal $eS = g^{-1}S$ is closed in S. By the same reason, the principal left ideal $Se = Sg$ is closed in S. The separate continuity of the semigroup operation yields a neighborhood $U_1 \subset S \setminus (g^{-1}S \cup Sg)$ of 1 such that $0 \notin (e \cdot U_1) \cap (U_1 \cdot e)$. We claim that $U_1 = \{1\}$. In the opposite case, U_1 contains some element $a \neq 1$, which can be written as $u^{-1}v$ for some positive words $u, v \in S^+$. Since $a \neq 1$ one of the words u, v is not empty. If u is not empty, then $a \in U_1 \subset S \setminus g^{-1}S$ implies that the word u^{-1} does not start with

g^{-1}. In this case $ea = g^{-1}gu^{-1}v = g^{-1} \cdot 0 = 0$, which contradicts the choice of the neighborhood $U_1 \ni a$. If the word v is not empty, then $a \in U_1 \subset S \setminus Sg$ implies that v does not end with g. In this case $ae = u^{-1}vg^{-1}g = 0$, again contradicting the choice of U_1. This contradiction shows that the unit 1 is an isolated point of S.

Now we can prove that each non-zero point $a \in S$ is isolated. Write a as $u^{-1}v$ for some positive words $u, v \in S^+$. Since $uav^{-1} = 1$, the separate continuity of the semigroup operation on S, yields an open neighborhood $O_a \subset S$ of a such that $uO_av^{-1} \subset U_1 = \{1\}$. By Lemma 1, the neighborhood O_a is finite and hence the singleton $\{a\} = O_a \setminus (O_a \setminus \{a\})$ is open, which means that the point a is isolated in S. □

Lemma 2 implies that the locally compact space S has a neighborhood base at zero, consisting of compact sets. It also implies the following useful lemma.

Lemma 3. *For any compact neighborhoods $U_0, V_0 \subset S$ of zero the set $U_0 \setminus V_0$ is finite.*

For an element $u \in S$ by $\mathcal{R}_u := \{x \in S : xS = uS\}$ we denote its *Green $\mathcal{R}$-class* in S. Here $uS = \{us : s \in S\}$ is the right principal ideal generated by the element u.

Lemma 4. *Every non-zero $\mathcal{R}$-class in S coincides with the $\mathcal{R}$-class $\mathcal{R}_{u^{-1}} = \mathcal{R}_{u^{-1}u}$ for some positive word $u \in S^+$.*

Proof. Each non-zero element of the semigroup P_λ can be written as $u^{-1}v$ for some positive words $u, v \in S^+$. Taking into account that $u^{-1}v \cdot v^{-1} = u^{-1}$, we conclude that $\mathcal{R}_{u^{-1}v} = \mathcal{R}_{u^{-1}} = \mathcal{R}_{u^{-1}u}$. □

In the following Lemmas 5–12 we assume that U_0 is any fixed compact neighborhood of zero in the semitopological monoid S. Since zero is a unique non-isolated point in S, the neighborhood U_0 is infinite.

Lemma 5. *The neighborhood U_0 has infinite intersection with some $\mathcal{R}$-class of S.*

Proof. To derive a contradiction, assume U_0 has finite intersection with each $\mathcal{R}$-class of the semigroup S. Taking into account that U_0 is infinite and applying Lemma 4, we can see that the set $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \cap U_0 \neq \varnothing\}$ is infinite. For every $u \in B$ denote by v_u a longest positive word in S^+ such that $u^{-1}v_u \in \mathcal{R}_{u^{-1}} \cap U_0$ (such word v_u exists as the set $\mathcal{R}_{u^{-1}} \cap U_0$ is finite). It follows that $A = \{u^{-1}v_u : u \in B\}$ is an infinite subset of U_0. Fix any element g of the generating set Λ of S. Since $0 \cdot g = 0$, we can use the separate continuity of the semigroup operation of S and find a compact neighborhood $V_0 \subseteq U_0$ of zero such that $V_0 \cdot g \subseteq U_0$. But then $V_0 \subseteq U_0 \setminus A$ which contradicts Lemma 3. □

Lemma 6. *The neighborhood U_0 has infinite intersection with each non-zero $\mathcal{R}$-class of the semigroup S.*

Proof. By Lemma 4, any non-zero $\mathcal{R}$-class of the semigroup $S = P_\lambda$ is of the form $\mathcal{R}_{v^{-1}}$ for some positive word $v \in S^+$. By Lemmas 4 and 5, for some element $u \in S^+$

the intersection $U_0 \cap \mathcal{R}_{u^{-1}}$ is infinite. Observe that $v^{-1}u \cdot \mathcal{R}_{u^{-1}} \subset \mathcal{R}_{v^{-1}}$. By the separate continuity of the semigroup operation at $0 = v^{-1}u \cdot 0$, there exists a neighborhood $V_0 \subset S$ of zero such that $v^{-1}u \cdot V_0 \subset U_0$. By Lemma 3, the difference $U_0 \setminus V_0$ is finite, which implies that the intersection $V_0 \cap \mathcal{R}_{u^{-1}}$ is infinite. Then the set $v^{-1}u \cdot (V_0 \cap \mathcal{R}_{u^{-1}}) \subset U_0 \cap \mathcal{R}_{v^{-1}}$ is infinite, too. □

Lemma 7. *If the generating set Λ is finite, then the neighborhood U_0 contains all but finitely many elements of the $\mathcal{R}$-class $\mathcal{R}_1 = \{x \in S : xS = S\}$.*

Proof. To derive a contradiction, assume that the set $A := \mathcal{R}_1 \setminus U_0$ is infinite. We claim that for every $g \in \Lambda$ the set $A_g = \{a \in A : ag \in U_0\}$ is finite. Indeed, suppose that A_g is infinite. By Proposition 1, $A_g \cdot g$ is an infinite subset of U_0. Since $0 \cdot g^{-1} = 0$, the separate continuity of the semigroup operation on S yields a compact neighborhood $V_0 \subseteq U_0$ of zero such that $V_0 \cdot g^{-1} \subseteq U_0$. Then $V_0 \subseteq U_0 \setminus (A_g \cdot g)$ which contradicts Lemma 3.

Let $A^* = A \setminus \bigcup_{g \in \Lambda} {\downarrow} A_g$ (we recall that ${\downarrow} A_g = \bigcup_{a \in A_g} {\downarrow} a$ where ${\downarrow} a$ is the set of all prefixes of the word a). It follows that A^* is a cofinite (and hence infinite) subset of A. Now we are going to show that A^* is a right ideal of $\mathcal{R}_1$. In the opposite case we could find elements $c \in \mathcal{R}_1$ and $v \in A^*$ such that $vc \notin A^*$. Let c^* be the longest prefix of c such that $vc^* \in A^*$ (the word c^* can be empty, in which case it is the unit of S). Then $vc^*g \notin A^*$ for some $g \in \Lambda$. Observe that $vc^* \in A^* \subset A \subset \mathcal{R}_1$ implies $vc^*g \in \mathcal{R}_1$. Assuming that $vc^*g \in U_0$, we conclude that $vc^* \in A_g \subset {\downarrow} A_g$, which contradicts the inclusion $vc^* \in A^*$. So, $vc^*g \notin U_0$ and hence $vc^*g \in A$. Then $vc^*g \notin A^*$ implies that $vc^*g \in {\downarrow} A_f$ for some $f \in \Lambda$ and thus $vc^* \in {\downarrow} A_f$, too. But this contradicts the inclusion $vc^* \in A^*$. The obtained contradiction implies that A^* is a right ideal of $\mathcal{R}_1$.

Let $u \in A^*$ be an arbitrary element. Since $u \cdot 0 = 0$, the separate continuity of the semigroup operation yields a compact neighborhood $V_0 \subset U_0$ of zero such that $u \cdot V_0 \subseteq U_0$. Proposition 1 and Lemma 6 imply that $u \cdot (V_0 \cap \mathcal{R}_1)$ is an infinite subset of $A^* \cap U_0 \subset A \cap U_0$. In particular, $A \cap U_0$ is not empty, which contradicts the definition of the set $A := \mathcal{R}_1 \setminus U_0$. □

Lemma 8. *If the cardinal $\lambda = |\Lambda|$ is finite, then the neighborhood U_0 contains all but finitely many elements of any $\mathcal{R}$-class $\mathcal{R}_x$, $x \in S$.*

Proof. The lemma is trivial if $x = 0$. So we assume that $x \neq 0$. By Lemma 4, $\mathcal{R}_x = \mathcal{R}_{u^{-1}}$ for some positive word $u \in S^+$. Since $u^{-1} \cdot 0 = 0$, the separate continuity of the semigroup operation yields an neighborhood $V_0 \subseteq U_0$ of zero such that $u^{-1} \cdot V_0 \subseteq U_0$. By Lemmas 3 and 7, $\mathcal{R}_1 \subset^* V_0$ (which means that $\mathcal{R}_1 \setminus V_0$ is finite). Then $\mathcal{R}_x = \mathcal{R}_{u^{-1}} = u^{-1} \cdot \mathcal{R}_1 \subset^* u^{-1} \cdot V_0 \subset U_0$, which means that U_0 contains all but finitely many points of the $\mathcal{R}$-class $\mathcal{R}_x$. □

The following lemma proves Main Theorem in case of finite cardinal $\lambda = |\Lambda|$.

Lemma 9. *If the cardinal λ is finite, then the set $S \setminus U_0$ is finite.*

Proof. To derive a contradiction, assume that $S \setminus U_0$ is infinite. By Lemma 8, for each $u \in S^+$ the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite. Since the complement $S \setminus U_0 = \bigcup_{u \in S^+} \mathcal{R}_{u^{-1}} \setminus U_0$ is infinite, the set $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \setminus U_0 \neq \varnothing\}$ is infinite, too. For every $u \in B$ denote by v_u the longest word in S^+ such that $u^{-1}v_u \in \mathcal{R}_{u^{-1}} \setminus U_0$. Then $C = \{u^{-1}v_u : u \in B\} \subset \mathcal{R}_{u^{-1}} \setminus U_0$ is infinite and by Proposition 1, for every $g \in \Lambda$ the set $C \cdot g$ is an infinite subset of U_0. Since $0 \cdot g^{-1} = 0$, the separate continuity of the semigroup operation yields a neighborhood $V_0 \subset U_0$ of zero such that $V_0 \cdot g^{-1} \subseteq U_0$. By Lemma 3, the set $U_0 \setminus V_0$ is finite. Since the set $Cg \subset U_0$ is infinite, there is an element $c \in C$ with $cg \in V_0$. Then $c = cgg^{-1} \in V_0 g^{-1} \subset U_0$, which contradicts the inclusion $C \subset \mathcal{R}_1 \setminus U_0$. □

Lemma 10. *The set $\mathcal{R}_1 \setminus U_0$ is finite.*

Proof. To derive a contradiction, assume that the complement $A := \mathcal{R}_1 \setminus U_0$ is infinite. By Lemma 6, the set $U_0 \cap \mathcal{R}_1$ is infinite.

For a finite subset $F \subset \Lambda$, let S_F be the smallest subsemigroup of S containing the set $F \cup F^{-1} \cup \{0, 1\}$. If $|F| \geq 2$, then S_F is a polycyclic monoid. Separately, we shall consider two cases.

1. First assume that for every finite subset $F \subset \Lambda$ the set $U_0 \cap S_F$ is finite. In this case for every point $g \in \Lambda$, consider the set $W_g = \{a \in U_0 \cap \mathcal{R}_1 : ag \notin U_0\}$. The separate continuity of the semigroup operation yields a neighborhood $V_0 \subset U_0$ of zero such that $V_0 \cdot g \subset U_0$. Lemma 3 implies that the set $W_g \subset U_0 \setminus V_0$ is finite and hence for every non-empty finite subset $F \subset \Lambda$ the set $U_F := (U_0 \cap \mathcal{R}_1) \setminus \bigcup_{g \in F} W_g$ is infinite. We claim that $U_F \cdot y \subseteq U_F$ for every $y \in S_F \cap \mathcal{R}_1$. In the opposite case, there exist elements $y \in S_F \cap \mathcal{R}_1$ and $x \in U_F$ such that $xy \notin U_F$. Let y^* be the longest prefix of y such that $xy^* \in U_F$ (note that y^* could be equal to 1). Then $xy^*g \notin U_F$ for some $g \in F$. Hence $xy^* \in W_g$ which contradicts the definition of $U_F \ni xy^*$. Hence $U_F \cdot y \subseteq U_F$ for each element $y \in S_F \cap \mathcal{R}_1$.

Fix any element $v \in U_F$ and find a finite subset $D \subset \Lambda$ such that $v \in S_D$, $F \subset D$ and $|D| \geq 2$. Proposition 1 implies that $v \cdot (S_F \cap \mathcal{R}_1)$ is an infinite subset of $U_F \cap S_D$, which contradicts our assumption.

2. Next, assume that for some finite subset $F \subset \Lambda$ the intersection $U_0 \cap S_F$ is infinite. For every $g \in F$ consider the subset $A_g := \{a \in A : ag \in U_0\}$ of the infinite set $A = \mathcal{R}_1 \setminus U_0$. The separate continuity of the semigroup operation yields a neighborhood $V_0 \subset S$ of zero such that $V_0 \cdot g^{-1} \subset U_0$. We claim that for every $a \in A_g$ we get $ag \notin V_0$. In the opposite case we would get $a = agg^{-1} \in V_0 \cdot g^{-1} \subset U_0$, which contradicts the inclusion $a \in A$. Then $A_g = \{a \in A : ag \in U_0 \setminus V_0\}$ and this set is finite by Lemmas 3 and 1. It follows that $A_F = A \setminus \bigcup_{g \in F} {\downarrow} A_g$ is a cofinite (and hence infinite) subset of A.

We claim that $A_F \cdot y \subseteq A_F$ for every $y \in S_F \cap \mathcal{R}_1$. In the opposite case, we can find elements $y \in S_F \cap \mathcal{R}_1$ and $x \in A_F$ such that $xy \notin A_F$. Let y^* be the longest prefix of y such that $xy^* \in A_F$ (note that y^* could be equal to 1). Then $xy^*g \notin A_F$ for some $g \in F$. It follows from $xy^* \in A_F \subset A = \mathcal{R}_1 \setminus U_0$ and $gg^{-1} = 1$ that $xy^*g \in \mathcal{R}_1$. Assuming that $xy^*g \in U_0$, we conclude that $xy^* \in A_g$, which contradicts the inclusion $xy^* \in A_F$. So, $xy^*g \in \mathcal{R}_1 \setminus U_0 = A$ and then $xy^*g \notin A_F$ implies that $xy^*g \in {\downarrow}A_h$ for some $h \in F$ and finally $xy^* \in {\downarrow}A_h$, which contradicts the inclusion $xy^* \in A_F$. This contradiction completes the proof of the inclusion $A_F \cdot y \subseteq A_F$ for each $y \in S_F \cap \mathcal{R}_1$.

Fix any element $v \in A_F$ and find a finite subset $D \subset \Lambda$ such that $v \in S_D$, $F \subset D$ and $|D| \geq 2$. The subset S_D contains the unique non-isolated point of the space S and hence is closed in S. The local compactness of S implies the local compactness of the polycyclic monoid S_D endowed with the subspace topology. Lemma 3 and our assumption guarantee that the semitopological polycyclic monoid S_D is not discrete. By Proposition 1, $v \cdot (S_F \cap \mathcal{R}_1)$ is an infinite subset of $A_F \cap S_D \subset S_D \setminus U_0$. But this contradicts Lemma 9 (applied to the locally compact polycyclic monoid S_D and the neighborhood $U_0 \cap S_D$ of zero in S_D). □

Lemma 11. *The neighborhood U_0 contains all but finitely many points of each $\mathcal{R}$-class in S.*

Proof. By Lemma 4, it suffices to check that for any $u \in S^+$ the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite. The separate continuity of the semigroup operation yields a compact neighborhood $V_0 \subseteq U_0$ of zero such that $u^{-1} \cdot V_0 \subseteq U_0$. By Lemmas 10 and 3, we get $\mathcal{R}_1 \subset^* V_0$. Then $\mathcal{R}_{u^{-1}} = u^{-1} \cdot \mathcal{R}_1 \subset^* u^{-1} \cdot V_0 \subset U_0$, which means that the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite. □

Our final lemma combined with Lemma 2 proves Main Theorem and shows that the semitopological polycyclic monoid S carries the topology of one-point compactification of the discrete space $S \setminus \{0\}$.

Lemma 12. *The complement $S \setminus U_0$ is finite and hence S is compact.*

Proof. To derive a contradiction, assume that the set $S \setminus U_0$ is infinite. By Lemma 11, for each $u \in S^+$ the set $\mathcal{R}_{u^{-1}} \setminus U_0$ is finite. Since $S = \bigcup_{u \in S^+} \mathcal{R}_{u^{-1}}$, the set $B = \{u \in S^+ : \mathcal{R}_{u^{-1}} \setminus U_0 \neq \varnothing\}$ is infinite. For every $u \in B$ denote by v_u the longest word in S^+ such that $u^{-1}v_u \in \mathcal{R}_{u^{-1}} \setminus U_0$. Then $C = \{u^{-1}v_u : u \in B\}$ is an infinite subset of $S \setminus U_0$. By Lemma 1, for any $g \in \Lambda$ the set $C \cdot g$ is infinite. The separate continuity of the semigroup operation yields a neighborhood $V_0 \subset U_0$ of zero such that $V_0 \cdot g^{-1} \subset U_0$. Then $V_0 \subset U_0 \setminus (C \cdot g)$ which contradicts Lemma 3. □

Acknowledgements

The author would like to express his thanks to Taras Banakh and Oleg Gutik for their fruitful comments and suggestions.

REFERENCES

1. T. Banakh, S. Dimitrova, O. Gutik, *The Rees-Suschkiewitsch Theorem for simple topological semigroups*, Mat. Stud. **31**:2 (2009), 211–218.
2. T. Banakh, I. Protasov, O. Sipacheva, *Topologization of sets endowed with an action of a monoid*, Topology Appl. **169** (2014) 161–174.
3. S. Bardyla, *On a semitopological α-bicyclic monoid*, Visn. L'viv. Univ., Ser. Mekh.-Mat. **81** (2016), (to appear).
4. S. Bardyla, O. Gutik, *On a semitopological polycyclic monoid*, Algebra Discr. Math. **21** (2016), no. 2, 163–183.
5. S. Bardyla, O. Gutik, *On a complete topological inverse polycyclic monoid,* Carpathian Math. Publ. (submitted), arXiv:1603.08147.
6. A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups*, Vols. I and II, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961 and 1967.
7. C. Eberhart, J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. **144** (1969), 115–126.
8. R. Engelking, *General Topology*, Heldermann, Berlin, 1989.
9. I.R. Fihel, O.V. Gutik, *On the closure of the extended bicyclic semigroup*, Carpathian Math. Publ. **3**:2, (2011) 131–157.
10. O. Gutik, *On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero*, Visn. L'viv. Univ., Ser. Mekh.-Mat. **80** (2015), 33–41.
11. D.G. Jones, *Polycyclic monoids and their generalizations*. PhD Thesis, Heriot-Watt University, 2011.
12. D.G. Jones, M.V. Lawson, *Graph inverse semigroups: Their characterization and completion*, J. Algebra **409** (2014), 444–473.
13. J.W. Hogan *Hausdorff topologies on the α-bicyclic semigroup*, Semigroup forum **36**:1 (1987), 189–209.
14. M. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries*, Singapore: World Scientific, 1998.
15. Z. Mesyan, J.D. Mitchell, M. Morayne, Y.H. Péresse, *Topological graph inverse semigroups*, Topology Appl. **208** (2016), 106–126.
16. M. Nivat, J.-F. Perrot, *Une généralisation du monoide bicyclique*, C. R. Acad. Sci., Paris, Sér. A **271** (1970), 824–827.
17. W. Ruppert, *Compact Semitopological Semigroups: An Intrinsic Theory*, Lect. Notes Math., **1079**, Springer, Berlin, 1984.
18. A.A. Selden, *A nonlocally compact nondiscrete topology for the α-bicyclic semigroup*, Semigroup forum **31** (1985), 372–374.
19. A. Weil, *L'integration dans les groupes lopologiques et ses applications*, Actualites Scientifiques No. **869**, Hermann, Paris, 1938.

Received 10.10.2016

Revised 11.11.2016