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FAST GROWING ENTIRE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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We investigate the fast growing entire solutions of linear differential equations. For that we introduce a general scale to measure the growth of entire functions of infinite order including arbitrary fast growth. We describe growth relations between entire coefficients and solutions of the linear differential equation $f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0$ in this scale. Obtained results contain those for iterated orders as a special case.

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Ми досліджуємо швидкозростаючі цілі розв'язки лінійних диференціальних рівнянь. Для цього ми вводимо узагальнену шкалу для вимірювання зростання цілих функцій нескінченного порядку та довільного швидкого зростання. Описано зв'язок між зростанням коефіцієнтів та розв'язків лінійного диференціального рівняння $f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0$ у цій шкалі. Отримані результати містять результати для ітераційного порядку як окремий випадок.

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1. Introduction and formulation of the main results

Let us consider the linear differential equations of the form

$$L(f) := f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0, \tag{1}$$

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where $a_0,\dots,a_{n-1}$ are entire functions, $n \geq 2$, $a_0 \not\equiv 0$. It is well-known that all solutions are entire in this case, moreover ([16]) all coefficients of (1) are polynomials if and only if all solutions have growth of finite order.

If the coefficients are transcendental and entire, then the solutions are of infinite order, in general. There are several scales to measure the growth of functions of infinite order (see e.g. [12, 14]). Many mathematicians, such as L. Kinnunen, J. Heittokangas, R. Korhonen, J. Rättyä, T.B. Cao, Z.X. Chen, K. Hamani, B. Belaïdi, and others used the iterated orders [9], [7], [3], [2], [6] to study the growth of solutions of (1). It is introduced as follows.

For $r \in [0, +\infty)$ define the iterations $\exp^{[1]} r = e^r$, $\exp^{[n+1]} r = \exp(\exp^{[n]} r)$, $n \in \mathbb{N}$, and for all sufficiently large r define $\ln^{[1]} r = \ln r$, $\ln^{[n+1]} r = \ln(\ln^{[n]} r)$, $n \in \mathbb{N}$. Also, $\exp^{[0]} r = r = \ln^{[0]} r$.

For $i \in \mathbb{N}$ the value

$$\sigma_i(f) = \limsup_{r\to+\infty} \frac{\ln^{[i]} T(r, f)}{\ln r}$$

is called *i-th iterated order* of a meromorphic function f, where

$$T(r, f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln^+ |f(re^{i\theta})|d\theta + N(r, f)$$

is the Nevanlinna characteristic of f ([5]) and $N(r, f)$ is the Nevanlinna counting function of poles.

Note that $\sigma_1(f)$ coincides with the usual concept of the order. In particular, the following estimate for order of the growth holds.

Theorem 1.1 ([16]). *Any solution f of the equation (1) with polynomial coefficients*

$$a_j(z) = \sum_{k=0}^{\alpha_j} c_{kj} z^k, \qquad j = 0, \dots, n-1,$$

has order $\sigma_1(f) \leq 1 + \max\limits_{0\leq j<n} \dfrac{\alpha_j}{n-j}$.

The *finiteness degree* of the order of a meromorphic function f ([9]) is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational;} \\ \min\{j \in \mathbb{N} \mid \sigma_j(f) < +\infty\}, & \text{for } f \text{ transcendental for which some} \\ & j \in \mathbb{N} \text{ with } \sigma_j(f) < +\infty \text{ exists;} \\ +\infty, & \text{for } f \text{ with } \sigma_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For entire solutions of (1) we introduce the following notation

$$\delta := \sup\{i(f) \mid L(f) = 0\}, \tag{2}$$

$$\gamma_j := \sup\{\sigma_j(f) \mid L(f) = 0\}, \quad j \in \mathbb{N}, \tag{3}$$

$$p := \max\{i(a_j) \mid j = 0, \dots, n-1\}. \tag{4}$$

If $0 < p < +\infty$, we also define

$$\varkappa := \max\{\sigma_p(a_j) \mid j = 0, \dots, n-1\}. \tag{5}$$

Here there are typical results that establish the relations between maximal growth of solutions and coefficients.

Theorem 1.2 ([9]). *Let p, δ, γ_p and $\varkappa$ be defined by* (2)–(5). *If $0 < p < +\infty$, then $\delta = p + 1$ and $\gamma_{p+1} = \varkappa$.*

Theorem 1.3 ([9]). *Let p be defined by* (4). *If $0 < p < +\infty$ and $j = \max\{k \mid i(a_k) = p,\ k = 0, \dots, n-1\}$, then* (1) *possesses at most j linearly independent solutions f with $i(f) \le p$.*

J. Lin, J. Tu and L.Z. Shi considered ([11]) a slightly more flexible scale to study the growth of solutions. Namely, they used the following definition of the order from [12] $(p, q \in \mathbb{N})$

$$\sigma_{[p,q]}(f) = \limsup_{r \to +\infty} \frac{\ln^{[p]} T(r, f)}{\ln^{[q]} r},$$

and obtained counterparts of results proved for iterated orders.

Both definitions of iterated orders and of $[p, q]$-orders have the disadvantage that they do not cover arbitrary growth, i.e. there exist functions of infinite $[p, q]$-order and pth iterated order for arbitrary $p \in \mathbb{N}$, i.e. of infinite degree.

Since we have not found an appropriate reference, we give an example for convenience of the reader.

Example 1.4. Similarly to the iterations of the exponent we define $2^{[1]}(x) = 2^x$, $2^{[n+1]}(x) = 2^{2^{[n]}(x)}$ for $n \in \mathbb{N}$. We consider the lacunary series

$$\psi(z) = \sum_{n=0}^{+\infty} 2^{-n 2^{[n]}(n)} z^{2^{[n]}(n)}. \tag{6}$$

It is easy to check that the radius of convergence of series (6) equals infinity, therefore ψ is an entire function.

We estimate from below the maximal term $\mu(r, \psi) = \max\{2^{-k 2^{[k]}(k)} r^{2^{[k]}(k)} : k \ge 0\}$, $|z| = r$ of series at the points $r_k = e^k$, $k \in \mathbb{N}$. From the definition of maximal term it follows that

$$\mu(r_n, \psi) \ge 2^{-n 2^{[n]}(n)} e^{n 2^{[n]}(n)} = r_n^{(1-\ln 2) 2^{[n]}(n)} =$$
$$= r_n^{(1-\ln 2) 2^{[n-2]}(4^n)} > r_n^{(1-\ln 2) 2^{[n-2]}(r_n)}. \tag{7}$$

It is clear that $\ln^{[j-1]} 2^{[j]}(r) \geq r$ as $r \to +\infty$ for every fixed $j \in \mathbb{N}$.
Hence, we have

$$\ln \mu(r_n, \psi) \geq (1 - \ln 2)2^{[n-2]}(r_n) \ln r_n \geq 2^{[n-2]}(r_n), \quad n \geq 3,$$

$$\ln^{[j]} \mu(r_n, \psi) \geq \ln^{[j-1]} 2^{[n-2]}(r_n) \geq 2^{[n-2-j]}(r_n), \quad n \to +\infty.$$

By Cauchy's inequality $\ln^{[j]} M(r_n, \psi) \geq \ln^{[j]} \mu(r_n, \psi) \geq 2^{[n-2-j]}(r_n)$ as $n \to +\infty$ hence $\sigma_j(\psi) = +\infty$ for all $j \in \mathbb{N}$.

We consider a more general scale, which does not have this disadvantage (cf. [14]).

Let φ be an increasing unbounded function on $[1, +\infty)$. We define the orders of the growth of an entire function f by

$$\widetilde{\sigma}^0_\varphi[f] = \limsup_{r\to+\infty} \frac{\varphi(M(r, f))}{\ln r}, \quad \widetilde{\sigma}^1_\varphi[f] = \limsup_{r\to+\infty} \frac{\varphi(\ln M(r, f))}{\ln r},$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$.

If g is meromorphic, then the orders are defined by

$$\sigma^0_\varphi[f] = \limsup_{r\to+\infty} \frac{\varphi(e^{T(r,g)})}{\ln r}, \quad \sigma^1_\varphi[f] = \limsup_{r\to+\infty} \frac{\varphi(T(r, g))}{\ln r}.$$

By Φ we define the class of positive unbounded increasing function on $[1, +\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.

$$\forall c > 0 : \qquad \frac{\varphi(e^{ct})}{\varphi(e^t)} \to 1, \qquad t \to +\infty.$$

We define also

$$\gamma_\varphi := \sup\{\sigma^1_\varphi[f] \mid L(f) = 0\},$$

$$\alpha_\varphi := \sup\{\sigma^0_\varphi[a_j] \mid j = 0, \dots, n-1\}, \quad j \in \mathbb{N}.$$

Remark 1.5. It follows from Proposition 3.1, which will be proved below, that

$$\gamma_\varphi = \sup\{\widetilde{\sigma}^1_\varphi[f] \mid L(f) = 0\}, \;\; \alpha_\varphi = \sup\{\widetilde{\sigma}^0_\varphi[f] \mid j = 0, \dots, n-1\}.$$

Remark 1.6. We note that in the case when $\varphi(r) = \ln^{[i]} r$, $i \in \mathbb{N}$, σ^1_φ-order coincides with ith iterated order. On the other hand, the function $\varphi(r) = \ln r$ does not belong to the class Φ, but in view of Theorem 1.1 the counterpart of the assertion of Theorem 1.8 does not hold in this case.

The next example shows that we can often successfully use σ_φ-order instead of $[p, q]$-order.

Example 1.7. Let f be an entire function such that

$$\ln^{[2]} T(r,f) = (\tau + o(1))(\ln r)^{\sigma},$$

where $\tau, \sigma > 0$. Such entire function exists due to [4], because $\exp^{[2]}(\tau(\ln r)^{\sigma})$ is convex in $\ln r$.

This function f has $[3,2]$-order σ, and moreover so called finite type τ of this order, i.e. there exists

$$\limsup_{r\to+\infty} \frac{\ln^{[2]} T(r,f)}{(\ln r)^{\sigma}} = \tau.$$

For $\varphi(r) = (\ln^{[2]} r)^{\frac{1}{\sigma}}$ there exists

$$\sigma^1_{\varphi}[f] = \limsup_{r\to+\infty} \frac{\varphi(T(r,f))}{\ln r} = \tau^{\frac{1}{\sigma}}.$$

We formulate the main results, which are counterparts of Theorems 1.2 and 1.3, respectively.

Theorem 1.8. *Let $\varphi \in \Phi$, then $\gamma_{\varphi} = \alpha_{\varphi}$.*

Theorem 1.9. *Let $\varphi \in \Phi$ and $j = \max\{k \mid \sigma^0_{\varphi}[a_k] \ge \beta, k = 0,\dots,n-1\}$, then (1) possesses at most j entire linearly independent solutions f with $\sigma^1_{\varphi}[f] < \beta$.*

If the last coefficient a_0 in (1) dominates, we can state more on the order of solutions.

Theorem 1.10. *Let $\varphi \in \Phi$, $a_0,\dots,a_{n-1}$ be entire functions such that $\sigma^0_{\varphi}[a_0] > \max\{\sigma^0_{\varphi}[a_j], j = 1,\dots,n-1\}$. Then all solutions $f \not\equiv 0$ of (1) satisfy*

$$\sigma^1_{\varphi}[f] = \sigma^0_{\varphi}[a_0].$$

Remark 1.11. The most general scale of the growth of entire functions is due to M. M. Sheremeta ([14]). His definition of (α,β)-order coincides with that $\tilde{\sigma}^1_{\alpha}$-order in the case $\beta(x) \equiv x$ with only difference that in Sheremeta's definition $\alpha(e^t)$ need not to be slowly growing, while $\alpha(t)$ is slowly growing. As we see from Remark 1.6, Theorem 1.8 is not valid for such α. The question whether counterparts of Theorems 1.8–1.10 hold for (α,β)-order with arbitrary $\alpha \in \Phi$ and increasing β satisfying $\beta((1+o(1))x) = (1+o(1))\beta(x)$, $x \to +\infty$, remains open.

2. Properties of functions from the class Φ

We give some properties of functions from the class Φ.

Proposition 2.1. *If $\varphi \in \Phi$, then*

$$\forall m > 0, \quad \forall k \geq 0: \qquad \frac{\varphi^{-1}(\ln x^m)}{x^k} \to +\infty, \qquad x \to +\infty; \tag{8}$$

$$\forall \delta > 0: \qquad \frac{\ln \varphi^{-1}((1+\delta)x)}{\ln \varphi^{-1}(x)} \to +\infty, \qquad x \to +\infty. \tag{9}$$

Remark 2.2. If φ is non-decreasing, (9) is equivalent to the definition of the class Φ.

Proof. We will prove from the contrary. Suppose that on some unbounded sequence $x_n = e^{t_n}$, $n \in \mathbb{N}$

$$\varphi^{-1}(\ln e^{t_n m}) < e^{t_n k}.$$

holds. Then $m t_n < \varphi(e^{t_n k})$.

Since by Karamata's Theorem ([13, p.10]), $\varphi(e^t) = t^{o(1)}$ as $t \to +\infty$. We obtain the contradiction which proves (8).

We now suppose that for $c > 0$ and $\delta > 0$ on an unbounded sequence $x_n = \varphi(e^{t_n})$, $n \to +\infty$, one has

$$\frac{\ln \varphi^{-1}((1+\delta)\varphi(e^{t_n}))}{\ln \varphi^{-1}(\varphi(e^{t_n}))} < c.$$

Then

$$\varphi^{-1}((1+\delta)\varphi(e^{t_n})) < e^{ct_n}, \quad 1+\delta < \frac{\varphi(e^{ct_n})}{\varphi(e^{t_n})}, \qquad n \to +\infty.$$

But it is impossible in view of (6). Therefore, property (9) is proved. □

3. Properties of meromorphic functions of finite σ_φ-order

This subsection contains basic properties of orders σ_φ^j, $\tilde{\sigma}_\varphi^j$, obtained using standard methods. We give proofs for completeness.

Proposition 3.1. *Let $\varphi \in \Phi$ and f be an entire function. Then*

$$\sigma_\varphi^j[f] = \widetilde{\sigma}_\varphi^j[f], \quad j = 0, 1.$$

Proof. We show the equality $\sigma_\varphi^1[f] = \widetilde{\sigma}_\varphi^1[f]$. The equality $\sigma_\varphi^0[f] = \widetilde{\sigma}_\varphi^0[f]$ can be proved similarly.

By monotonicity of the function φ and the known inequality ([5])

$$T(r, f) \leq \ln M(r, f) \leq \frac{R+r}{R-r} T(R, f), \quad 0 < r < R, \tag{10}$$

we have $\sigma_\varphi^1[f] \leq \widetilde{\sigma}_\varphi^1[f]$.

We now prove the converse inequality.

We choose $R = kr$, $k > 1$, and estimate the numerator. It follows from (10) that as $r \to +\infty$

$$\frac{\varphi(\ln M(r,f))}{\ln r} \le \frac{\varphi\left(\frac{R+r}{R-r}T(R,f)\right)}{\ln r} = \frac{\varphi\left(\frac{k+1}{k-1}T(kr,f)\right)}{\ln r} \le \frac{(1+o(1))\varphi(T(kr,f))}{\ln kr - \ln k}.$$

In fact, the last inequality holds because (2) implies

$$\forall c > 1: \qquad \varphi(ct) \le \varphi(t^c) \le (1+o(1))\varphi(t), \quad t \to +\infty. \tag{11}$$

Therefore, $\widetilde{\sigma}^1_\varphi[f] \le \limsup_{r\to+\infty} \dfrac{\varphi(T(kr,f))}{\ln kr} = \sigma^1_\varphi[f]$.

The last inequality proves the required equality. □

Henceforth we consider only $\sigma^1_\varphi[f]$ and $\sigma^0_\varphi[f]$ for meromorphic and entire functions f.

We now show that the operations of addition and multiplication cannot increase the order.

Proposition 3.2. *Let f_1, f_2 be meromorphic functions in $\mathbb{C}$, $\varphi \in \Phi$. Then*

1) $\sigma^j_\varphi[f_1 + f_2] \le \max\{\sigma^j_\varphi[f_1], \sigma^j_\varphi[f_2]\}$, $j = 0, 1$.
2) $\sigma^j_\varphi[f_1 f_2] \le \max\{\sigma^j_\varphi[f_1], \sigma^j_\varphi[f_2]\}$, $j = 0, 1$.
3) $\sigma^j_\varphi\left[\frac{1}{f_1}\right] = \sigma^j_\varphi[f_1]$, $j = 0, 1$, $f_1 \not\equiv 0$.

Proof. Let $\sigma^1_\varphi[f_1] = \alpha$, $\sigma^1_\varphi[f_2] = \beta$. Without loss of generality, we may suppose that $\alpha \le \beta$. The definition of σ^1_φ-order implies that for any $\varepsilon > 0$ and for all $r \ge r_0$

$$T(r, f_j) = O(\varphi^{-1}(\ln r^{\beta+\varepsilon})), \quad j = 1, 2.$$

The properties of the Nevanlinna characteristic ([5]) yield

$$T(r, f_1 + f_2) \le T(r, f_1) + T(r, f_2) + O(1) = O(\varphi^{-1}(\ln r^{\beta+\varepsilon})), \quad r \to +\infty.$$

By the above inequality and (11), we have $\varphi(T(r, f_1 + f_2)) \le (\beta + \varepsilon)\ln r$. By arbitrariness of ε, we obtain $\sigma^1_\varphi[f_1 + f_2] \le \beta = \max\{\sigma^1_\varphi[f_1], \sigma^1_\varphi[f_2]\}$.

Properties 2 and 3 can be proved similarly. The proofs for σ^0_φ-order are analogous. □

To prove the next assertion we need the following lemma.

Lemma 3.3 ([1]). *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \le h(r)$ outside an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Proposition 3.4. *Let f be a meromorphic function and $\varphi \in \Phi$. Then*

$$\sigma_{\varphi}^{j}[f'] \le \sigma_{\varphi}^{j}[f], \quad j = 0, 1.$$

Proof. We denote $\sigma_{\varphi}^{1}[f] = \alpha$. The definition of the σ_{φ}^{1}-order implies that for any $\varepsilon > 0$ and for all $r \ge r_0$ as $r \to +\infty$ we have $T(r, f) = O(\varphi^{-1}(\ln r^{\alpha+\varepsilon}))$.

By the Lemma of the logarithmic derivative ([5, Chap. 3])

$$m\Big(r, \frac{f'}{f}\Big) = O\big(\ln T(r, f) + \ln r\big), \quad r \notin E,$$

where $E \subset [0, +\infty)$ is a set of finite linear measure.

Using the last estimate, properties of Nevanlinna's characteristics and (8), we get

$$\begin{aligned} T(r, f') = m(r, f') + N(r, f') \le m\Big(r, \frac{f'}{f}\Big) + m(r, f) + 2N(r, f) \le \\ \le m\Big(r, \frac{f'}{f}\Big) + 2T(r, f) = O\big(\ln \varphi^{-1}(\ln r^{\alpha+\varepsilon}) + \ln r\big) + O\big(\varphi^{-1}(\ln r^{\alpha+\varepsilon})\big) = \\ = O\big(\varphi^{-1}(\ln r^{\alpha+\varepsilon})\big), \quad r \notin E. \end{aligned}$$

By Lemma 3.3,

$$T(r, f') = O\big(\varphi^{-1}(\ln(2r)^{\alpha+\varepsilon})\big) = O\big(\varphi^{-1}(\ln r^{\alpha+2\varepsilon})\big), \qquad r > 0.$$

Therefore, in view of (9) we have that $\varphi(T(r, f')) \le (\alpha + 3\varepsilon)\ln r$ as $r \to +\infty$. By arbitrariness of ε, we finally obtain $\sigma_{\varphi}^{1}[f'] \le \alpha = \sigma_{\varphi}^{1}[f]$, $\sigma_{\varphi}^{0}[f'] \le \sigma_{\varphi}^{0}[f]$. □

Lemma 3.5. *Let f be a meromorphic function such that $0 < \sigma_{\varphi}^{0}[f] := \sigma_0 < +\infty$. Then $\forall \mu < \sigma_0$ exists a set $E \in [0; +\infty)$ of infinite logarithmic measure on $[0; +\infty)$ such that $\varphi(e^{T(r,f)}) > \mu \ln r$ for all $r \in E$.*

Proof. The definition of the upper limit implies that there exists a sequence $(R_j)_{j=1}^{+\infty}$ satisfying

$$\big(1 + \tfrac{1}{j}\big)R_j < R_{j+1} \quad \text{and} \quad \lim_{j\to+\infty} \frac{e^{T(R_j,f)}}{\ln R_j} = \sigma_0.$$

From the last equality it follows that for any ε $(0 < \varepsilon < \sigma_0 - \mu)$ there exists an integer j_1 such that for $j \ge j_1$

$$\varphi(e^{T(R_j,f)}) > (\sigma_0 - \varepsilon)\ln R_j. \tag{12}$$

Since $\mu < \sigma_0 - \varepsilon$, there exists an integer j_2 such that for $j \ge j_2$ we have

$$\Big(\frac{\sigma_0 - \varepsilon}{\mu} - 1\Big)\ln R_j > \ln\Big(1 + \frac{1}{j}\Big), \quad \frac{\sigma_0 - \varepsilon}{\mu} \cdot \frac{\ln R_j}{\ln(1 + \frac{1}{j})R_j} > 1.$$

By the last inequality and the inequality (12) for $j \geq j_3 = \max\{j_1, j_2\}$ and for any $R \in \big[R_j, (1+\frac{1}{j})R_j\big]$ we obtain

$$\varphi(e^{T(r,f)}) \geq \varphi(e^{T(R_j,f)}) > (\sigma_0-\varepsilon)\ln R_j = \frac{\sigma_0-\varepsilon}{\mu}\mu\frac{\ln R_j}{\ln R}\ln R \geq$$
$$\geq \frac{\sigma_0-\varepsilon}{\mu}\frac{\ln R_j}{\ln(1+\frac{1}{j})R_j}\mu\ln R > \mu\ln R. \quad (13)$$

We denote $E = \bigcup\limits_{j=j_3}^{+\infty}[R_j, (1+\frac{1}{j})R_j]$. It is easy to show that E is set of infinite logarithmic measure:

$$m_l E := \int\limits_E \frac{dr}{r} = \sum_{j=j_3}^{+\infty}\int\limits_{R_j}^{(1+\frac{1}{j})R_j}\frac{dr}{r} = \sum_{j=j_3}^{+\infty}\ln\Big(1+\frac{1}{j}\Big) = +\infty.$$

□

Lemma 3.6. *Let f be a meromorphic function of order $\sigma_\varphi^1[f] = \sigma$ and $k \in \mathbb{N}$, and $\varphi \in \Phi$. Then $\forall\varepsilon > 0$*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big)$$

outside, possibly, an exceptional set E of finite linear measure.

Proof. First, let $k = 1$. The definition of σ_φ^1-order implies that for any $\varepsilon > 0$ for all $r \geq r_0$ as $r \to +\infty$ we have $T(r, f) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big)$. Again, by the Lemma of the logarithmic derivative, in view of (8)

$$m\Big(r, \frac{f'}{f}\Big) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon}) + \ln r\big) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big), \quad r \notin E, \quad (14)$$

where E is a set of finite linear measure.

Next, assume that

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big), \quad r \notin E$$

for some $k \in \mathbb{N}$. Since $N(r, f^{(k)}) \leq (k+1)N(r, f)$, we deduce

$$T(r, f^{(k)}) = m(r, f^{(k)}) + N(r, f^{(k)}) \leq m\Big(r, \frac{f^{(k)}}{f}\Big) + m(r, f) + (k+1)N(r, f) \leq$$
$$\leq (k+1)T(r, f) + O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big) = O\big(\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big). \quad (15)$$

By (14) we obtain $m\big(r, \frac{f^{(k+1)}}{f^{(k)}}\big) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big), \quad r \notin E$, and hence,

$$m\Big(r, \frac{f^{(k+1)}}{f}\Big) \leq m\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) + m\Big(r, \frac{f^{(k)}}{f}\Big) = O\big(\ln\varphi^{-1}(\ln r^{\sigma+\varepsilon})\big), \quad r \notin E.$$

□

4. Proofs of the main results

Proof of Theorem 1.8. First, we show that $\alpha_\varphi \le \gamma_\varphi$. If $\gamma_\varphi = +\infty$, it is trivial.

We assume that $\gamma_\varphi < +\infty$. Let $f_1, \dots, f_n$ be a solution base for (1) of finite σ^1_φ-order. Properties of the Wronsky determinant imply that $W = W(f_1, \dots, f_n) \neq 0$.

Propositions 3.2 and 3.4 imply that W is of finite σ^1_φ-order. By properties of the Wronsky determinant ([10])

$$a_{n-q}(z) = -W_{n-q}(f_1, \dots, f_n) \cdot W^{-1}, \quad q \in \{1, \dots, n\},$$

where

$$W_j(f_1, \dots, f_n) = \begin{vmatrix} f_1 & \dots & f_n \\ \vdots & \vdots & \vdots \\ f_1^{(j-1)} & \dots & f_n^{(j-1)} \\ f_1^{(n)} & \dots & f_n^{(n)} \\ f_1^{(j+1)} & \dots & f_n^{(j+1)} \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

In view of Proposition 3.2 we can conclude that all coefficients $a_0, \dots, a_{n-1}$ are of finite σ^1_φ-order.

By Lemma 3.6

$$m\Big(r, \frac{f_i^{(k)}}{f_i}\Big) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi + \varepsilon})\big), \quad r \notin E,$$

where $k \ge 1$, $i \in \{1, \dots, n\}$, E is a set of finite linear measure.

We now apply the standard order reduction procedure ([10]). Let us denote

$$\nu_1(z) := \frac{d}{dz}\left(\frac{f(z)}{f_1(z)}\right),$$

$a_n = 1$, and $\nu_1^{(-1)} := \dfrac{f}{f_1}$, i.e., $\big(\nu_1^{(-1)}\big)' = \nu_1$. Hence,

$$f^{(k)} = \sum_{m=0}^{k} \binom{k}{m} f_1^{(m)} \nu_1^{(k-1-m)}, \quad k = 0, \dots, n. \tag{16}$$

Substituting (16) into (1) and using the fact that f_1 solves (1), we obtain

$$\nu_1^{(n-1)} + a_{1,n-2}(z)\nu_1^{(n-2)} + \dots + a_{1,0}(z)\nu_1 = 0, \tag{17}$$

where

$$a_{1,j} = a_{j+1} + \sum_{m=1}^{n-j-1} \binom{j+1+m}{m} a_{j+1+m} \frac{f_1^{(m)}}{f_1}$$

for $j = 0, \dots, n-2$. The meromorphic functions

$$v_{1,j}(z) = \frac{d}{dz}\left(\frac{f_{j+1}(z)}{f_1(z)}\right), \quad j = 1, \dots, n-1, \tag{18}$$

form a solution base to (17). By $\gamma_\varphi < +\infty$ and Proposition 3.4, the functions $v_{1,j}$ are of finite σ_φ^1-order.

We now show that

$$m(r, a_{1,j}) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big), \quad r \notin E, \quad j = 0, \dots, n-2 \tag{19}$$

imply

$$m(r, a_i) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big), \quad r \notin E, \quad i = 0, \dots, n-1. \tag{20}$$

We prove it by induction on i following [10]. By equality (17) for $j = n-2$ we have $a_{1,n-2} = a_{n-1} + n\frac{f'}{f}$. By Lemma 3.6 and (19)

$$m(r, a_{n-1}) \le m(r, a_{1,n-2}) + m\Big(r, \frac{f_1'}{f_1}\Big) + O(1) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big).$$

We assume that

$$m(r, a_i) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big), \quad i = n-1, \dots, n-k. \tag{21}$$

Since

$$a_{1,n-(k+2)} = a_{n-(k+1)} + \sum_{m=1}^{k+1} \binom{m+n-k-1}{m} a_{m+n-k-1} \frac{f_1^{(m)}}{f_1},$$

we have

$$m(r, a_{n-(k+1)}) \le m(r, a_{1,n-(k+2)}) + m(r, a_{n-1}) + \dots + m(r, a_{n-k}) + \\ + m\Big(r, \frac{f'}{f}\Big) + \dots + m\Big(r, \frac{f_1^{(k+1)}}{f_1}\Big) + O(1) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big), \quad r \notin E, \tag{22}$$

by Lemma 3.6, (19) and (21).

We may now proceed as above the order reduction procedure for (17). On each reduction step, we obtain a solution base of meromorphic functions of finite σ_φ^1-order according to (18), and the implication (19)$\Rightarrow$(20) remains valid. Hence, we finally obtain an equation of the form $u' + A(z)u = 0$. Since u is of finite σ_φ^1-order we obtain

$$m(r, A) = m\Big(r, \frac{u'}{u}\Big) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big), \quad r \notin E.$$

Observing the reasoning corresponding to (19) and (20) in each reduction step, we see that

$$m(r, a_j) = O\big(\ln \varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big), \quad r \notin E,$$

holds for $j = 0, \dots, n-1$. Since the coefficients a_j are entire, we have $m(r, a_j) = T(r, a_j) = O\big(\ln\varphi^{-1}(\ln r^{\gamma_\varphi+\varepsilon})\big)$ outside, possibly, an exceptional set E of finite logarithmic measure. By Lemma 3.3 and Proposition 2.1

$$T(r, a_j) = O\big(\ln\varphi^{-1}(\ln(2r)^{\gamma_\varphi+\varepsilon})\big) \le \ln\varphi^{-1}(\ln r^{\gamma_\varphi+2\varepsilon}), \qquad r \to +\infty.$$

Hence, $\dfrac{\varphi(e^{T(r,a_j)})}{\ln r} \le \gamma_\varphi + 2\varepsilon$. By arbitrariness of ε we obtain that $\alpha_\varphi \le \gamma_\varphi$.

We now prove the converse inequality under the assumption that $\alpha_\varphi < +\infty$. We need the following assertion.

Lemma 4.1 ([10, p.10]). *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_0$ with $a_n \ne 0$ be a polynomial. Then all zeros of $P(z)$ lie in the discs $D(0, r)$ of radius*

$$r \le 1 + \max_{1\le k\le n-1}\Big(\Big|\frac{a_k}{a_n}\Big|\Big).$$

We recall also the definition of the central index and the maximal term.

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $z = re^{i\theta}$ be an entire function, then $\mu(r, f) = \max\{|a_n|r^n : n \ge 0\}$ is the maximal term, and $\nu(r, f) = \max\{n : |a_n|r^n = \mu(r, f)\}$ is the central index of f.

Theorem 4.2 ([10, p.10]). *Let g be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and z such that $|z| = r$ and $|g(z)| > M(r, g)\nu(r, g)^{-\frac{1}{4}+\delta}$. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure such that*

$$g^{(m)}(z) = \Big(\frac{\nu(r, g)}{z}\Big)^m \big(1 + o(1)\big)g(z)$$

holds for all $m \ge 0$ and all $r \notin F$.

By Theorem 4.2, for some set $F \subset \mathbb{R}_+$ of finite logarithmic measure the following equality

$$f^{(i)}(z) = \Big(\frac{\nu(r)}{z}\Big)^i \big(1 + o(1)\big)f(z) \tag{23}$$

holds for $i = 0, \dots, n$ and for $|z| = r \in F$, chosen as in Theorem 4.2. Substituting (23) into (1) we obtain

$$\Big(\frac{\nu(r)}{z}\Big)^n + a_{n-1}(z)\Big(\frac{\nu(r)}{z}\Big)^{n-1}(1+o(1)) + \dots + a_1(z)\frac{\nu(r)}{z}(1+o(1)) + a_0(z)(1+o(1)) = 0,$$

or

$$\nu(r)^n + za_{n-1}(z)\nu(r)^{n-1}(1+o(1)) + \dots + z^{n-1}a_1(z)\nu(r)(1+o(1)) + z^n a_0(z)(1+o(1)) = 0.$$

The definition of σ^1_φ-order yields that for any $\varepsilon > 0$ and for all $r \ge r_0$

$$e^{T(r,a_j)} < \varphi^{-1}(\ln r^{\alpha_\varphi+\varepsilon}), \qquad r \to +\infty.$$

By Lemma 4.1 in view of (8), we have for any $\varepsilon > 0$

$$\nu(r,f) \le 1 + \max_{0\le j\le n-1} |z^{n-j}a_j(z)(1+o(1))| \le 1 + \max_{0\le j\le n-1} 2r^{n-j}\varphi^{-1}(\ln r^{\alpha_\varphi+\varepsilon}) \le$$
$$\le 1 + 2r^n\varphi^{-1}(\ln r^{\alpha_\varphi+\varepsilon}) \le \varphi^{-1}(\ln r^{\alpha_\varphi+2\varepsilon}), \quad r \notin F.$$

By the inequalities (4.12) and (4.15) [8, p.36–37], we obtain for each $\varepsilon > 0$

$$T(r,f) \le \ln M(r,f) \le \ln \mu(r,f) + \ln(\nu(2r,f)+2) \le$$
$$\le \nu(r,f)\ln r + \ln(2\nu(2r,f)) \le \varphi^{-1}(\ln r^{\alpha_\varphi+2\varepsilon})\ln r + \ln(2\varphi^{-1}(\ln(2r)^{\alpha_\varphi+\varepsilon})) \le$$
$$\le \varphi^{-1}(\ln r^{\alpha_\varphi+3\varepsilon}) + \ln 2 + \ln\varphi^{-1}(\ln(2r)^{\alpha_\varphi+2\varepsilon}) \le \varphi^{-1}(\ln r^{\alpha_\varphi+4\varepsilon}).$$

Hence, $\varphi(T(r,f))/\ln r \le \alpha_\varphi + 4\varepsilon$. By arbitrariness of ε we obtain that $\gamma_\varphi \le \alpha_\varphi$. Thus, the theorem is proved under the assumption that $\max\{\alpha_\varphi, \gamma_\varphi\} < +\infty$.

If only one of α_φ or γ_φ is finite, then by the proved we obtain a contradiction. Therefore, either $\alpha_\varphi = \gamma_\varphi < +\infty$, or $\alpha_\varphi = \gamma_\varphi = +\infty$. □

Proof of Theorem 1.9. By our assumptions there exist two numbers β_1 and β_2 such that $\sigma^0_\varphi[a_j] \ge \beta$ and $\sigma^0_\varphi[a_l] \le \beta_1 < \beta_2 < \beta$ for $l = j+1, \dots, n-1$.

Let $f_1, \dots, f_{j+1}$ be linearly independent solutions of (1) such that $\sigma^1_\varphi[f_i] < \beta$, $i = 1, \dots, j+1$. If $j = n-1$, then all $f_1, \dots, f_n$ are of σ^1_φ-order smaller than β, contradicting Theorem 1.8, because $\sup\{\sigma^0_\varphi[f] \mid L(f) = 0\} = \beta$. Hence, $j < n-1$.

Let now apply the same order reduction procedure as in the proof of Theorem 1.8. Let us use the notation ν_0 instead of f, and $a_{0,0}, \dots, a_{0,n-1}$ instead of $a_0, \dots, a_{n-1}$. In the general reduction step, we obtain an equation of the form

$$\nu_k^{(n-k)} + a_{k,n-k-1}(z)\nu_k^{(n-k-1)} + \dots + a_{k,0}(z)\nu_k = 0, \tag{24}$$

where

$$a_{k,l} = a_{k-1,l+1} + \sum_{m=1}^{n-l-k} \binom{l+1+m}{m} a_{k-1,l+1+m}\frac{\nu^{(m)}_{k-1,1}}{\nu_{k-1,1}}, \tag{25}$$

and the functions

$$\nu_{k,l}(z) = \frac{d}{dz}\Big(\frac{\nu_{k-1,l+1}(z)}{\nu_{k-1,1}(z)}\Big),\ l = 1, \dots, n-k, \quad \nu_0 = f, \quad \nu_k(z) = \frac{d}{dz}\Big(\frac{\nu_{k-1}(z)}{\nu_{0,k-1}(z)}\Big),$$

determine a solutions base of (24) in terms of the preceding solution base. We may express (1) and the jth reduction steps by the following Table. The rows correspond to (24) for $\nu_0, \dots, \nu_j$, i.e., the first row corresponds to (1), and the columns from n to 0 give the coefficients of these equations, while the last column lists the solutions with $\sigma^1_\varphi[f] < \beta$.

	n	n-1	...	j	j-1	...	0	$\sigma_\varphi^1[f] < \beta$
ν_0	1	$a_{0,n-1}$	...	$\mathbf{a_{0,j}}$	$a_{0,j-1}$	...	$a_{0,0}$	$\nu_{0,1}, \dots, \nu_{0,j+1}$
ν_1		1	...	$a_{1,j}$	$\mathbf{a_{1,j-1}}$	...	$a_{1,0}$	$\nu_{1,1}, \dots, \nu_{1,j}$
.				.	.	.	.	.
.				.	.	.	.	.
.				.	.	.	.	.
ν_{j-1}				$a_{j-1,j}$	$a_{j-1,j-1}$	...	$a_{j-1,0}$	$\nu_{j-1,1}, \nu_{j-1,1}$
ν_j				$a_{j,j}$	$a_{j,j-1}$	...	$\mathbf{a_{j,0}}$	$\nu_{j,1}$

By Lemma 3.6 and by (25), we see that in the second row, corresponding to the first reduction step, $m(r, a_{1,l}) = O(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon}))$, $r \notin E$, holds for $l = j, \dots, n-2$, while $\beta_1 + \varepsilon < \beta$ and $m(r, a_{1,j-1}) \neq O(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon}))$, $r \notin E$.

Similarly, in each reduction step (25) implies that

$$m(r, a_{k,l}) = O(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon})), \qquad r \notin E \tag{26}$$

when $l = j+1-k, \dots, n-(k+1)$, i.e., for all coefficients to the left from the boldface coefficient $a_{k,j-k}$, while $m(r, a_{k,j-k}) \neq O(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon}))$, $r \notin E$ for $k = 1, \dots, j$. In particular, $m(r, a_{j,0}) \neq O(\ln \varphi^{-1}(\ln r^{\beta+\varepsilon}))$ for $r \notin E$. We now apply Lemma 3.5 to the coefficient $a_{j,0}$ with the constant β_2, and obtain that $T(r, a_{j,0}) > \ln \varphi^{-1}(\ln r^{\beta})$ and $r \to +\infty$, $r \in F$, where F is a set of infinite logarithmic measure.

On the other hand, after the jth reduction step, we have by (25)

$$a_{j,0} = -\frac{\nu_{j,1}^{(n-j)}}{\nu_{j,1}} - a_{j,n-j-1}\frac{\nu_{j,1}^{(n-j-1)}}{\nu_{j,1}} - \cdots - a_{j,1}\frac{\nu'_{j,1}}{\nu_{j,1}},$$

therefore $m(r, a_{j,0}) = O\big(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon})\big)$, $r \notin E$, by (26) and Lemma 3.6. Since $\sigma_\varphi^0[\nu_{j,1}] < \beta_1$, in view of Propositions 3.2 and 3.4 we have that

$$N(r, a_{j,0}) = O\big(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon})\big), \quad r \notin E,$$

where E is a set of finite linear measure. Therefore, $T(r, a_{j,0}) = O\big(\ln \varphi^{-1}(\ln r^{\beta_1+\varepsilon})\big)$, $r \notin E$. By Lemma 3.3,

$$T(r, a_{j,0}) = O\big(\ln \varphi^{-1}(\ln (2r)^{\beta_1+\varepsilon})\big) = \ln \varphi^{-1}(\ln r^{\beta_1+2\varepsilon}).$$

Choosing ε so that $\beta_1 + 2\varepsilon < \beta_2$, we obtain the contradiction with our assumption that $\beta_1 < \beta$. Hence, there exist at most j linearly independent solutions (1) with $\sigma_\varphi^1[f] < \beta$. □

Proof of Theorem 1.10. Let f be a nontrivial solution of (1). We denote $\sigma_1 := \sigma_\varphi^1[f]$ and $\sigma_0 := \sigma_\varphi^0[a_0]$. The inequality $\sigma_0 \leq \sigma_1$ follows from Theorem 1.9 when $j = 0$ and $\beta = \sigma_0$.

To prove the conserve inequality we need the following theorem.

Theorem 4.3 ([7]). *Let f be the solutions of* (1) *in* $\mathbb{C}$, *where* $0 < R < +\infty$ *and let* $1 \leq p < +\infty$. *Then for all* $0 < r < R$

$$m_p(r, f)^p \leq C\Big(\sum_{j=0}^{n-1} \int_0^{2\pi} \int_0^r |a_j(se^{i\theta})|^{\frac{p}{n-j}} ds d\theta + 1\Big),$$

where $C > 0$ *is a constant which depends on* p *and the initial value of* f *in a point* z_0, *where* $a_j \neq 0$ *for some* $j = 0, \dots, n-1$, *and where*

$$m_p(r, f)^p = \frac{1}{2\pi} \int_0^{2\pi} (|\ln^+ |f(re^{i\theta})||)^p d\theta.$$

It follows from Theorem 4.3 with $p = 1$, (8) and the definition of σ_φ^0-order that for sufficiently large r we have

$$\begin{aligned} m(r, f) &\leq C\Big(\sum_{j=0}^{n-1} \int_0^{2\pi} \int_0^r |a_j(se^{i\theta})|^{\frac{1}{n-j}} ds d\theta + 1\Big) \leq \\ &\leq C\Big(n \max_{0 \leq j \leq n-1} \int_0^{2\pi} \int_0^r |a_j(se^{i\theta})|^{\frac{1}{n-j}} ds d\theta + 1\Big) \leq \\ &\leq \widetilde{C} \max_{0 \leq j \leq n-1} \int_0^r (\varphi^{-1}((\sigma_0 + \varepsilon) \ln s))^{\frac{1}{n-j}} ds \leq \widetilde{C} \int_0^r \varphi^{-1}((\sigma_0 + \varepsilon) \ln s) ds \leq \\ &\leq \widetilde{C} r \varphi^{-1}((\sigma_0 + \varepsilon) \ln r) \leq \varphi^{-1}((\sigma_0 + 2\varepsilon) \ln r). \end{aligned}$$

Since f is an entire function, we have

$$T(r, f) \leq \varphi^{-1}((\sigma_0 + 2\varepsilon) \ln r), \quad \frac{\varphi(T(r, f))}{\ln r} \leq \sigma_0 + 2\varepsilon.$$

By arbitrariness of ε we have that $\sigma_1 \leq \sigma_0$. Thus, Theorem 1.10 is proved. □

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