HARDY TYPE SPACES ON REDUCED HEISENBERG GROUPS

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The Hardy space of complex functions defined on the Schrödinger orbit of reduced $(2d + 1)$-Heisenberg group, generated by the Gauss density function, is investigated. The Cauchy type integral formula is established and radial boundary values for analytic extensions are decribed.

1 Main results

The Hardy type spaces for irreducible regular representations of locally compact groups were introduced in [1]. In this work we concentrate on an important similar case of such spaces, defined by the Schrödinger representation of reduced $(2d+1)$-Heisenberg group $\mathbb{H}_{2d+1}$. To be more precise, the Hardy type space $\mathcal{H}^2_\mu$ consists of complex functions which are defined on the unitary orbit G (under the Schrödinger representation $\mathbb{H}_{2d+1} \ni (x, y, \tau) \longmapsto U_{x,y,\tau}$ over $L^2(\mathbb{R}^d)$) of the Gauss density function $h \in L^2(\mathbb{R}^d)$. At that $\mathcal{H}^2_\mu$ is defined to be the closure in $L^2_\mu(G)$ of all Hilbert-Schmidt polynomials over $L^2(\mathbb{R}^d)$, where μ means an invariant measure on G which

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is uniquely determined by the Haar measure $dx\,dy\,d\tau$ on $\mathbb{H}_{2d+1}$. We establish the Cauchy type formula

$$C[f](\xi) = \int_{\mathbb{H}_{2d+1}} C(\xi, U_{x,y,\tau}h)(f \circ U_{x,y,\tau})(h)\,dx\,dy\,d\tau, \qquad \xi \in B_{L^2(\mathbb{R}^d)}, \quad (1)$$

which for each function $f \in \mathcal{H}^2_\mu$ produces its unique analytic extension $C[f]$ on the open unit ball $B_{L^2(\mathbb{R}^d)}$ in $L^2(\mathbb{R}^d)$. It is proved that for every function $f \in \mathcal{H}^2_\mu$ the radial boundary values of analytic extension $C[f]$ on the orbit G are equal to f in some sense.

2 Reduced $(2d+1)$-Heisenberg group and its Schrödinger representation

Let us consider the reduced Heisenberg group $\mathbb{H}_{2d+1} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ with the multiplication

$$\left(x, y, e^{\mathrm{i}\vartheta}\right)\left(u, v, e^{\mathrm{i}\eta}\right) = \left(x+u, y+v, e^{\mathrm{i}(\vartheta+\eta)} e^{\frac{\mathrm{i}}{2}(x\cdot v - y\cdot u)}\right), \quad x\cdot y = \sum_{j=1}^{d} x_j y_j,$$

for all $x, y, v, u \in \mathbb{R}^d$ and $\vartheta, \eta \in \mathbb{T} := \left\{e^{\mathrm{i}\vartheta} \colon \vartheta \in [0, 2\pi)\right\}$, where $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ and $\mathrm{i} = \sqrt{-1}$. The Haar measure on $\mathbb{H}_{2d+1}$ coincides with the Lebesque measure and has the form $dx\,dy\,d\tau$, where $dx := dx_1 \ldots dx_d$, $dy := dy_1 \ldots dy_d$, $d\tau = d\vartheta/2\pi$ with $\tau = e^{\mathrm{i}\vartheta} \in \mathbb{T}$. We refer to [2] about Heisenberg groups.

In order to define the Schrodinger representation of $\mathbb{H}_{2d+1}$ we need the space $L^2(\mathbb{R}^d)$ of complex functions $\xi \colon \mathbb{R}^d \ni (t_1, \ldots, t_d) = t \longmapsto \xi(t)$ with the scalar product $\langle \xi \mid \zeta \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \xi(t)\bar{\zeta}(t)\,dt$ and the norm $\|\xi\|_{L^2(\mathbb{R}^d)} = \langle \xi \mid \xi \rangle^{1/2}_{L^2(\mathbb{R}^d)}$, where $dt := dt_1 \ldots dt_d$.

The Schrödinger representation U from $\mathbb{H}_{2d+1}$ into $\mathscr{L}\left[L^2(\mathbb{R}^d)\right]$ has the form

$$U_{x,y,\tau} \colon \psi(t_1, \ldots, t_d) \longmapsto \tau e^{\frac{\mathrm{i}}{2} x\cdot y} \psi_1(t_1 + x_1) e^{\mathrm{i} y_1 t_1} \ldots \psi_d(t_d + x_d) e^{\mathrm{i} y_d t_d}$$

for all function $\psi = \psi_1 \otimes \ldots \otimes \psi_d \in L^2(\mathbb{R}^d)$ with $\psi_1, \ldots, \psi_d \in L^2(\mathbb{R})$ and $(t_1, \ldots, t_d)$, $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

In order to continue we need the symmetric Fock space over the space $L^2(\mathbb{R}^d)$. Consider its hilbertian n-th tensor power $\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)$ with the norm $\|\omega\|_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)} = \langle \omega \mid \omega\rangle^{1/2}_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)}$, where

$$\big\langle \xi_1 \otimes \ldots \otimes \xi_n \mid \zeta_1 \otimes \ldots \otimes \zeta_n \big\rangle_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)} = \langle \xi_1 \mid \zeta_1\rangle_{L^2(\mathbb{R}^d)} \cdots \langle \xi_d \mid \zeta_d\rangle_{L^2(\mathbb{R}^d)}$$

denotes the scalar product on $\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)$ defined on the total subset of functions $\omega = \xi_1 \otimes \ldots \otimes \xi_n \in \otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)$ with $\xi_1, \ldots, \xi_n \in L^2(\mathbb{R}^d)$. We denote by $\mathcal{F}_n\left[L^2(\mathbb{R}^d)\right]$ the codomain of the orthogonal projector

$$P_n \colon \otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d) \ni \xi_1 \otimes \ldots \otimes \xi_n \longmapsto \frac{1}{n!} \sum_{\sigma} \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(n)},$$

where σ runs through all n-elements permutations. We denote $\xi^{\otimes n} := P_n(\xi_1 \otimes \ldots \otimes \xi_n)$ if $\xi_1 = \ldots = \xi_n$. Clearly, functions from $\mathcal{F}_n\left[L^2(\mathbb{R}^d)\right]$ are symmetric under the permutation of d-dimensional variables. The symmetric Fock space is defined to be the orthogonal sum

$$\mathcal{F} := \bigoplus_{n\in\mathbb{Z}_+} \mathcal{F}_n\left[L^2(\mathbb{R}^d)\right] = \mathbb{C} \oplus L^2(\mathbb{R}^d) \oplus \mathcal{F}_2\left[L^2(\mathbb{R}^d)\right] \oplus \ldots$$

with the scalar product $\langle \psi \mid \omega\rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \langle \psi_n \mid \omega_n\rangle_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)}$ and the norm $\|\psi\|_{\mathcal{F}} = \langle \psi \mid \psi\rangle^{1/2}_{\mathcal{F}}$ for all $\psi = \sum_{n=0}^{\infty} \psi_n$, $\omega = \sum_{n=0}^{\infty} \omega_n \in \mathcal{F}$ and $\psi_n, \omega_n \in \mathcal{F}_n\left[L^2(\mathbb{R}^d)\right]$.

To construct the orthogonal basis in $\mathcal{F}$ we first consider the Hilbert space $L^2(\mathbb{R})$ of quadratically integrable complex functions of one variable $s \in \mathbb{R}$. In $L^2(\mathbb{R})$ we fix the orthonormal basis

$$\varphi_j(s) = \frac{e^{-s^2/2}}{\sqrt[4]{\pi}} \frac{H_j(s)}{\sqrt{2^j j!}}, \qquad H_j(s) = (-1)^j e^{s^2} \frac{d^j}{ds^j} e^{-s^2}, \qquad s \in \mathbb{R}, \quad j \in \mathbb{Z}_+,$$

where H_j means the Hermitean polynomials. Then the orthonormal basis of $L^2(\mathbb{R}^d)$ forms the system $\left\{\varphi_{j_1} \otimes \ldots \otimes \varphi_{j_d} \colon j_1, \ldots, j_d \in \mathbb{Z}_+\right\}$ (see [3]). Now we consider the d-block indexes subset in $\mathbb{Z}_+^{dn}$ of the form

$$Z_+^{dn} := \Big\{[\alpha] := [(\alpha_1), \ldots, (\alpha_n)] \colon (\alpha_j) \in \mathbb{Z}_+^d,\ j \neq i \Longrightarrow (\alpha_j) \neq (\alpha_i),\ \forall j, i\Big\}$$

with $(\alpha_j) := (\alpha_j^1, \ldots, \alpha_j^d) \in \mathbb{Z}_+^d$ and $j, i = 1, \ldots, n$. In the subspace $\mathcal{F}_n\left[L^2(\mathbb{R}^d)\right]$ the following system forms an orthogonal basis,

$$\Phi_n := \Big\{P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big) \colon (k) := (k_1, \ldots, k_n) \in \mathbb{Z}_+^n,\ |(k)| = n\Big\},$$

where $\varphi_{(\alpha_j)} := \varphi_{\alpha_j^1} \otimes \ldots \otimes \varphi_{\alpha_j^d} \in L^2(\mathbb{R}^d)$, $[\alpha] \in Z_+^{dn}$ and $|(k)| := k_1 + \ldots + k_n$. Clearly, the system

$$\Phi = \Big\{ \Big(0, \ldots, 0, P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big), 0, 0 \ldots \Big) \colon [\alpha] \in Z_+^{dn},\ n \in \mathbb{Z}_+ \Big\}$$

forms an orthogonal basis in the symmetric Fock space $\mathcal{F}$ (see [3]). Remind that

$$\Big\| P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big) \Big\|_{\mathcal{F}}^2 = \frac{k_1! \ldots k_n!}{n!}, \qquad |(k)| = n.$$

Now we consider the Gauss density function $h = h_1 \otimes \ldots \otimes h_d \in L^2(\mathbb{R}^d)$, where every function $h_j(t_j) = \pi^{-1/4} e^{-t_j^2/2}$, $j = 1, \ldots, d$, of the variable $t_j \in \mathbb{R}$ belongs to $L^2(\mathbb{R})$, hence,

$$h \colon \mathbb{R}^d \ni t = (t_1, \ldots, t_d) \longmapsto h(t_1, \ldots, t_d) = \pi^{-d/4} e^{-(t_1^2 + \ldots + t_d^2)/2}.$$

It is easy to see that $\|h\|_{L^2(\mathbb{R}^d)} = 1$, so h belongs to the unit sphere $S_{L^2(\mathbb{R}^d)}$ in $L^2(\mathbb{R}^d)$. Consider its orbit under the Schrödinger representation

$$\begin{aligned} G &:= \big\{ U_{x,y,\tau} h \colon (x, y, \tau) \in \mathbb{H}_{2d+1} \big\} = \\ &= \Big\{ g_{x,y,\tau}(t) := \pi^{-\frac{d}{2}} \tau e^{\frac{\mathrm{i}}{2} x \cdot y} e^{-\frac{(t_1+x_1)^2 + \ldots + (t_d+x_d)^2}{2}} e^{\mathrm{i}(y_1 t_1 + \ldots + y_d t_d)} \Big\}, \end{aligned}$$

which consists of complex functions $g_{x,y,\tau} \colon \mathbb{R}^d \ni t \longmapsto g_{x,y,\tau}(t)$ belonging to the unit sphere in $L^2(\mathbb{R}^d)$ and subsequently means the Gauss orbit.

To define on G a $(\mathbb{H}_{2d+1})$-invariant measure let the closed unit ball $B_{L^2(\mathbb{R}^d)} \cup S_{L^2(\mathbb{R}^d)}$ be endowed with the weak topology of $L^2(\mathbb{R}^d)$, in which it is a compact. Since $\mathbb{H}_{2d+1}$ is a second countable locally compact group, its Gauss orbit G is a Borel subset in this compact. Recall that a Borel measure μ on the orbit G means $(\mathbb{H}_{2d+1})$-invariant if

$$\int_G (f \circ U_{x,y,\tau})\,(g)\, d\mu(g) = \int_G f(g)\, d\mu(g), \quad f \in L^1_\mu(G),\ (x, y, \tau) \in \mathbb{H}_{2d+1}.$$

Theorem 2.1. *On the Gauss orbit G the following equality*

$$\int_G f(g)\, d\mu(g) = \int_{\mathbb{H}_{2d+1}} (f \circ U_{x,y,\tau})(h)\, dx\, dy\, d\tau, \qquad f \in L^1_\mu(G), \tag{2}$$

uniquely defines a $(\mathbb{H}_{2d+1})$-invariant measure μ which has the following decomposition

$$\int_G f(g)\, d\mu(g) = \frac{1}{2\pi} \int_G d\mu(g) \int_0^{2\pi} f(e^{\mathrm{i}\vartheta} g)\, d\vartheta. \tag{3}$$

Proof. First recall (see e.g., [4]) that for any locally compact second countable group $\mathfrak{G}$ with a Haar measure χ and its compact subgroup $\mathfrak{G}_0$ with the Haar measure ς the equality

$$\int_{\mathfrak{G}/\mathfrak{G}_0} d\mu(v) \int_{\mathfrak{G}_0} f(vu)\, d\varsigma(u) = \int_{\mathfrak{G}} f(g)\, d\chi(g), \qquad f \in L^1_\chi(\mathfrak{G})$$

holds. Put $\mathfrak{G} = \mathbb{H}_{2d+1}$. Now let us equip the Gauss orbit G with the weak topology of $L^2(\mathbb{R}^d)$. Then we can identify the Gauss orbit G with the topological factor-space $\mathbb{H}_{2d+1}/\mathfrak{G}_0$, $\mathfrak{G}_0 := \{(x, y, \tau) \in \mathbb{H}_{2d+1} \colon U_{x,y,\tau} h = h\}$ is a stationary subgroup in $\mathbb{H}_{2d+1}$ under the Schrödinger representation. The stationary subgroup $\mathfrak{G}_0$ exactly coincides with the group unit $(0, \dots, 0, 1)$ in $\mathbb{H}_{2d+1}$. Hence, the above equality takes the form (2). The formula (3) is a consequence of (2) and Fubini's theorem (see [5]). □

3 Polynomial orthogonal systems on orbit

For any element $\psi_n \in \mathcal{F}_n \left[L^2(\mathbb{R}^d)\right]$ uniquely assists the Hermitean form $\psi_n^* := \langle \cdot \mid \psi_n \rangle_{\otimes^n_{\mathfrak{h}} L^2(\mathbb{R}^d)}$ which belongs to the Hermitean dual $\mathcal{F}_n^* \left[L^2(\mathbb{R}^d)\right]$. We can identify this form with the n-homogeneous Hilbert-Schmidt polynomial $\psi_n^* \colon L^2(\mathbb{R}^d) \ni \xi \to \psi_n^*(\xi) := \langle \xi^{\otimes n} \mid \psi_n \rangle_{\otimes^n_{\mathfrak{h}} L^2(\mathbb{R}^d)}$. Now for each ψ_n^* with $\psi_n \in \mathcal{F}_n \left[L^2(\mathbb{R}^d)\right]$ we assign the complex function

$$h_n(\psi_n) \colon G \ni g \longmapsto \left\langle g^{\otimes n} \mid \psi_n \right\rangle_{\otimes^n_{\mathfrak{h}} L^2(\mathbb{R}^d)}$$

of the variable $g = U_{x,y,\tau} h$ with $(x, y, \tau) \in \mathbb{H}_{2d+1}$ belonging to the Gauss orbit G and the mapping $h_n \colon \mathcal{F}_n \left[L^2(\mathbb{R}^d)\right] \ni \psi_n \longmapsto h_n(\psi_n) \in L^2_\mu(G)$. The following axillary statements show that the mapping h_n is well defined.

Lemma 3.1. *For any $n \in \mathbb{N}$ and $(k) \in \mathbb{Z}^n_+$ such that $|(k)| = n$, and any $[(\alpha_1), \dots, (\alpha_n)] \in Z^{dn}_+$ the inequality*

$$\int_{\mathbb{H}_{2d+1}} \left| \left\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \mid P_n\left(\varphi^{\otimes k_1}_{(\alpha_1)} \otimes \ldots \otimes \varphi^{\otimes k_n}_{(\alpha_n)}\right) \right\rangle_{\mathcal{F}} \right|^2 dx\, dy\, d\tau \le \left(\frac{2\pi}{n}\right)^d$$

holds, which transforms into the equality for $(\alpha_1) = (0, \dots, 0) \in \mathbb{Z}^d_+$ and $(k) = (n, 0, \dots, 0)$.

Proof. Let us use the following equality $\prod_{j=1}^{n}\left\langle U_{x,y,\tau}h \mid \varphi_{(\alpha_j)}\right\rangle_{L^2(\mathbb{R}^d)}^{k_j} = \left\langle (U_{x,y,\tau}h)^{\otimes n} \mid P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1}\otimes\ldots\otimes\varphi_{(\alpha_n)}^{\otimes k_n}\big)\right\rangle_{\mathcal{F}}$. Since

$$\left\langle U_{x,y,\tau}h \mid \varphi_{(j)}\right\rangle_{L^2_{\mathbb{R}^d}} = \tau e^{\frac{i}{2}x\cdot y}\pi^{\frac{d}{2}}\prod_{l=1}^{d}\int_{\mathbb{R}} e^{iy_lt_l}e^{-\frac{(t_l+x_l)^2}{2}}\varphi_{j_l}(t_l)dt_l =$$
$$= \tau e^{\frac{i}{2}x\cdot y}\prod_{l=1}^{d}\frac{(-1)^{j_l}(x_l-iy_l)^{j_l}}{\sqrt{2^{j_l}\ j_l!}}e^{-(x_l^2+2ix_ly_l+y_l^2)/4},$$

we have the sequence of equalities

$$\left|\left\langle \big(U_{x,y,\tau}h\big)^{\otimes n} \mid P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1}\otimes\ldots\otimes\varphi_{(\alpha_n)}^{\otimes k_n}\big)\right\rangle_{\mathcal{F}}\right|^2 =$$
$$=\left(\prod_{l=1}^{d}\frac{e^{-\frac{x_l^2+y_l^2}{2}}(x_l^2+y_l^2)^{\alpha_1^l}}{2^{\alpha_1^l}\alpha_1^l!}\right)^{k_1}\cdots\left(\prod_{l=1}^{d}\frac{e^{-\frac{x_l^2+y_l^2}{2}}(x_l^2+y_l^2)^{\alpha_n^l}}{2^{\alpha_n^l}\alpha_n^l!}\right)^{k_n} =$$
$$= e^{-\frac{n(x_1^2+y_1^2)}{2}}\prod_{m=1}^{n}\left(\frac{(x_1^2+y_1^2)^{\alpha_m^1}}{2^{\alpha_m^1}\alpha_m^1!}\right)^{k_m}\cdots e^{-\frac{n(x_d^2+y_d^2)}{2}}\prod_{m=1}^{n}\left(\frac{(x_d^2+y_d^2)^{\alpha_m^d}}{2^{\alpha_m^d}\alpha_m^d!}\right)^{k_m}.$$

Now using the facts that

$$\int_0^{+\infty}e^{-nq}\prod_{l=1}^{n}\Big(\frac{q^{j_l}}{j_l!}\Big)^{k_l}dq = \prod_{l=1}^{n}\frac{m!}{(j_l!)^{k_l}}\int_0^{+\infty}e^{-nq}\frac{q^m}{m!}\,dq =$$
$$=\prod_{l=1}^{n}\frac{m!}{(j_l!)^{k_l}}\frac{1}{n^m}\int_0^{+\infty}e^{-nq}\frac{(qn)^m}{m!}\,dq \le \frac{1}{n}$$

with $m=\sum_{l=1}^{n}j_lk_l$ and that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f\Big(\frac{p^2+s^2}{2}\Big)\,dp\,ds = 4\int_0^{+\infty}\int_0^{\pi/2}f(q)dqd\vartheta = 2\pi\int_0^{+\infty}f(q)\,dq$$

with $p^2=2q\cdot\cos^2\vartheta$ and $s^2=2q\cdot\sin^2\vartheta$, we finally obtain

$$\int_{\mathbb{H}_{2d+1}}\left|\left\langle \big(U_{x,y,\tau}h\big)^{\otimes n} \mid P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1}\otimes\ldots\otimes\varphi_{(\alpha_n)}^{\otimes k_n}\big)\right\rangle_{\mathcal{F}}\right|^2 dx\,dy\,d\tau =$$

$$= \int_{\mathbb{H}_{2d+1}} \prod_{j=1}^{d} e^{-\frac{n(x_j^2+y_j^2)}{2}} \prod_{m=1}^{n} \left(\frac{(x_j^2+y_j^2)^{\alpha_m^j}}{2^{\alpha_m^j} \alpha_m^j!} \right)^{k_m} dx\, dy\, d\tau \leq \left(\frac{2\pi}{n}\right)^d.$$

If $(\alpha_1) = (0, \ldots, 0) \in \mathbb{Z}_+^d$ and $(k) = (n, 0, \ldots, 0)$ then the above inequality transforms to the equality. □

The next statement gives an estimation for any $\psi_n^* \in \mathcal{F}_n^* \left[L^2(\mathbb{R}^d)\right]$.

Lemma 3.2. *For any $\psi_n \in \mathcal{F}_n \left[L^2(\mathbb{R}^d)\right]$ the following inequality holds*

$$\int_{\mathbb{H}_{2d+1}} \left| \left\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \mid \psi_n \right\rangle_{\mathcal{F}} \right|^2 dx\, dy\, d\tau \leq n! \left(\frac{2\pi}{n}\right)^d \|\psi_n\|^2_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)}.$$

Proof. Since $\Big\{P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big) \colon (k_1, \ldots, k_n) \in \mathbb{Z}_+^n, |(k)| = n, [(\alpha_1), \ldots, (\alpha_n)] \in Z_+^{dn}\Big\}$ forms the orthogonal basis in $\mathcal{F}_n \left[L^2(\mathbb{R}^d)\right]$, we can consider the Fourier decomposition of ψ_n:

$$\psi_n = \sum_{\alpha \in Z_+^{dn},\ |(k)|=n} \beta_{\alpha,k} P_n\Big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\Big) \sqrt{\frac{n!}{k_1! \ldots k_n!}}$$

with $\|\psi_n\|^2_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)} = \sum |\beta_{\alpha,k}|^2$, where $\alpha = [(\alpha_1), \ldots, (\alpha_n)]$ and $(k) = (k_1, \ldots, k_n)$. It follows that

$$\int_{\mathbb{H}_{2d+1}} \left| \left\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \mid \psi_n \right\rangle_{\mathcal{F}} \right|^2 dx\, dy\, d\tau \leq$$
$$\leq n! \int_{\mathbb{H}_{2d+1}} \bigg(\sum |\beta_{\alpha,k}| \Big| \Big\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \Big| P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big) \Big\rangle_{\mathcal{F}} \Big| \bigg)^2 dx\, dy\, d\tau =$$
$$= n! \sum_{\alpha,k,i,m} |\beta_{\alpha,k}| |\beta_{i,m}| \int_{\mathbb{H}_{2d+1}} \Big| \Big\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \Big| P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big) \Big\rangle_{\mathcal{F}} \Big| \times$$
$$\times \Big| \Big\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \mid P_n\big(\varphi_{(i_1)}^{\otimes m_1} \otimes \ldots \otimes \varphi_{(i_n)}^{\otimes m_n}\big) \Big\rangle_{\mathcal{F}} \Big| dx\, dy\, d\tau.$$

Using the Cauchy-Schwartz inequality for the integral we get that

$$\int_{\mathbb{H}_{2d+1}} \Big| \Big\langle \left(U_{x,y,\tau} h\right)^{\otimes n} \mid P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1} \otimes \ldots \otimes \varphi_{(\alpha_n)}^{\otimes k_n}\big) \Big\rangle_{\mathcal{F}} \Big| \times$$

$$\times\left|\Big\langle\langle\big(U_{x,y,\tau}h\big)^{\otimes n}\mid P_n\big(\varphi_{(i_1)}^{\otimes m_1}\otimes\ldots\otimes\varphi_{(i_n)}^{\otimes m_n}\big)\Big\rangle_{\mathcal{F}}\right|dx\,dy\,d\tau\leq$$
$$\leq\left(\int_{\mathbb{H}_{2d+1}}\left|\Big\langle\big(U_{x,y,\tau}h\big)^{\otimes n}\mid P_n\big(\varphi_{(\alpha_1)}^{\otimes k_1}\otimes\ldots\otimes\varphi_{(\alpha_n)}^{\otimes k_n}\big)\Big\rangle_{\mathcal{F}}\right|^2dx\,dy\,d\tau\right)^{1/2}\times$$
$$\times\left(\int_{\mathbb{H}_{2d+1}}\left|\Big\langle\big(U_{x,y,\tau}h\big)^{\otimes n}\mid P_n\big(\varphi_{(i_1)}^{\otimes m_1}\otimes\ldots\otimes\varphi_{(i_n)}^{\otimes m_n}\big)\Big\rangle_{\mathcal{F}}\right|^2dx\,dy\,d\tau\right)^{1/2}\leq$$
$$\leq\left(\frac{2\pi}{n}\right)^d.$$

Finally, using the Cauchy-Schwartz inequality one more time, i.e.

$$\sum_{\alpha,k,i,m}|\beta_{\alpha,k}||\beta_{i,m}|\leq\Big(\sum_{\alpha,k}|\beta_{\alpha,k}|^2\Big)^{1/2}\Big(\sum_{i,m}|\beta_{i,m}|^2\Big)^{1/2}=\|\psi_n\|^2_{\otimes_{\mathfrak{h}}^n L^2(\mathbb{R}^d)},$$

we achieve the required inequality. □

Consider the following closed subspaces and their hilbertian orthogonal sum $F_n:=\mathcal{F}_n\left[L^2(\mathbb{R}^d)\right]\ominus\ker h_n$, $F:=\mathbb{C}\oplus F_1\oplus F_2\oplus\ldots$, where $\ker h_n$ means the kernel of h_n. Now let us introduce the denotations $\widetilde{h}_n:=h_n/\|h_n\|$ and $\widetilde{\psi}_n:=\widetilde{h}_n(\psi_n)$ and consider the corresponding linear mapping

$$\widetilde{h}\colon F\ni\psi=\sum_{n\in\mathbb{Z}_+}\psi_n\longrightarrow\widetilde{\psi}:=\sum_{n\in\mathbb{Z}_+}\widetilde{\psi}_n.$$

Let $\mathcal{H}_n^2:=\widetilde{h}_n(F_n)$ and $\mathcal{H}_\mu^2:=\widetilde{h}(F)$ mean codomains in $L^2_\mu(G)$ of the mapping $\widetilde{h}_n$ and $\widetilde{h}$, respectively.

Theorem 3.1. *The mappings $\widetilde{h}$ and $\widetilde{h}_n$ have the following properties:*
(i) $\widetilde{h}_n$ *is an isometry between F_n and its codomain $\mathcal{H}_n^2$.*
(ii) $\widetilde{h}$ *is an isometry between F and $\mathcal{H}_\mu^2$.*
(iii) *the orthogonal decomposition $\mathcal{H}_\mu^2=\mathbb{C}\oplus\mathcal{H}_1^2\oplus\mathcal{H}_2^2\oplus\mathcal{H}_3^2\oplus\ldots$ holds.*

Proof. Lemma 3.2 implies that the operator h_n is bounded. It follows that

$$\int_G h_n(\psi_n)\overline{h_n(\omega_n)}\,d\mu=\int_G(\psi_n^*\circ U_{x,y,\tau})(h)\overline{(\omega_n^*\circ U_{x,y,\tau})(h)}\,d\mu(U_{x,y,\tau}h)$$

is an Hermitean continuous form on F_n, which is linear by ω_n and antilinear by ψ_n. So, there exists a bounded operator $A_n\in\mathscr{L}(F_n)$ for which $\langle\omega_n \mid A_n\psi_n\rangle_{\mathcal{F}} = \int_G h_n(\psi_n)\overline{h_n(\omega_n)}\,d\mu$. Using the same technique as in [1] we show that A_n commutates with the diagonal nth tensor power of Schrödinger's representation $\big\{U^{\otimes n}_{\tilde{x},\tilde{y},\tilde{\tau}} = U_{\tilde{x},\tilde{y},\tilde{\tau}}\otimes\ldots\otimes U_{\tilde{x},\tilde{y},\tilde{\tau}}\colon(\tilde{x},\tilde{y},\tilde{\tau})\in\mathbb{H}_{2d+1}\big\}$. Applying the $(\mathbb{H}_{2d+1})$-invariancy of the measure μ on the Gauss orbit G we obtain

$$\begin{aligned}
&\langle\omega_n \mid \big(A_n\circ U_{\tilde{x},\tilde{y},\tilde{\tau}}\big)\psi_n\rangle_{\mathcal{F}} = \\
&= \int_G \langle(U_{x,y,\tau}h)^{\otimes n} \mid U^{\otimes n}_{\tilde{x},\tilde{y},\tilde{\tau}}\psi_n\rangle_{\mathcal{F}}\overline{\langle(U_{x,y,\tau}h)^{\otimes n} \mid \omega_n\rangle_{\mathcal{F}}}\,d\mu(U_{x,y,\tau}h) = \\
&= \int_G \Big\langle(U_{x,y,\tau}h)^{\otimes n} \mid \psi_n\Big\rangle_{\mathcal{F}}\overline{\Big\langle(U_{x,y,\tau}h)^{\otimes n} \mid U^{\otimes n}_{(-\tilde{x},-\tilde{y},\tilde{\tau}^{-1})}\omega_n\Big\rangle_{\mathcal{F}}}d\mu(U_{x,y,\tau}h) = \\
&= \Big\langle\omega_n \mid (U^{\otimes n}_{\tilde{x},\tilde{y},\tilde{\tau}}\circ A_n)\psi_n\Big\rangle_{\mathcal{F}}.
\end{aligned}$$

Since for any $n\in\mathbb{N}$ the set $\big\{(U_{x,y,\tau}h)^{\otimes n} : (x,y,\tau)\in\mathbb{H}_{2d+1}\big\}$ is total in F_n due to its definition, the representations $U^{\otimes n}_{\tilde{x},\tilde{y},\tilde{\tau}}$ are irreducible over F_n. Via to the well-known property [6, Theorem 21.30] the operator A_n is proportional to the identity operator 1_{F_n} on F_n i.e., $A_n\mid_{F_n} = \aleph^{-2}1_{F_n}$ for some $\aleph^2\in\mathbb{C}$. Hence, we have

$$\langle\omega_n \mid \psi_n\rangle_{\mathcal{F}} = \aleph^2\int_G h_n(\psi_n)\overline{h_n(\omega_n)}d\mu,\ \|h_n\| = \sup_{\|\psi_n\|_{\mathcal{F}=1}}\|h_n(\psi_n)\|_{L^2_\mu} = \frac{1}{\aleph_n}. \quad (4)$$

Finally, applying Theorem 2.1 for all $\psi_n\in F_n$ and $\omega_m\in F_m$ we get

$$\begin{aligned}
\textstyle\int_G h_n(\psi_n)\overline{h_m(\omega_m)}\,d\mu &= \frac{1}{2\pi}\int_G h_n(\psi_n)\overline{h_m(\omega_m)}\,d\mu\int_0^{2\pi}e^{i(n-m)\vartheta}\,d\vartheta = \\
&= \begin{cases} 0 & : n\neq m \\ \langle\omega_n \mid \psi_n\rangle_{\mathcal{F}} & : n=m. \end{cases}
\end{aligned}$$

Hence $\widetilde{h}_n(\psi_n)\perp\widetilde{h}_m(\omega_m)$ if $n\neq m$ and the orthogonal decomposition (iii) holds. □

4 Cauchy type formula for Gauss orbit

Note that the lemmas directly imply the estimation $\|h_n\|\leq\sqrt{n!\left(\frac{2\pi}{n}\right)^d}$ and the equality

$$\big\|h_n(h^{\otimes n})\big\|^2_{L^2_\mu(G)} = \int_{\mathbb{H}_{2d+1}}\big|\big\langle(U_{x,y,\tau}h)^{\otimes n} \mid h^{\otimes n}\big\rangle_{\mathcal{F}}\big|^2 dx\,dy\,d\tau = \left(\frac{2\pi}{n}\right)^d.$$

Though finding the exact value of $\|h_n\|$ is not an easy task we can give another estimation for $\|h_n\|$ which will be useful for $\aleph_n$. It easy to see that $h^{\otimes n} \in F_n$ and $\|h^{\otimes n}\|_{\mathcal{F}} = 1$. It follows that the following estimation holds

$$\|h_n\| = \sup_{\|\psi_n\|_{\mathcal{F}}=1} \big\|h_n(\psi_n)\big\|_{L^2_\mu(G)} \geq \big\|h_n(h^{\otimes n})\big\|_{L^2_\mu(G)} = \Big(\frac{2\pi}{n}\Big)^{d/2}.$$

From $\left(\frac{2\pi}{n}\right)^{d/2} \leq \|h_n\| \leq (n!)^{1/2}\left(\frac{2\pi}{n}\right)^{d/2}$ it follows that $\sqrt{\frac{1}{n!}\left(\frac{n}{2\pi}\right)^d} \leq \aleph_n \leq \sqrt{\left(\frac{n}{2\pi}\right)^d}$. The fact that $\lim\limits_{n\to\infty} \sqrt[n]{\aleph_n^2} \leq \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{n}{2\pi}\right)^d} = 1$ justifies that we can mean

$$\begin{aligned} C\big(\xi, U_{x,y,\tau}h\big) &= \sum_{n\in\mathbb{Z}_+} \aleph_n^2 \big\langle \xi \mid U_{x,y,\tau}h\big\rangle^n_{L^2(\mathbb{R}^d)} = \sum_{n\in\mathbb{Z}_+} \aleph_n^2 \big\langle \xi^{\otimes n} \mid (U_{x,y,\tau}h)^{\otimes n}\big\rangle_{\mathcal{F}} = \\ &= 1 + \sum_{n\in\mathbb{N}} \aleph_n^2 \left(\frac{\tau e^{\frac{\mathrm{i}}{2}x\cdot y}}{\pi^{d/4}} \prod_{l=1}^d \int_{\mathbb{R}} \xi_l(t_l) e^{\mathrm{i}y_l t_l - (t_l - x_l)^2/2}\, dt_l \right)^n \end{aligned} \tag{5}$$

with $\xi \in B_{L^2(\mathbb{R}^d)}$ and $(x,y,\tau) \in \mathbb{H}_{2d+1}$, as a generalization of the Cauchy kernel. Since $U_{x,y,\tau}h \in S_{L^2(\mathbb{R}^d)}$ for all $(x,y,\tau) \in \mathbb{H}_{2d+1}$ and above power series is convergent for all $\|\xi\|_{L^2(\mathbb{R}^d)} < 1$, the kernel $C(\xi,\cdot)$ is an analytic $L^\infty(\mathbb{H}_{2d+1})$-valued function by the variable $\xi \in B_{L^2(\mathbb{R}^d)}$ (see [5]).

Theorem 4.1. *The integral operator*

$$C[f](\xi) = \int_{\mathbb{H}_{2d+1}} C(\xi, U_{x,y,\tau}h)(f \circ U_{x,y,\tau})(h)\, dx\, dy\, d\tau, \quad f \in \mathcal{H}^2_\mu,\ \xi \in B_{L^2(\mathbb{R}^d)},$$

belongs to $\mathscr{L}(\mathcal{H}^2_\mu)$. *The function* $C_r[f]\colon G \ni \xi \longmapsto C[f](r\xi)$ *belongs to* $\mathcal{H}^2_\mu$ *and*

$$\|f\|_{L^2_\mu(G)} = \sup_{r\in[0;1)} \left(\int_G |C[f](r\xi)|^2 d\mu(\xi) \right)^{1/2}.$$

For any $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2_\mu$ *with* $f_n \in \mathcal{H}^2_\mu$ *the integral transform* $C[f]$ *is a unique analytic extension of* f *on the open ball* $B_{L^2(\mathbb{R}^d)}$ *for which its radial boundary values on* G *are equal to* f *in the following sense*

$$\lim_{r\to 1} \int_G \big|C_r[f] - f\big|^2 d\mu = 0, \qquad r \in [0,1).$$

Proof. Use the short notation $\varphi^{\otimes(k)}_{[\alpha]} := P_n\big(\varphi^{\otimes k_1}_{(\alpha_1)}\otimes\ldots\otimes\varphi^{\otimes k_n}_{(\alpha_n)}\big)$ with $|(k)| = n$. All such elements $\varphi^{\otimes(k)}_{[\alpha]}$ have been previously identified with Φ_n. For $\varphi^{\otimes(k)}_{[\alpha]} \in \Phi_n$ we denote $\widetilde{\varphi}^{(k)}_{[\alpha]} := \widetilde{h}_n\big(\varphi^{\otimes(k)}_{[\alpha]}\big)$. Substituting elements $\omega_n = \varphi^{\otimes(k)}_{[\alpha]}$ and $\psi_n = \varphi^{\otimes(k')}_{[\alpha']}$ from Φ_n with different indexes in the equality (4) we get

$$\int_G \widetilde{\varphi}^{(k)}_{[\alpha]}\overline{\widetilde{\varphi}^{(k')}_{[\alpha']}}\,d\mu = \big\langle \varphi^{\otimes(k)}_{[\alpha]} \mid \varphi^{\otimes(k')}_{[\alpha']}\big\rangle_{\mathcal{F}} = 0.$$

So, the system $\widetilde{\varphi}^{(k)}_{[\alpha]}$ with all $\varphi^{\otimes(k)}_{[\alpha]} \in \Phi_n$ forms an orthonormal basis in $\mathcal{H}^2_n$. We can write the Fourier expansion $\xi^{\otimes n} = \sum_{\varphi^{\otimes(k)}_{[\alpha]}\in\Phi_n} \big\langle \xi^{\otimes n} \mid \varphi^{\otimes(k)}_{[\alpha]}\big\rangle_{\mathcal{F}}\varphi^{\otimes(k)}_{[\alpha]}$ for any element $\xi^{\otimes n} \in \mathcal{F}_n\big[L^2(\mathbb{R}^d)\big]$. Using this we have

$$C_n\big(\xi, U_{x,y,\tau}h\big) := \aleph^2_n\langle \xi^{\otimes n} \mid (U_{x,y,\tau}h)^{\otimes n}\rangle_{\mathcal{F}} = r^n \sum_{\varphi^{\otimes(k)}_{[\alpha]}\in\Phi_n} \widetilde{\varphi}^{(k)}_{[\alpha]}(\xi/r)\overline{\widetilde{\varphi}^{(k)}_{[\alpha]}}\big(U_{x,y,\tau}h\big),$$

where $r = \|\xi\|_{L^2(\mathbb{R}^d)}$. It follows that

$$\begin{aligned} C\big(\xi, U_{x,y,\tau}h\big) &= \sum_{n\in\mathbb{Z}_+} r^n \sum_{\varphi^{\otimes(k)}_{[\alpha]}\in\Phi_n} \widetilde{\varphi}^{(k)}_{[\alpha]}(\xi/r)\overline{\widetilde{\varphi}^{(k)}_{[\alpha]}}\big(U_{x,y,\tau}h\big) = \\ &= \sum_{n\in\mathbb{Z}_+} r^n C_n\big(\xi/r, U_{x,y,\tau}h\big). \end{aligned}$$

Now Theorem 3.1 implies that

$$\int_G \widetilde{\varphi}^{(k)}_{[\alpha]}\big(U_{x,y,\tau}h\big)C_n\big(\xi/r, U_{x,y,\tau}h\big)\,d\mu\big(U_{x,y,\tau}h\big) = \widetilde{\varphi}^{(k)}_{[\alpha]}\big(\xi/r\big).$$

Since $\widetilde{\varphi}^{(k)}_{[\alpha]}$ with all $\varphi^{\otimes(k)}_{[\alpha]} \in \Phi_n$ form an orthonormal basis in $\mathcal{H}^2_n$, the integral operator with kernel C_n produces the identity mapping over $\mathcal{H}^2_n$.

Let $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2_\mu$ with $f_n \in \mathcal{H}^2_n$. Using that $f_n \perp C_m$ if $n \neq m$ in $L^2_\mu(G)$ we obtain

$$\begin{aligned} f(\xi) &= \sum_{n\in\mathbb{Z}_+} \int_G C_n\big(\xi, U_{x,y,\tau}h\big) f_n\big(U_{x,y,\tau}h\big)\,d\mu\big(U_{x,y,\tau}h\big) = \\ &= \int_G C\big(\xi, U_{x,y,\tau}h\big) f\big(U_{x,y,\tau}h\big)\,d\mu\big(U_{x,y,\tau}h\big) \end{aligned}$$

for all $\xi \in G$. It follows that the series $C[f](\xi) = \sum_{n\in\mathbb{Z}_+} C[f_n](\xi)$ with

$$C[f_n](\xi) = \int_G C_n\big(\xi, U_{x,y,\tau}h\big) f_n\big(U_{x,y,\tau}h\big)\, d\mu\big(U_{x,y,\tau}h\big) = \\ = r^n \int_G C_n\big(\xi/r, U_{x,y,\tau}h\big) f_n\big(U_{x,y,\tau}h\big)\, d\mu\big(U_{x,y,\tau}h\big) = r^n f_n\big(\xi/r\big) = f_n(\xi)$$

is convergent in $\mathcal{H}^2_\mu$ by the variable $\xi/r \in G$, uniformly by $r \in [0,\varepsilon]$ with $0 < \varepsilon < 1$. Since $C_m \perp f_n$ and $f_m \perp f_n$ if $n \neq m$ in $L^2_\mu(G)$, we have

$$\big\|C_r[f]\big\|^2_{L^2_\mu(G)} = \int_G \Big| \sum_{n\in\mathbb{Z}_+} r^n \int_G C_n\big(\xi, U_{x,y,\tau}h\big) f_n\big(U_{x,y,\tau}h\big)\, d\mu\big(U_{x,y,\tau}h\big)\Big|^2 d\mu(\xi) = \\ = \int_G \Big| \sum_{n\in\mathbb{Z}_+} r^n f_n(\xi)\Big|^2 d\mu(\xi) = \Big\| \sum_{n\in\mathbb{Z}_+} r^n f_n \Big\|^2_{L^2_\mu(G)} = \sum_{n\in\mathbb{Z}_+} r^{2n} \|f_n\|^2_{L^2_\mu(G)}$$

for all $r < 1$. It follows that

$$\sup_{r\in[0.1)} \int_G \big|C[f](r\xi)\big|^2 d\mu(\xi) = \sup_{r\in[0,1)} \sum_{\mathbb{Z}_+} r^{2n} \|f_n\|^2_{L^2_\mu(G)} = \|f\|^2_{L^2_\mu(G)}.$$

We can apply the Cauchy-Schwarz inequality which implies

$$\big\|C_r[f]\big\|^2_{L^2_\mu(G)} \leq \frac{1}{\sqrt{1-r^2}} \Big(\sum_{n\in\mathbb{Z}_+} \|f_n\|^2_{L^2_\mu(G)} \Big)^{1/2} = \frac{\|f_n\|_{L^2_\mu(G)}}{\sqrt{1-r^2}}$$

for all $f \in \mathcal{H}^2_\mu$. Therefore the operator $C[f]$ belongs to $\mathscr{L}(\mathcal{H}^2_\mu)$.

Now we will use that $C(\xi,\cdot)$ is an analytic $L^\infty(\mathbb{H}_{2d+1})$-valued function by $\xi \in B_{L^2(\mathbb{R}^d)}$. Then in view of [7, Theorem 3.1.2] the function $C[f]$ is also analytic by $\xi \in B_{L^2(\mathbb{R}^d)}$. Applying the orthogonal property once again, we have

$$\int_G \big|C_r[f] - f\big|^2 d\mu = \sum_{n\in\mathbb{Z}_+} (r^{2n} - 1) \|f_n\|^2_{L^2_\mu(G)} \to 0$$

if $r \to 1$. Thus, the theorem is completely proved. □

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ПРОСТОРИ ХАРДІ НА ЗВЕДЕНИХ ГРУПАХ ГЕЙЗЕНБЕРГА

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Розглядається простір Харді комплексних функцій, визначених на орбіті Шредінгера зведеної $(2d+1)$-вимірної групи Гейзенберга, породженої функцією Гаусса. Наведена інтегральна формула типу Коші та доведено існування граничних значень для аналітичних продовжень.