GROUPS WITH MANY ČERNIKOV-BY-NILPOTENT SUBGROUPS

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We prove that a non-perfect group is a minimal non-"Černikov-by-nilpotent" group if and only if it is a Heineken-Mohamed type group.

Introduction

Let χ be a class of groups closed under subgroups. A minimal non-χ group G is a group which is not a χ-group, while all proper subgroups of G are χ. Belyaev and Sesekin (see [1] and [2]) have determined the minimal non-"finite-by-abelian" groups. Bruno and Phillips [3] have classified infinite groups in which every proper subgroup is finite-by-"nilpotent of class c". Later Otal and Peña [4] have extended this class of groups by replacing the term "finite group" with "Černikov group" and have considered the locally graded groups in which every proper subgroup is Černikov-by-nilpotent.

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In particular, it was proved that a locally graded group G is Černikov-by-"nilpotent of class c" if and only if all proper subgroups of G are Černikov-by-"nilpotent of class c" ($c \geq 1$ is an integer). Xu [5] has investigated the minimal non-"finite-by-nilpotent" groups.

In this paper we study the groups with many Černikov-by-nilpotent subgroups. We concerned the non-perfect groups in which all proper subgroups are Černikov-by-nilpotent and prove the following

Theorem. *Let G be a non-perfect group. Then G is a minimal non-"Černikov-by-nilpotent" group if and only if it is a Heineken-Mohamed type group.*

Minimal non-nilpotent groups with subnormal proper subgroups (so-called the Heineken-Mohamed type groups) are constructed, for example, in [6].

Throughout this paper p is a prime, $\mathbb{Z}$ is the ring of integers. For a group G, we denote by $Z(G)$ the centre, by $G', G'', \ldots, G^{(n)}, \ldots$ the members of the derived series, by $\gamma_1(G) = G, \gamma_2(G), \ldots, \gamma_m(G), \ldots$ the members of the lower central series of G. Moreover $\mathbb{Z}G$ means the group ring of G over $\mathbb{Z}$ and $H \lhd G$ means that H is a normal subgroup in G. Recall that a group G is non-perfect if $G' \neq G$.

We shall also use other standard terminology from [7] and [8].

1 Minimal non-"Černikov-by-nilpotent" groups

From Theorem 1 of [4] it follows that a locally graded group G with Černikov-by-abelian proper subgroups is Černikov-by-abelian. Recall that a group G is called indecomposable if any two proper subgroups of G generate a proper subgroup in G, and is called decomposable otherwise.

In the sequel we shall need the next lemmas.

Lemma 1.1. *Let G be a non-perfect group. If all proper normal subgroups of G are Černikov-by-nilpotent, then one of the following statements holds:*

(1) *G is a Černikov-by-nilpotent group;*

(2) *G/G' is a quasicyclic group and G' is a torsion subgroup;*

(3) *G/G' is a cyclic group.*

Proof. If the quotient group G/G' is decomposable, then $G = AB$ is a product of two proper normal subgroups A and B. Since A and B are Černikov-by-nilpotent, G is such as well.

Now let G/G' be an indecomposable group. Then by Lemma 2 of [9] G/G' is a quasicyclic p-group or a cyclic p-group for some prime p.

Suppose that G is not a Černikov-by-nilpotent group and the quotient group G/G' is quasicyclic. We need to show that the derived subgroup G' is torsion. For this, assume by contrary that G' is non-torsion. Then without restricting of generality we can assume that G' is an abelian torsion-free subgroup.

Let q be a prime different from p. By Lemma 2.3 of [10] there exists a proper $\mathbb{Z}[G/G']$-submodule N in G' such that $\overline{G'} = G'/N$ is a q-group. If a is any element of G and $\overline{G} = G/N$, then $\overline{G'}\langle\overline{a}\rangle$ is an abelian group, and this gives a contradiction Hence G' is a torsion subgroup. The proof is completed. □

Lemma 1.2. *Let G be a non-perfect group with Černikov-by-nilpotent proper subgroups. If G/G' is a cyclic p-group, then G is a Černikov-by-nilpotent group or the derived subgroup G' is torsion.*

Proof. Let G be a non-"Černikov-by-nilpotent" group. Suppose that the result is false. Then without loss of generality we may assume that G' is an abelian torsion-free subgroup. Obviously $G = G'\langle a\rangle$ for some element $a \in G$, where $a^{p^n} \in G'$ for some positive integer n. Let q and r be the distinct primes different from p. By Lemma 2.3 of [10] there exists a proper $\mathbb{Z}[G/G']$-submodule M of G' such that

$$\overline{G'} = G'/M = \overline{A} \times \overline{B}$$

is a group direct product of a non-trivial q-subgroup $\overline{A}$ and a non-trivial r-subgroup $\overline{B}$. Let A (and respectively B) be an inverse image of $\overline{A}$ (and respectively $\overline{B}$) in G. Then $A\langle a\rangle$ and $B\langle a\rangle$ are two proper subgroups of G and therefore there are positive integers k and s such that $\gamma_k(A\langle a\rangle) \leq A$, $\gamma_s(B\langle a\rangle) \leq B$ and $\gamma_k(A\langle a\rangle)$, $\gamma_s(B\langle a\rangle)$ are Černikov. This gives that G is Černikov-by-nilpotent, which is a contradiction. □

Lemma 1.3. *Let G be a group with the quasicyclic quotient group G/G'. Then G is a minimal non-"Černikov-by-nilpotent" group if and only if G is a Heineken-Mohamed type group.*

Proof. ($\Leftarrow$) is clear.

($\Rightarrow$) It is now easy to see that G does not contain a proper subgroup of finite index. Let S be a proper subgroup of G, $\overline{G} = G/G''$ and $\overline{S} = SG''/G''$. If $\overline{S}$ is proper in $\overline{G}$, then $\overline{S}$ (and consequently S) is a nilpotent group. Assume that $\overline{S} = \overline{G}$. Then there exists a positive integer k such that $\gamma_k(\overline{G})$ is a Černikov group and so $\gamma_k(\overline{G}) \leq Z(\overline{G})$. This means that $\overline{G}$ is a nilpotent group, a contradiction. □

Lemma 1.4. *If G is a Černikov-by-nilpotent group, then G is Černikov or the derived subgroup G' is of infinite index in G.*

Proof. By the hypothesis, there is a positive integer m such that $\gamma_m(G)$ is a Černikov subgroup. If $\overline{G} = G/\gamma_m(G)$ is a finite group, then G is Černikov. Suppose that $\overline{G}$ is an infinite group. Then the quotient group $\overline{G}/\overline{G}'$ is also infinite and this completes the proof. □

Newman and Wiegold [11] have determined the structure of minimal non-nilpotent groups with maximal subgroups (see also Theorem 3.1 from [12]). By Theorem 2.5 of [12] every infinite soluble minimal non-nilpotent group is either of Heineken-Mohamed type or contains a maximal subgroup. Theorem 3.1 of [12] yields that a minimal non-nilpotent group with a maximal subgroup is a hypercentral Černikov-by-nilpotent group.

Lemma 1.5. *Let G be a group with the cyclic quotient p-group G/G'. If all proper subgroups of G are Černikov-by-nilpotent, then G is Černikov-by-nilpotent.*

Proof. Let G be as given. By hypothesis, $G = G'\langle a\rangle$ for some element $a \in G$, where $a^{p^n} \in G$ for some positive integer n. Assume that G is not a Černikov-by-nilpotent group. Then in view of Lemma 1.2 the derived subgroup G' is a torsion subgroup.

Next we prove that G' does not contains a proper subgroup of a finite index. Assume to the contrary, i.e. let H be a proper G-invariant subgroup of finite index in G'. Then $H\langle a\rangle$ is a proper subgroup of G and so there is a positive integer s such that $\gamma_s(H\langle a\rangle)$ is a Černikov subgroup. Since G' is not Černikov, we obtain in view of Lemma 1.4 that G'' is of infinite index in G'. Put $\overline{G} = G/G''$. Assume that $\overline{H}\langle\overline{a}\rangle = (H\langle a\rangle G'')/G''$ is a Černikov group, then $\overline{G}$ is a Černikov group as well. Inasmuch as $\gamma_m(G')$ is a Černikov subgroup for some positive integer m, $\gamma_m(G') \leq G''$ and $G'/\gamma_m(G')$ is a nilpotent group, we see in view of Corollary (see [13], p. 19) that $G'/\gamma_m(G')$ (and consequently G') is a Černikov group, a contradiction. By Lemma 1.4 this means that $(\overline{H}\langle\overline{a}\rangle)'$ is of infinite index in $\overline{H}\langle\overline{a}\rangle$. But then $\overline{G}/(\overline{H}\langle\overline{a}\rangle)'$ is a central-by-finite group, a contradiction with Theorem 4.12 of [8]. Hence G' is a $\mathcal{F}$-perfect group. Moreover, $G'/\gamma_m(G')$ is a divisible abelian group by Theorem 2.2 of [7]. Hence $G'' = \gamma_m(G')$. By Theorem 3.29 of [8] $G' = C_{G'}(G'')$ and therefore G' is a divisible abelian group.

Let D be a quasicyclic subgroup of G' such that $D \nsubseteq Z(G)$ and $T = \langle D, a\rangle$. Assume that $T \neq G$. Then $\gamma_k(T)$ is a Černikov subgroup for some positive integer k and $\widehat{T} = T/\gamma_k(T)$ is a nilpotent group. Inasmuch $\widehat{D} \subseteq Z(\widehat{T})$, we deduce that $\widehat{T}$ is an abelian group and therefore $T' = \gamma_k(T)$. But then $T'\langle a\rangle$ is Černikov and so

$$D \leq C_T(T'\langle a\rangle).$$

This yields that T is an abelian group, a contradiction. Thus we conclude that $T = G$ and $\langle D^x \mid x \in G\rangle = G'$. Since $C_G(D)$ is of finite index in G, G' is a Černikov group. This final contradiction proves the lemma. □

Proof of Theorem. Follows from Lemmas 1.1, 1.3 and 1.5.

□

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ГРУПИ, БАГАТІ НІЛЬПОТЕНТНИМИ-НАД-ЧЕРНІКОВСЬКИМИ ПІДГРУПАМИ

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Доведено, що недосконала група є мінімальною не нільпотентною-над-черніковською тоді і тільки тоді, коли вона група типу Хайнекена-Мохамеда.