THE PROBABILITY OF ABSENCE ZEROS IN THE DISC FOR SOME RANDOM ANALYTIC FUNCTIONS

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We consider random analytic function $f(z,\omega)=\sum\limits_{n=0}^{+\infty}\phi_n a_n z^n$, where $a_n>0$ such that $\lim\limits_{n\to+\infty}\sqrt[n]{a_n}=1$, $\phi_n=\phi_n(\omega)$ are standard complex gaussian random variables in probability space of Steinhaus $(\Omega,\mathcal{A},P)$. If $\{a_n\}$ is log-concave and $\lim\limits_{r\to1-0}(1-r)\ln\ln M(r)=+\infty$, then $(\exists E(\varepsilon,f)=E\subset(0,1))(\forall r\in(0,1)\backslash E)$

$$\overline{\lim_{r\to1-0}}\frac{1}{1-r}\mathrm{meas}(E\cap[r,1))=0 \text{ and } \ln^- P(\{\omega\colon f(z,\omega)\neq0\})=S(r)+$$

$$+o(S(r))\ (r\to1-0),$$

where $S(r)=2\sum\limits_{n\colon a_nr^n\geq1}\ln(a_nr^n)$.

1 Introduction and main definitions

We consider the random analytic function

$$f(z,\omega)=\sum_{n=0}^{+\infty}\phi_n a_n z^n, \tag{1}$$

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where $a_n > 0$ such that $\lim\limits_{n\to+\infty} \sqrt[n]{a_n} = 1$, $\phi_n = \phi_n(\omega)$ are standard complex gaussian random variables in probability space of Steinhaus $(\Omega, \mathcal{A}, P)$, where $\Omega = [0,1]$, $\mathcal{A}$ is σ-algebra of measurable subsets of Ω by Lebesgue and P is Lebesgue measure on the line. The probability density function for random variable ϕ_n ([1]) $p_{\phi_n}(z) = e^{-|z|^2}/\pi,\ z \in \mathbb{C}$.

We will use the following notations (see [2])

$$\mathcal{N}_x(r) = \{n \in \mathbb{Z}_+ : \ln a_n r^n \geq (1-x)\ln \mu(r)\},\quad \mathcal{N}_{m,m+1}(r) = \\ = \mathcal{N}_{m+1}(r)/\mathcal{N}_m(r),$$

$$N_x(r) = \#\mathcal{N}_x(r),\quad N_{m,m+1}(r) = \#\mathcal{N}_{m,m+1}(r),$$

$$N_1(r) = \#\mathcal{N}_1(r) = \#\{n \in \mathbb{Z}_+ : \ln(a_n r^n) \geq 0\},\quad p_0(r) = \\ = \ln^- P(\{\omega\colon f(z,\omega) \neq 0 \text{ for } |z| \leq r\}),$$

$$S(r) = \ln\Big(\prod_{n\in\mathcal{N}_1(r)} (a_n r^n)^2\Big) = 2\sum_{n\in\mathcal{N}_1(r)} \ln(a_n r^n),$$

where $\#E$ means the quantity of elements of the set E.

We called, that sequence $\{a_n\}$ is *log-concave*, if $\exists a(t) \in C^2(0,+\infty)$ such that $a(n) = a_n$ and $\ln a(t)$ is concave.

For random entire functions A. Nishry ([2]) has proved, that $p_0(r) = S(r) + o(S(r))$, $r \to +\infty$ ($r \notin E$, $\int_E dr/r < +\infty$), when $\{a_n\}$ is log-concave. We will prove analogues of this statement for random analytic functions in the unit disc.

Also we assume, that $a_0 = 1$. (If $a_0 \neq 1$, then we will consider the function $f(z,\omega)/a_0$. In the case when $a_0 = 0$, $P(\{\omega\colon f(z,\omega) \neq 0 \text{ for } |z| \leq r\}) \equiv 0$).

Denote by L the class of positive increasing to $+\infty$ functions on $[0,+\infty)$. Let L_0 be the class of nonnegative nonincreasing on $[0,+\infty)$ functions v such that

$$\int_0^{+\infty} \frac{dv(y)}{y} < +\infty.$$

We denote the class functions $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ with radii of convergence $R(f) = 1$, such that $(\exists r_0 \in (0,1))$ $(\forall r > r_0)$ $(\exists b > 0)\colon \nu(r) \geq \Phi_1(b/(1-r))$ for $\Phi_1 \in L$ by $S_1^*(\Phi_1)$.

The following statement is a consequence of the theorem proved in [3].

Theorem 1. ([3]) *If* $v \in L_0$, $f \in S_1^*(\Phi_1)$ *and*

$$\lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\int\limits_0^{\Phi_1(t)}\Big(\int\limits_u^{+\infty}\frac{dv(x)}{x}\Big)du=0,$$

then $(\exists E\in(0,1))\ (\forall r\in(0,1)\backslash E)\ (\forall n\geq\nu(r))$

$$|a_n|r^n\leq\mu(r)\exp\Big(-\int_\nu^n\frac{n-t}{t}dv(t)\Big),$$

where E *is the set of zero density, that is (here* meas *is Lebesgue's measure on the line)*

$$DE=\overline{\lim_{r\to1-0}}\frac{1}{1-r}\mathrm{meas}(E\cap[r,1))=0.$$

Lemma 1. $(\forall\varepsilon>0)\ (\exists E\subset(0,1))\ (\forall r\in(0,1)\backslash E)$

$$M(r)<\frac{\mu(r)}{1-r}\Big(\ln\frac{\mu(r)}{1-r}\ln\frac{1}{1-r}\Big)^{\frac12+\varepsilon},\ \nu(r)<\frac{1}{1-r}\ln\frac{\mu(r)}{1-r}\ln_2^{1+\varepsilon}\frac{\mu(r)}{1-r}\ \textit{and}$$

$$\int\limits_E\frac{dr}{1-r}<+\infty.$$

Denote by $\ln_k x$ the k-th iteration of the logarithm, and let E_1 be the set of finite logarithmic measure; denote by E the set for which $DE=0$.

Lemma 2. *Let* $f(x)$, $g(x)$ *be positive increasing functions,* $f\in C^1(0,1)$, $g\in C(0,+\infty)$, $\int_1^{+\infty}\frac{dt}{g(t)}<+\infty$ *and* $\exists t_0\in(0,1)\colon f(t_0)\geq1$. *Then outside the set of finite logarithmic measure* $E\subset(t_0,1)\colon\ (1-t)f'(t)\leq g(f(t))$.

***Proof of the lemma* 1.** Let $f(r)=\ln\frac{\mu(r)}{1-r}$. Since $r(\ln\mu(r)+\ln\frac{1}{1-r})'_r=\nu(r)+\frac{r}{1-r}$, we will get by lemma 2 that $\frac{1-r}{r}r\frac{d}{dr}f(r)<g(f(r))$, $\frac{1-r}{r}(\nu(r)+\frac{r}{1-r})<g(\ln\frac{\mu(r)}{1-r})$, that is $\nu(r)<\frac{r}{1-r}(g(\ln\frac{\mu(r)}{1-r})-1)<\frac{1}{1-r}g(\ln\frac{\mu(r)}{1-r})$. It remains to choose $g(x)=x\ln^{1+\varepsilon}x$. □

Function $h(t)=\ln a(t)+t\ln r$ is concave, because the function $a(t)$ is log-concave. Let $N'(r)$ be a larger root of the equation $h(t)=0$.

If the line satisfies the following equation

$$y(t)=\frac{\ln\mu(r)}{N_1'(r)-\nu(r)}(N_1'(r)-t),$$

then $y(\nu(r))=\ln\mu(r)$, $y(N_1'(r))=0$. Since $h(t)$ is concave, we have $(\forall t>N_1'(r))\ h(t)<y(t)$. Also $(N_1(r)-2)\ln\mu(r)\leq S(r)\leq2N_1(r)\ln\mu(r)$.

Lemma 3. *If* $x \geq 1$, *then* $N_x(r) \leq xN_1(r) + 1$.

Proof. Since $h(t) < y(t)$ for $t > N_1'(r)$ we get

$$N_x(r) = \#\{n\colon h(n) \geq (1-x)\ln\mu(r)\} \leq \#\Big\{n\colon \frac{\ln\mu(r)}{N_1'(r)-\nu(r)}\times$$
$$\times(N_1'(r)-n) \geq (1-x)\ln\mu(r)\Big\} = \#\Big\{n\colon \frac{1}{N_1'(r)-\nu(r)}(N_1'(r)-n) \geq$$
$$\geq (1-x)\Big\},$$

therefore $N_1'(r) - n \geq (1-x)(N_1'(r)-\nu(r))$; $n \leq N_1'(r) - (1-x)(N_1'(r) - \nu(r))$,

$$N_x(r) \leq N_1'(r) + (x-1)(N_1'(r)-\nu(r)) + 1 = xN_1'(r) - (x-1)\nu(r) + 1 \leq$$
$$\leq xN_1'(r) + 1 \leq xN_1(r) + 1.$$

□

Remark 1. $N_{m,m+1}(r) \leq N_{m+1}(r) - N_m(r) \leq (m+1)N_1(r) + 1 - N_1(r) = mN_1(r) + 1$.

Lemma 4. *If*

$$\lim_{r\to 1-0}(1-r)\ln\ln M(r) = +\infty, \qquad (*)$$

then $(\forall\varepsilon > 0)$ $(\exists E(\varepsilon, f) = E \subset (0,1))$ $(\forall r \in (0,1)\backslash E)$

$$N_1(r) \leq \ln\mu(r)\ln_2^{4+\varepsilon}\mu(r) \text{ and } DE = 0.$$

Proof. Notice that using lemma 2 $1/(1-r) = o(\ln\ln\mu(r))$ as $r \to 1-0$ we obtain $\nu(r) < \ln\mu(r)\ln_2^{2+\varepsilon}\mu(r)$. Since $\mu(r) \to +\infty$ i $\nu(r) = r\frac{d}{dr}\ln\mu(r)$, we have $(\forall r > r_1 > r_0)$ $(\exists C > 0)$

$$\ln\mu(r) - \ln\mu(r_0) = \int_{r_0}^{r}\frac{\nu(t)dt}{t} \leq$$
$$\leq \nu(r)(\ln r - \ln r_0),\ \nu(r) \geq \frac{\ln\mu(r) - \ln\mu(r_0)}{\ln r - \ln r_0} \geq \frac{C\ln\mu(r)}{-\ln r_0}.$$

Therefore, $(\exists r_1 \in (0,1))(\exists C_1 > 0)(\forall r > r_1)$

$$C_1\ln\mu(r) < \nu(r) < \ln\mu(r)\ln_2^{2+\varepsilon}\mu(r). \qquad (2)$$

We choose $\Phi_1(t) = C_1 \ln\mu(1-1/t) \in L$ $[\Phi_1(1/(1-r)) = C_1\ln\mu(r)]$ when $t > t_0$, $f \in S_0^*(\Phi_1)$, because $\varliminf\limits_{r\to 1-0} \frac{\nu(r)}{C_1 \ln\mu(r)} \geq 1$. Taking into account

$$\lim_{r\to 1-0}(1-r)\ln_2 M(r) = +\infty \text{ or } \lim_{r\to 1-0}(1-r)\ln_2\mu(r) = +\infty,$$

we get $\lim\limits_{r\to 1-0}(1-r)(\ln\Phi_1(\frac{1}{1-r}) - \ln C_1) = +\infty$, $\lim\limits_{t\to+\infty}\frac{\ln\Phi_1(t)}{t} = +\infty$.

We may put

$$v(x) = \begin{cases} \int\limits_{e^2}^{x} \frac{dt}{\ln^2 t}, & \text{if } x \geq e^2; \\ 0, & \text{if } 0 \leq x \leq e^2. \end{cases}$$

It's obvious, that $v \in L_0$ and providing theorem 1 $\Phi(t) = t\Phi_1(t)$, we have

$$\lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\int\limits_0^{\Phi_1(t)}\Big(\int\limits_u^{+\infty}\frac{dv(x)}{x}\Big)du = \lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\Big[\int\limits_0^{e^2}\Big(\int\limits_u^{+\infty}\frac{dv(x)}{x}\Big)du+$$
$$+\int\limits_{e^2}^{\Phi_1(t)}\Big(\int\limits_u^{+\infty}\frac{dv(x)}{x}\Big)du\Big] = \lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\Big[\int\limits_0^{e^2}\Big(\int\limits_{e^2}^{+\infty}\frac{dx}{x\ln^2 x}\Big)du+$$
$$+\int\limits_{e^2}^{\Phi_1(t)}\Big(\int\limits_u^{+\infty}\frac{dx}{x\ln^2 x}\Big)du\Big] = \lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\Big[\int\limits_0^{e^2}\Big(-\frac{1}{\ln x}\Big)\Big|_{e^2}^{+\infty}du+$$
$$+\int\limits_{e^2}^{\Phi_1(t)}\frac{du}{\ln u}\Big] \leq 2\lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\Big[\frac{e^2}{4}+\int\limits_{e^2}^{\Phi_1(t)}\frac{(\ln u-1)du}{\ln^2 u}\Big] =$$
$$= 2\lim_{t\to+\infty}\frac{t}{\Phi_1(t)}\Big[\frac{e^2}{4}+\frac{\Phi_1(t)}{\ln\Phi_1(t)}-\frac{e^2}{2}\Big] \leq 3\lim_{t\to+\infty}\frac{t}{\ln\Phi_1(t)} = 0.$$

Outside the set E of finite logarithmic measure using the theorem 1 we will obtain

$$\ln a_n r^n \leq \ln\mu(r) - \int_\nu^n \frac{n-t}{t}dv(t) = \ln\mu(r) - n\int_\nu^n\frac{dt}{t\ln^2 t} + \int_\nu^n\frac{dt}{\ln^2 t} \leq$$
$$\leq \ln\mu(r) - n\int_{\ln\nu}^{\ln n}\frac{dz}{z^2} + \frac{n}{\ln^2\nu} = \ln\mu(r) - n\Big(\frac{1}{\ln\nu}-\frac{1}{\ln n}\Big) + \frac{n}{\ln^2\nu}.$$

Now we put $n = [C_1^{-1}\nu(r)\ln^2\nu(r)]$, where C_1 is constant from inequality (2). Then $(\exists r_1 \in (0,1))$ $(\forall r \in (r_1,1)\backslash E)$

$$\ln a_n r^n \le \ln\mu(r) - C_1^{-1}\Big(\nu(r)\ln^2\nu(r)\frac{2\ln_2\nu(r) - \ln C_1}{\ln\nu(r)(\ln\nu(r) + 2\ln_2\nu(r) - \ln C_1)} -$$

$$-\nu(r)\Big) \le \ln\mu(r) - C_1^{-1}(\nu(r)\ln_2\nu(r) - \nu(r)) \le \ln\mu(r) - C_1^{-1}\nu(r) < 0.$$

So,

$$N_1(r) \le C_1^{-1}\nu(r)\ln^2\nu(r) \le C_1^{-1}\ln\mu(r)\ln_2^{2+\varepsilon}\mu(r)\times$$

$$\times[\ln_2\mu(r) + (2+\varepsilon)\ln_3\mu(r)]^2 \le \ln\mu(r)\times\ln_2^{4+2\varepsilon_1}\mu(r).$$

□

Also we need lower bound for $N_1(r)$.

Lemma 5. *If condition (*) holds, then* $(\exists r_1 \in (0,1))(\forall r > r_1)$ $N_1(r) \ge \nu(r) \ge C_1\ln\mu(r)$.

2 Upper bound for $\mathbf{p_0(r)}$

If a is standard gaussian random variable, then

$$P(|a| \ge \lambda) = \frac{1}{\pi}\iint_{|w|\ge\lambda} e^{-|w|^2}dS = \frac{1}{\pi}\int_0^{2\pi}\int_t^{+\infty} te^{-t^2}dtd\theta =$$

$$= \int_{\lambda^2}^{+\infty} e^{-t}dt = \exp(-\lambda^2),$$

$$P(|a| \le \lambda) = 1 - e^{-\lambda^2} = \lambda^2 - \frac{\lambda^4}{2!} + \ldots \in [\lambda^2/2, \lambda^2] \text{ as } \lambda < 1. \qquad (3)$$

Theorem 2. *If condition (*) holds, then* $(\exists C > 0)(\forall r \in (0,1)\backslash E)$

$$p_0(r) \le S(r) + CN_1(r)\ln N_1(r).$$

We will use the following events

$$\Omega_r^{(1)} = \Big\{\omega\colon |\phi_0(\omega)| \ge \sqrt{N_1(r) + 9\sqrt{N_1(r)}}\Big\},$$
$$\Omega_r^{(2)} = \Big\{\omega\colon |\phi_n(\omega)| \le \frac{(a_n r^n)^{-1}}{\sqrt{N_1(r)}},\ n \in \mathcal{N}_1(r)\Big\},$$
$$\Omega_r^{(4)} = \Big\{\omega\colon |\phi_n(\omega)| \le N_{1,2}^{-1}(r),\ n \in \mathcal{N}_{1,2}(r)\Big\},$$
$$\Omega_r^{(4)} = \Big\{\omega\colon |\phi_n(\omega)| \le \frac{\mu(r)^{m-1}}{N_{m,m+1}(r)m^2},\quad n \in \mathcal{N}_{m,m+1}(r),\quad m \ge 2\Big\},$$
$$\Omega_r = \bigcap_{j=1}^{4} \Omega_r^{(j)}.$$

Lemma 6. $\Omega_r \subset \{\omega\colon (\forall z \in r\mathbb{D})\colon f(z,\omega) \ne 0\}$.

Proof. To see that $f(z,\omega)$ has no zeros inside $r\mathbb{D}$, we note that

$$|f(z,\omega)| \ge |\,\phi_0| - \sum_{n=1}^{+\infty} |\phi_n| a_n r^n. \tag{4}$$

$$\sum_{n\in\mathcal{N}_1(r)\setminus\{0\}} |\phi_n| a_n r^n \le \sum_{n\in\mathcal{N}_1(r)} N_1(r)^{-1/2} = \sqrt{N_1(r)}.$$

$$\sum_{n\notin\mathcal{N}_1(r)} |\phi_n| a_n r^n = \sum_{m=1}^{+\infty}\Big[\sum_{n\in\mathcal{N}_{m,m+1}(r)} |\phi_n| a_n r^n\Big] \le$$

$$\le \sum_{m=1}^{+\infty}\Big[\sum_{n\in\mathcal{N}_{m,m+1}(r)} (N_{m,m+1}(r))^{-1} m^{-2}\Big] = \sum_{m=1}^{+\infty} \frac{1}{m^2} \le 2.$$

From (4) we obtain

$$|f(z,\omega)| \ge \sqrt{N_1(r) + 9\sqrt{N_1(r)}} - \sqrt{N_1(r)} - 2 \ge$$

$$\ge \frac{9}{\sqrt{1 + 9/\sqrt{N_1(r)}} + 1} - 2 \ge \frac{9}{1+\sqrt{10}} - 2 > 0.$$

□

Proof of the theorem 3. $P(\Omega_r^{(1)}) = \exp(-N_1(r) - 9\sqrt{N_1(r)})$. Since $a_n r^n \geq 1$ when $n \in \mathcal{N}_1(r)$, we get

$$P(\Omega_r^{(1)}) = \exp(-N_1(r) - 9\sqrt{N_1(r)}),\ P(\Omega_r^{(2)}) \geq \prod_{n\in\mathcal{N}_1(r)\setminus\{0\}} \frac{(a_n r^n)^{-2}}{2N_1(r)} \geq$$

$$\geq \Big(\prod_{n\in\mathcal{N}_1(r)} \frac{1}{(a_n r^n)^2}\Big)\Big(\frac{1}{2N_1(r)}\Big)^{N_1(r)} \geq \exp(-S(r) - CN_1(r)\ln N_1(r)).$$

$$P(\Omega_r^{(3)}) \geq \Big(\frac{1}{2N_{1,2}(r)^2}\Big)^{N_{1,2}(r)} \geq$$

$$\geq \exp(-CN_{1,2}(r)\ln N_{1,2}(r)) \geq \exp(-CN_1(r)\ln N_1(r)),$$

$$P(\Omega_r^{(4)}) = 1 - \exp\Big(-\frac{\mu(r)^{2(m-1)}}{N^2_{m,m+1}(r)m^4}\Big).$$

Using inequality $P(\{\omega\colon \forall n\colon |\phi_n(\omega)| \leq A_n\}) = 1 - P(\{\omega\colon \exists N\colon |\phi_n(\omega)| > A_n\}) \geq 1 - \sum_{n=1}^{+\infty} P(\{\omega\colon |\phi_n(\omega)| > A_n\})$ we obtain for $r > r_0$

$$P(\Omega_r^{(4)}) \geq 1 - \sum_{m=2}^{+\infty} N_{m,m+1}(r)\exp\Big(-\frac{\mu(r)^{2(m-1)}}{N^2_{m,m+1}(r)m^4}\Big) \geq$$

$$\geq 1 - CN_1(r)\sum_{m=2}^{+\infty} m\exp\Big(-\frac{\mu(r)^{2m-2}}{N_1^2(r)m^6}\Big) \geq$$

$$\geq 1 - CN_1(r)\sum_{m=2}^{+\infty} m\exp\Big(-\frac{\mu(r)^{2m-3}}{m^6}\Big) \geq$$

$$\geq 1 - CN_1(r)\exp(-\ln\mu(r)) = 1 - \exp(-C\mu(r) + C_1\ln N_1(r)) \geq$$

$$\geq 1 - \exp(-C\mu(r)).$$

Therefore, $P(\Omega_r^{(4)}) \geq 1 - \exp(-C\mu(r))$.

Since ϕ_n are independent, events $\{\Omega_r^{(i)}\}_{i=1}^4$ are also independent.

$$P(\Omega_r) = P(\Omega_r^{(1)})P(\Omega_r^{(2)})P(\Omega_r^{(3)})P(\Omega_r^{(4)}) \geq \exp(-S(r) - CN_1(r)\ln N_1(r)).$$

□

3 Lower bound for $\mathbf{p_0(r)}$

Now we denote $M(r) = \sum\limits_{n=0}^{+\infty} a_n r^n,\ \mathcal{M}(r,\omega) = \max\limits_{|z|\leq r}|f(z,\omega)|$.

Lemma 7. *If* $0<\delta\leq 1/2$, *then* $(\exists E_1\in(0,1))$ $(\forall r\in(0,1)\backslash E_1)$ $(\exists C>0)$

$$\ln P\Big(\Big\{\omega\colon \frac{\ln\mathcal{M}(r,\omega)}{\ln M(r)}\geq 1+\delta\Big\}\Big)\leq -C\mu(r)^{2\delta}.$$

Proof. We consider $\Omega_r^*=\Omega_r^{(4)}\cap\Omega_r^{(5)}$, where $\Omega_r^{(5)}=\{\omega\colon |\phi_n(\omega)|\leq\mu^\delta(r),\ n\in\mathcal{N}_2(r)\}$. Then $P(\{\omega\colon |\phi(\omega)|>\mu^\delta(r)\})=\exp(-\mu^{2\delta}(r))$. By lemma 6 we obtain $P(\Omega_r^{(4)})\geq 1-\exp(-C\mu(r))$. So, $P(\overline{\Omega_r^*})\leq\exp(-C\mu(r))+N_2(r)\exp(-\mu^{2\delta}(r))\leq\exp(-C\mu^{2\delta}(r))$. Now we remark that

$$|f(z,\omega)|\leq\sum_{n\in\mathcal{N}_2(r)}|\phi_n|a_nr^n+\sum_{n\in\mathcal{N}_2^c(r)}|\phi_n|a_nr^n\leq\mu^\delta(r)\sum_{n=0}^{+\infty}a_nr^n+2=$$

$$=\mu^\delta(r)M(r)+2\leq M^{1+\delta}(r).$$

□

Lemma 8 ([2])**.** $\ln P(\{\omega\colon \ln\mathcal{M}(r,\omega)\leq 0\})\leq -S(r)$.

We use the following notations $\delta(r)=N_1(r)^{-\alpha}\in(0,1),\ \alpha>0,\ k(r)=1-\delta(r)$.

Denote by $P(z,z_j)$ Poisson kernel for the unit disc

$$\frac{|z|^2-|z_j|^2}{|z-z_j|^2}=\frac{r^2-k^2(r)r^2}{r^2-2k(r)r^2\cos(\theta-\phi)+k^2(r)r^2},$$

where $z=re^{i\theta}$, $\{z_j\}_{j=0}^{N_1(r)-1}\in kr\mathbb{T}$, $z_j=k(r)r\exp(\frac{2\pi ij}{N_1(r)})=k(r)re^{i\varphi}$ $\{j\in 1,...,N_1(r)\}$.

Formulation of the following lemma we find in [2]. We produce it with complete proof, having corrected some inaccuracies in its proof in [2].

Lemma 9. $(\exists E_1\in(0,1))$ $(\forall r\in(0,1)\backslash E_1)$

$$P\Big(\Big\{\omega\colon \frac{2\pi}{N_1(r)}\sum_{j=0}^{N_1(r)-1}\ln|f(z_j,\omega)|=\int_0^{2\pi}\ln|f(z,\omega)|d\theta+$$

$$+\frac{\widetilde{C}(r)}{\delta^4(r)N_1(r)}\ln\mu(r),\ \textit{where}\ |\widetilde{C}(r)|\leq 100\pi\Big\}\Big)\geq 1-2\exp(-S(k(r)r)).$$

Proof. We suppose that $f(z,\omega)$ has no zeros in $r\mathbb{D}$. Then $\ln|f(z,\omega)|$ is a harmonic function in $r\mathbb{D}$ and $\ln|f(z_j,\omega)| = \frac{1}{2\pi}\int_0^{2\pi} P(z,z_j)\ln|f(z,\omega)|d\theta$.

$$\frac{2\pi}{N_1(r)}\sum_{j=0}^{N_1(r)-1}\ln|f(z_j,\omega)| = \int_0^{2\pi}\Big(\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)\Big)\ln|f(z,\omega)|d\theta =$$
$$= \int_0^{2\pi}\ln|f(z,\omega)|d\theta + \int_0^{2\pi}\Big(\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)-1\Big)\ln|f(z,\omega)|d\theta.$$

Last expression can be estimated by

$$\Big|\int_0^{2\pi}\Big(\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)-1\Big)\ln|f(z,\omega)|d\theta\Big| \le$$
$$\le \max_{z\in r\mathbb{T}}\Big|\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)-1\Big|\int_0^{2\pi}|\ln|f(z,\omega)||d\theta.$$

Now we put $z=re^{i\theta}, w=k(r)re^{i\varphi}$ and $\int_0^{2\pi}P(z,w)d\varphi=1$. Then we split the circle $kr\mathbb{T}$ into a union of $N_1(r)$ disjoint arcs I_j of equal angular measure $\mu(I_j)=2\pi/N_1(r)$ centred at the z_j's.

$$2\pi = \frac{2\pi}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j) + \sum_{j=0}^{N_1(r)-1}\int_{I_j}(P(z,w)-P(z,z_j))d\varphi,$$
$$|P(z,w)-P(z,z_j)| = \Big|\frac{|z|^2-|w|^2}{|z-w|^2} - \frac{|z|^2-|z_j|^2}{|z-z_j|^2}\Big| =$$
$$= \delta^2(r)r^2\Big|\frac{1}{|z-w|^2} - \frac{1}{|z-z_j|^2}\Big| =$$
$$= \delta^2(r)r^2\frac{(|z-z_j|+|z-w|)\cdot||z-z_j|-|z-w||}{|z-w|^2|z-z_j|^2} \le \delta^2(r)r^2\frac{4r\cdot|z_j-w|}{(|z|-|w|)^4} \le$$
$$\le 4r^3\delta^2(r)\cdot\frac{2\pi r}{N_1(r)}\frac{1}{\delta^4(r)r^4} = \frac{8\pi}{\delta^2(r)N_1(r)}.$$

$$-\frac{4}{\delta^2(r)N_1(r)} \le \frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)-1 \le \frac{4}{\delta^2(r)N_1(r)}. \qquad (5)$$

Using lemma 8, we may suppose that there is a point $a\in kr\mathbb{T}\ \ln|f(a,\omega)| \ge 0$ (discarding an exceptional event of probability at most $\exp(-S(kr))$).

Then

$$0 \le \int_0^{2\pi} P(z,a)\ln|f(z,\omega)|d\theta = \int_0^{2\pi} P(z,a)(\ln^+|f(z,\omega)| - \ln^-|f(z,\omega)|)d\theta,$$
$$\int_0^{2\pi} P(z,a)\ln^-|f(z,\omega)|d\theta \le \int_0^{2\pi} P(z,a)\ln^+|f(z,\omega)|d\theta.$$

For $|z| = r$ i $|a| = kr$ we have

$$\frac{\delta(r)}{2} \le \frac{1-(1-\delta(r))}{1+(1-\delta(r))} \le P(z,a) \le \frac{1+(1-\delta(r))}{1-(1-\delta(r))} \le \frac{2}{\delta(r)}.$$

By lemma 7 with probability at least $1-\exp(-C\mu(r))$, we get $\ln \mathcal{M}(r) \le 1.5\ln M(r)$, $\int_0^{2\pi}\ln^+|f(z,\omega)|d\theta \le 3\pi\ln M(r)$.

$$\int_0^{2\pi} P(z,a)\ln^-|f(z,\omega)|d\theta \le \int_0^{2\pi} P(z,a)\ln^+|f(z,\omega)|d\theta \le$$
$$\le \frac{2}{\delta(r)}\int_0^{2\pi}\ln^+|f(z,\omega)|d\theta \le \frac{6\pi}{\delta(r)}\ln M(r).$$
$$\int_0^{2\pi}\ln^-|f(z,\omega)|d\theta \le \frac{12\pi}{\delta^2(r)}\ln M(r),\ \max_{z\in r\mathbb{T}}\Big|\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)-1\Big| \le$$
$$\le \frac{4}{\delta^2(r)N_1(r)},$$
$$\int_0^{2\pi}\ln^+|f(z,\omega)|d\theta \le 3\pi\ln M(r),$$
$$\int_0^{2\pi}|\ln|f(z,\omega)||d\theta \le \Big(\frac{12\pi}{\delta^2(r)} + 3\pi\Big)\ln M(r) \le \frac{16\pi}{\delta^2(r)}\ln M(r).$$

Using theorem 1 with $h(r) = (1-r)^{-1}$ from [4], classical Wiman's inequality holds by condition (*), that is $(\forall r \in (0,1)\backslash E_1)$: $\ln M(r) \le$

$\ln\mu(r)+(\frac12+\varepsilon)\ln_2\mu(r)\le\frac{100}{64}\ln\mu(r)$.

$$\Big|\Big(\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}P(z,z_j)-1\Big)\int\limits_0^{2\pi}|\ln|f(z,\omega)||d\theta\Big|\le$$
$$\le\frac{4}{\delta^2(r)N_1(r)}\cdot\frac{16\pi}{\delta^2(r)}\ln M(r)\le\frac{100\pi}{\delta^4(r)N_1(r)}\ln\mu(r),$$
$$\int_0^{2\pi}\ln|f(z,\omega)|d\theta-\frac{100\pi}{\delta^4(r)N_1(r)}\ln\mu(r)\le$$
$$\le\frac{2\pi}{N_1(r)}\sum_{j=0}^{N_1(r)-1}\ln|f(z_j,\omega)|\le\int_0^{2\pi}\ln|f(z,\omega)|d\theta+\frac{100\pi}{\delta^4(r)N_1(r)}\ln\mu(r).$$

With probability at least $1-2\exp(-S(k(r)r))$ $(\forall r\in(0,1)\backslash E_1)$

$$\frac{1}{N_1(r)}\sum_{j=0}^{N_1(r)-1}\ln|f(z_j,\omega)|=\int_0^{2\pi}\ln|f(z,\omega)|d\theta+\frac{\widetilde{C}(r)}{\delta^4(r)N_1(r)}\ln\mu(r),$$

where $|\widetilde{C}(r)|\le 100\pi$. □

Since $f(z,\omega)=\sum\limits_{n=0}^{+\infty}\phi_na_nz^n$, then covariance matrix $\Sigma=(\Sigma_{i,j})$ of random vector $\{f(z_1),...,f(z_n)\}$

$$\sum\nolimits_{ij}=\mathrm{Cov}(f(z_i,\omega),f(z_j,\omega))=\mathbf{E}\Big[\Big(\sum_{n=0}^{+\infty}\phi_na_nz_i^n\Big)\cdot\overline{\sum_{n=0}^{+\infty}\phi_na_nz_j^n}\Big]=$$

$$=\sum_{k=0}^{+\infty}a_k^2(z_i\cdot\overline{z}_j)^k. \tag{6}$$

The density function of such random vector ([5])

$$\xi\mapsto\frac{1}{\pi^n|\Sigma|}\cdot\exp(-\xi^T\Sigma^{-1}\xi).$$

We denote for $(r \notin E_1)$ the following events $(\xi_j = f(z_j))$

$$\mathcal{A}' = \Big\{\xi \in \mathbb{C}^N \colon e^{2\pi} \prod_{j=1}^{N_1(r)} |\xi_j| \le \exp(2N_1(r) \ln_2 \mu(r) + 100\pi\delta^{-4}(r) \ln \mu(r))\Big\},$$

$$\mathcal{B} = \Big\{\omega \colon \frac{2\pi}{N_1(r)} \sum_{j=0}^{N_1(r)-1} \ln|f(z_j, \omega)| = \int_0^{2\pi} \ln|f(z,\omega)| d\theta + \\ + \frac{\widetilde{C}(r)}{\delta^4(r) N_1(r)} \ln \mu(r), \text{ where } |\widetilde{C}(r)| \le 100\pi\Big\}.$$

Lemma 10. $(\exists E_1 \in (0,1))(r \notin E_1)$ $P\Big(\Big\{\omega \colon \int_0^{2\pi} \ln|f(z,\omega)| d\theta \le 2\ln_2 \mu(r)\Big\}\Big) \le P(\mathcal{A}') + P(\overline{\mathcal{B}})$.

Proof. We may consider $\overline{\mathcal{A}'} = \{\xi \in \mathbb{C}^N \colon e^{2\pi} \prod_{j=1}^{N_1(r)} |\xi_j| \ge \exp(2N_1(r) \times \ln_2 \mu(r) + 100\pi\delta^{-4}(r) \times \times \ln \mu(r))\}$, $\mathcal{B} = \{\xi \in \mathbb{C}^N \colon e^{2\pi} \prod_{j=1}^{N_1(r)} |\xi_j| = \exp(N_1(r) \int\limits_0^{2\pi} \ln|f(z,\omega)| d\theta + \widetilde{C}(r) \delta^{-4}(r) \ln \mu(r)), \text{ where } |\widetilde{C}(r)| \le 100\pi\}$. Then

$$\overline{\mathcal{A}'} \cap \mathcal{B} = \Big\{\xi \in \mathbb{C}^N \colon \exp\Big(N_1(r) \int\limits_0^{2\pi} \ln|f(z,\omega)| d\theta + \widetilde{C}(r)\delta^{-4}(r) \ln \mu(r)\Big) \ge \\ \ge \exp(2N_1(r) \ln_2 \mu(r) + 100\pi\delta^{-4}(r) \ln \mu(r))\Big\}.$$

So, the event $\{\omega \colon \int_0^{2\pi} \ln|f(z,\omega)| d\theta \ge 2\ln_2 \mu(r)\} \subset \overline{\mathcal{A}'} \cap \mathcal{B}$. Then

$$P\Big(\Big\{\omega \colon \int_0^{2\pi} \ln|f(z,\omega)| d\theta \le 2\ln_2 \mu(r)\Big\}\Big) \le P(\overline{\overline{\mathcal{A}'} \cap \mathcal{B}}) = \\ = P(\mathcal{A}' \cup \overline{\mathcal{B}}) \le P(\mathcal{A}') + P(\overline{\mathcal{B}}) \le P(\mathcal{A}') + P(\overline{\mathcal{B}}).$$

□

Lemma 11 ([2])**.** *Let* Σ *be covariance matrix* (6), *then* $\ln(\det \Sigma) \ge S(kr)$.

For $(r \notin E_1)$ we denote by $\mathcal{A}$ the following event

$$\mathcal{A} = \{\xi \in \mathbb{C}^N \colon \xi \in \mathcal{A}' \text{ i } |\xi_j| \le M^2(r), 0 \le j \le N-1\},$$

$$I = \pi^{-N} \mathrm{vol}_{\mathbb{C}^{\mathbb{N}}}(\mathcal{A}) = \pi^{-N} \int_{\mathcal{A}} dV.$$

We will use the following lemma from [6].

Lemma 12 ([6])**.** *If* $s > 0,\ t > 0,\ N \in \mathcal{N}^+\colon \ln(t^N/s) \ge N$ *and* $\mathcal{C}_N = \mathcal{C}_{\mathcal{N}}(t,s) = \{r = (r_1, ..., r_N)\colon 0 \le r_j \le t, \prod_{j=1}^{N} r_j \le s\}$, *then*

$$\mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}_N) \le \frac{s}{(N-1)!} \ln^N(t^N/s).$$

Lemma 13. $(\forall r \in (0,1)\backslash E_1)$ *and* $\delta^{-4}(r) = o(N_1(r))$ $(r \to 1-0)$

$$\ln I \le 7N_1(r)\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r).$$

Proof. Suppose now that in lemma 12 $N = N_1(r), s = \exp(2N\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r))$, $t = M^2(r)$. We want to translate the integral I into an integral in $\mathbb{R}^n$, using the change of variables $\xi = r_j\cos\theta_j + ir_j\sin\theta_j$.

$$\det J = \begin{vmatrix} \cos\theta_1 & -r_1\sin\theta_1 & 0 & 0 & \dots \\ \sin\theta_1 & r_1\cos\theta_1 & 0 & 0 & \dots \\ 0 & 0 & \cos\theta_2 & -r_2\sin\theta_2 & \dots \\ 0 & 0 & \sin\theta_2 & r_2\cos\theta_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = \prod_{j=1}^{N} r_j.$$

$$I' = \pi^{-N}\int_D d\theta \int_{\mathcal{C}} \prod_{j=1}^{N} r_j dr = \pi^{-N}(2\pi)^N \int_{\mathcal{C}} \prod_{j=1}^{N} r_j dr = 2^N \int_{\mathcal{C}} \prod_{j=1}^{N} r_j dr,$$

where $D = [0, 2\pi]^N$, $\mathcal{C} = \{r = (r_1, ..., r_N)\colon 0 \le r_j \le t, \prod_{j=1}^{N} r_j \le s\}$.

Therefore $I' \le 2^N \cdot s \cdot \mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}_N)$. We check the following condition $\ln(t^N/s) \ge N$, that is $2N_1(r)\ln M(r) - 2N_1(r)\ln_2\mu(r) - 100\pi\delta^{-4}(r)\ln\mu(r) \ge N_1(r)$, which is satisfied when $r > r_1$ and $\delta^{-4}(r) = o(N_1(r))$ $(r \to 1-0)$. Thus, using lemma 12 we get

$$I' \le 2^N s\frac{N}{N!} s\ln^N\frac{t^N}{s} \le \frac{s^2 e^{2N}}{N^N}\ln^N\frac{t^N}{s} =$$
$$= \exp(2\ln s + 2N - N\ln N + N(\ln_2 t + \ln N) - N\ln_2 s) =$$
$$= \exp(2\ln s + 2N + N\ln_2 t - N\ln_2 s).$$
$$\ln I' \le 2\ln s + 2N + N\ln_2 t - N\ln_2 s \le 4N_1(r)\ln_2\mu(r) +$$
$$+100\pi\delta^{-4}(r)\ln\mu(r) + 2N_1(r) + N_1(r)\ln_2\mu(r) -$$
$$-N_1(r)\ln[2N_1(r)\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r)] \le$$
$$\le 7N_1(r)\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r).$$

□

Lemma 14. $(\exists E_1 \subset (0,1))$ $(\forall r \in (0,1)\backslash E_1)$ $(\exists C > 0)$ $(\delta^{-4}(r) = o(N_1(r))$ $(r \to 1-0))$

$$P(\mathcal{A}'\backslash\mathcal{A}) \leq \exp(-C\ln\mu(r)), \quad P(\mathcal{A}) \leq$$

$$\leq \exp(-S(kr) + 7N_1(r)\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r)).$$

Proof. From lemma 7 implies the first inequality $P(\mathcal{A}'\backslash\mathcal{A}) = P(\{\xi\colon |\xi| \leq M^2(r)\}) = P(\{\omega\colon |f(z_j,\omega)| \leq M^2(r)\}) \leq P(\{\omega\colon \mathcal{M}(r,\omega) \leq M^2(r)\}) \leq \exp(-C\ln\mu(r))$.

Since covariance matrix is positive definite, then inverse matrix Σ^{-1} is also positive definite, that is $\forall\xi \in R^N_+\colon -\xi^T\Sigma^{-1}\xi > 0$.

$$P(\mathcal{A}) = \int_{\mathcal{A}} \frac{1}{\pi^N|\Sigma|} \cdot \exp(-\xi^T\Sigma^{-1}\xi)dV \leq \frac{\mathrm{vol}_{\mathbb{C}^N}(\mathcal{A})}{\pi^N|\Sigma|} \leq$$
$$\leq \exp(-S(kr) + 7N_1(r)\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r)).$$

□

Theorem 3. *If condition* (*) *holds, then* $(\forall r \in (0,1)\backslash E)(\delta^{-4}(r) = o(N_1(r))$ $(r \to 1-0))$

$$p_0(r) \geq S(k(r)r) - 7N_1(r)\ln_2\mu(r) - 100\pi\delta^{-4}(r)\ln\mu(r).$$

Proof. We may suppose that $f(z,\omega)$ has no zeros in $r\mathbb{D}$. Then function $\ln|f(z,\omega)|$ is harmonic in $r\mathbb{D}$. Thus

$$\int_0^{2\pi} \ln|f(z,\omega)|d\theta = \ln|f(0,\omega)|.$$
$$P(\{\omega\colon \ln|f(0,\omega)| \geq 2\ln_2\mu(r)\}) = P(\{\omega\colon |\phi_0(\omega)| \geq \ln^2\mu(r)\}) =$$
$$= \exp(-\ln^4\mu(r)).$$

Let $G_1 = \{\omega\colon \int_0^{2\pi} \ln|f(z,\omega)|d\theta \leq 2\ln_2\mu(r)\}$, $G_2 = \{\omega\colon \ln|f(0,\omega)| \geq 2\ln_2\mu(r)\}$, then $\overline{G}_1 \cap \overline{G}_2 = \{\omega\colon \ln|f(0,\omega)| < 2\ln_2\mu(r) < \int_0^{2\pi} \ln|f(z,\omega)|d\theta\}$.

So,

$$p_0(r) = P\Big(\Big\{\omega\colon \int_0^{2\pi} \ln|f(z,\omega)|d\theta = \ln|f(0,\omega)|\Big\}\Big) \le P(G_1) + P(G_2) =$$
$$= P(\{\omega\colon \ln|f(0,\omega)| \ge 2\ln_2\mu(r)\}) + P\Big(\Big\{\omega\colon \int_0^{2\pi} \ln|f(z,\omega)|d\theta \le$$
$$\le 2\ln_2\mu(r)\Big\}) \le \exp(-\ln^4\mu(r)) + P(\mathcal{A}') + P(\overline{\mathcal{B}}) =$$
$$= \exp(-\ln^4\mu(r)) + P(\mathcal{A}) + P(\mathcal{A}'\backslash\mathcal{A}) + P(\overline{\mathcal{B}}) \le \exp(-\ln^4\mu(r)) +$$
$$+ \exp(-S(k(r)r) + 7N_1(r)\ln_2\mu(r) + 100\pi\delta^{-4}(r)\ln\mu(r)) +$$
$$+ \exp(-\ln\mu(r)) + 2\exp(-S(k(r)r)).$$

Therefore $p_0(r) \ge S(k(r)r) - 7N_1(r)\ln_2\mu(r) - 100\pi\delta^{-4}(r)\ln\mu(r))$. □

4 Main result

Theorem 4. *If* $\{a_n\}$ *is log-concave and* $\lim\limits_{r\to 1-0}(1-r)\ln\ln M(r) = +\infty$, *then* $(\forall\varepsilon>0)$ $(\exists E_1(\varepsilon,f)=E_1\subset(0,1))$ $(\forall r\in(0,1)\backslash E_1)$

$$DE_1 = 0 \text{ and } p_0(r) = S(r) + o(S(r)) \quad (r\to 1-0),$$

or

$$S(r) - S^{9/10}(r)\ln^{18/5+\varepsilon}S(r) \le p_0(r) \le S(r) + \sqrt{S(r)}\ln^{3+\varepsilon}S(r).$$

Proof. At first we will estimate $N_1(r)\ln N_1(r)$. $S(r) \le \ln^2\mu(r)\ln_2^{4+\varepsilon}\mu(r) \le \ln^2\mu(r)\times \times \ln^{4+\varepsilon}S(r)$. Then $\sqrt{S(r)} \le \ln\mu(r)\ln^{2+\varepsilon/2}S(r)$.

$$N_1(r) \le 2\frac{S(r)}{\ln\mu(r)} = 2\frac{\sqrt{S(r)}\sqrt{S(r)}}{\ln\mu(r)} \le \sqrt{S(r)}\ln^{2+\varepsilon/2}S(r). \tag{7}$$

$$\ln N_1(r) \le \ln S(r) \ \Rightarrow N_1(r)\ln N_1(r) \le \sqrt{S(r)}\ln^{3+\varepsilon/2}S(r). \tag{8}$$

Now we will investigate the function $S(r)$. We remark that $S(r) = 2\sum_{n\in\mathcal{N}_1(r)}(\ln a_n + n\ln r)$. Then $S(r)$ is continuous in (0,1). Let $\{r_n\}$ be points where $S'(r)$ doesn't exist. $(\forall k\in\mathbb{N})$ $(r\ne r_n)(\forall r>r_0)$

$$S'(r) = 2\sum_{n\in\mathcal{N}_1(r)}\frac{n}{r} \le \frac{2}{r_0}\Big(\sum_{n\in\mathcal{N}_1(r)} n\Big) = O(N_1^2(r)) \text{ if } r\to 1-0,$$

$$S''(r) = -2 \sum_{n\in\mathcal{N}_1(r)} \frac{n}{r^2} < 0.$$

If we denote $\Delta_k = S'_+(r_k) - S'_-(r_k) = N_1(r_k)/r_k$, then for $r > r_1$

$$S(r) - S(k(r)r) \leq \Big(S'(k(r)r) + \sum_{n\in\mathcal{N}_1(r)\setminus\mathcal{N}_1(k(r)r)} \Delta_n\Big)(r - k(r)r) \leq$$

$$\leq \Big(S'(k(r)r) + \frac{1}{r_1} \sum_{n\in\mathcal{N}_1(r)\setminus\mathcal{N}_1(k(r)r)} N_1(r)\Big)(r - k(r)r) \leq$$

$$\leq C(N_1^2(k(r)r) + N_1^2(r))(r - k(r)r) \leq CN_1^2(r)\delta(r)r \leq C\delta(r)N_1^2(r).$$

Therefore, $S(r') \geq S(r) - C\delta(r)N_1^2(r)$. From theorem 4 we get

$$p_0(r) \geq S(r) - C\delta(r)N_1^2(r) - 7N_1(r)\ln_2 \mu(r) - 100\pi\delta^{-4}(r)\ln\mu(r).$$

If we choose $\delta(r) = N_1(r)^{-1/5}$, then

$$p_0(r) \geq S(r) - C_1N_1^{9/5}(r) - C_2N_1^{4/5}(r)\ln\mu(r) \geq S(r) - CN_1^{9/5}(r).$$

Using inequalities (7) and (8) we obtain

$$S(r) - S^{9/10}(r)\ln^{18/5+\varepsilon} S(r) \leq p_0(r) \leq S(r) + \sqrt{S(r)}\ln^{3+\varepsilon} S(r).$$

□

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ЙМОВІРНІСТЬ ВІДСУТНОСТІ НУЛІВ ДЛЯ ДЕЯКИХ АНАЛІТИЧНИХ ФУНКЦІЙ

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Ми розглянемо випадкові аналітичні функції $f(z,\omega)=\sum\limits_{n=0}^{+\infty}\phi_n a_n z^n$, де $a_n>0$ такі, що $\lim\limits_{n\to+\infty}\sqrt[n]{a_n}=1$, $\phi_n=\phi_n(\omega)$ — комплексні гаусові випадкові величини задані на ймовірнісному просторі Штейнгауса (Ω, $\mathcal{A}, P$). Якщо $\{a_n\}$ — логарифмічно ввігнута і $\lim\limits_{r\to1-0}(1-r)\ln\ln M(r)=+\infty$, тоді $(\exists E(\varepsilon,f)=E\subset(0,1))(\forall r\in(0,1)\backslash E)$

$$\overline{\lim_{r\to1-0}}\frac{1}{1-r}\mathrm{meas}(E\cap[r,1))=0 \text{ і } \ln^{-}P(\{\omega\colon f(z,\omega)\neq0\})=S(r)+o(S(r))$$

$(r\to1-0)$, де $S(r)=2\sum\limits_{n\colon a_nr^n\geq1}\ln(a_nr^n)$.