L-IDEMPOTENT LINEAR OPERATORS BETWEEN PREDICATE SEMIMODULES, DUAL PAIRS AND CONJUGATE OPERATORS

©*2011 Oleh NYKYFORCHYN, Oksana MYKYTSEY*

Vasyl Stefanyk Precarpathian National University,
57 Shevchenka Str., Ivano-Frankivsk 76018, Ukraine

e-mail: *oleh.nyk@gmail.com, oksana39@if.ua*

Received 21 August 2011

It is shown that the sets of monotonic predicates on domains with values in a completely distributive quantale are free completely distributive idempotent semimodules over these domains. Idempotent dual pairs and conjugate operators are also constructed.

Introduction

Domain theory is a branch of mathematics which was founded by Dana Scott to apply methods of order theory, topology, logic to computer science, first of all to denotational semantics of lambda caclulus. Its key idea is to represent "partial" or "incomplete" information on the state of a system or on the result of computations as an element of partially ordered set, in which the elements are ordered by increasing of precision or specialization. On mathematical aspects of domain theory, see the perfect book "Continuous Lattices and Domains" [1], which is a successor to the famous "Compendium on Continuous Lattices".

УДК: 515.12; MSC 2000: 03B52, 06B35, 68T37
Key words and phrases: monotone predicate, Scott continuity, polarity, conjugate linear operator, idempotent semimodule

This theory is naturally linked with fuzzy sets and fuzzy predicates [2], which are in fact mappings from/to domains, although in most publications the considered scale is a subset of the set of reals. The aim of the present paper is to uncover some relations between lattice-valued monotonic predicates on domains, idempotent linear algebra and idempotent functional analysis.

1 Semimodules of monotonic predicates

Throughout this paper, if f, g are functions with a common domain, α is a constant, and $*$ is a binary operation, then we denote by $f * g$, $\alpha * f$ and $f * \alpha$ the functions with the same domain obtained by pointwise application of the operation $*$ (provided it is defined for the corresponding values). In the sequel $\sup_p$ and $\inf_p$ for a family of functions with a common domain to a poset will denote the pointwise suprema and infima, respectively. For a subset A of a poset X, we denote by $A\uparrow$ the subset $\{x \in X \mid a \leqslant x \text{ for some } a \in A\}$. The least and the greatest elements of a poset (if they exist) are commonly denoted by 0 and 1, respectively.

See [1] for basic definitions and facts on partially ordered sets, including continuous semilattices and lattices. Here we shall recall only few definitions. A poset is *directed complete* (*dcpo* for short) if it has suprema for all directed non-empty sets. An element a *approximates* b or is *way below* b in a poset X, which is written as $a \ll b$, if, for each directed subset $C \subset X$ such that $b \leqslant \sup C$, there is $c \in C$ such that $a \leqslant c$. If such is valid for all (not necessarily directed) subsets $C \subset X$, then a is said to be *way-way below* b, written $a \lll b$. A poset X is called *continuous* if, for each $b \in X$, the set of all $a \ll b$ is directed and has b as its lowest upper bound. A directed complete continuous poset is called a *domain.* A *continuous semilattice* is a domain that has pairwise meets. The *Scott topology* on a poset X is the least topology such that all lower sets C that are closed under directed suprema are closed. A mapping between *dcpo*s is Scott continuous, i.e. continuous w.r.t. the Scott topology on the both sets, if and only if it is isotone and preserves all suprema of directed sets. The *lower topology* on X is the least topology such that the sets $\{a\}\uparrow$ are closed for all $a \in X$. The join, i.e. the least topology that contains the Scott and the lower topologies, is called the *Lawson topology.*

We shall also use basic notions of denotational semantics of programming languages. Consider a system or a state of a computational process. All possible (probably incomplete) portions of information we can have about it form a *domain of computation* D [3]. This set carries a partial order $\leqslant$ which represents a hierarchy of information or knowledge: the more information contains an element (i.e. the more specific/restrictive it is), the higher it is. It is also often required, that there is a least element $0 \in D$ (no information at all), and for all a and b in D there is a meet $a \wedge b$, which, e.g. can be (but not necessarily is) treated as "a or b is true". See the latter reference for more details, in particular for explanation why it is natural to require that D is a continuous meet-semilattice with a least element.

In the sequel L will be a completely distributive lattice [1]. By a result of Raney [4], a complete lattice is completely distributive if and only if each element is the supremum of all elements way-way below it. This is equivalent to L being a compact Hausdorff distributive Lawson lattice with some topology (which in this case coincides with the Lawson topology) [1, Proposition VII-2.8]. A topological lattice is said to be Lawson if at each point it possesses a local base consisting of sublattices. Then the same is true for L^{op}. We denote by 0, 1, $\oplus$, and $\otimes$ the bottom element, the top element, the join, and the meet in L, respectively. The elements of this (arbitrary, but fixed throughout the paper) lattice will be used to express truth values. The operation $\oplus$ is the disjunction, but the conjuction does not necessarily coincide with $\otimes$.

Following [5], for a semilattice D we call the elements of the set $[D \to L^{op}]^{op}$ *L-fuzzy monotonic predicates* on D (here $[A \to B]$ stands for the set of all Scott continuous mappings from A to B). For $m \in [D \to L^{op}]^{op}$ and $a \in D$, we regard $m(a)$ as the truth value of a, hence it is required that $m(b) \leqslant m(a)$ for all $a \leqslant b$. The second ^{op} means that we order the fuzzy predicates pointwisely, i.e. $m_1 \leqslant m_2$ iff $m_1(a) \leqslant m_2(a)$ in L (not in L^{op} !) for all $a \in D$. We denote $\underline{M}_{[L]}D = [D \to L^{op}]^{op}$, and, for D with a least element 0, consider also the subset $M_{[L]}D \subset \underline{M}_{[L]}D$ of all *normalized* predicates that take $0 \in D$ (no information) to $1 \in L$ (complete truth). Observe that $M_{[L]}D$ is a complete sublattice of $\subset \underline{M}_{[L]}D$.

It follows from [6, Theorem 4] (although called "folklore knowledge" in [5]) that, for a domain D and a completely distributive lattice L, the set

$[D \to L^{op}]$ is a completely distributive lattice, hence this is also valid for $\underline{M}_{[L]}D$. If D possesses a least element, then $M_{[L]}D$ is a completely distributive lattice as well.

For an element $d_0 \in D$, we denote by $\eta_{[L]}D(d_0)$ the function $D \to L$ that sends each $d \in D$ to 1 if $d \leqslant d_0$ and to 0 otherwise. It is easy to see that $\eta_{[L]}D(d_0) \in M_{[L]}D \subset \underline{M}_{[L]}D$, and $\delta_L^D = \eta_{[L]}D(0)$ is a least element of $M_{[L]}D$.

Lemma 1.1 ([7], 1.1)**.** *The mapping $\eta_{[L]}D : D \to \underline{M}_{[L]}D$ is Scott continuous and lower continuous.*

Remark. *For D with a bottom element, $M_{[L]}D$ is a complete sublattice of $\underline{M}_{[L]}D$, hence we obtain that $\eta_{[L]}D$ is Scott and lower continuous also as a mapping $: D \to M_{[L]}D$.*

Therefore we consider D as a subspace both of $M_{[L]}D$ and $\underline{M}_{[L]}D$ w.r.t. the Scott and the lower, hence w.r.t. the Lawson topologies on the both sets. If D is a continuous semilattice, it is also a lower subsemilattice of $M_{[L]}D$ and $\underline{M}_{[L]}D$.

Infima and finite suprema in the complete lattices $\underline{M}_{[L]}D$ and $M_{[L]}D$ of functions are taken pointwise, whereas arbitrary suprema are described by the following easy, but useful statement. For a function $f : D \to L$, let

$$f^u(b) = \inf\{f(a) \mid a \in D, a \ll b\}, \text{ for all } b \in D.$$

Observe that f^u is always a monotonic predicate. Moreover [8, Lemma I.4]:

Lemma 1.2. *For an antitone function $f : D \to L$, the function f^u is the least monotonic predicate f' such that $f \leqslant f'$ pointwise.*

Hence, for a family $\mathcal{F} \subset \underline{M}_{[L]}D$ (or $\mathcal{F} \subset M_{[L]}D$), we have $\inf \mathcal{F} = \inf_p \mathcal{F}$, $\sup \mathcal{F} = (\sup_p \mathcal{F})^u$. For finite $\mathcal{F}$, the latter ^{u} can be dropped.

We use notation $\bar{\oplus}$ and $\bar{\otimes}$ for joins and meets in $\underline{M}_{[L]}D$ and $M_{[L]}D$.

In the sequel we shall additionally require that L be a *unital quantale* [9], i.e. there exists a binary operation $* : L \times L \to L$ such that 1 is a two-sided unit and $*$ in infinitely distributive w.r.t. supremum in both variables, which is equivalent to being continuous w.r.t. the Scott topology on L. Observe that, for such $*$, its infinite distributivity also w.r.t.

infimum means the continuity w.r.t. the Lawson topology on L. Recall that we treat $\oplus$ as a disjunction, and $*$ will be a (possibly noncommutative) conjunction in an L-valued fuzzy logic [10, 11]. The Boolean case is obtained for $L = \{0, 1\}$, $\oplus = \vee$ and $* = \wedge$.

Recall that a (left idempotent) $(L, \oplus, *)$-semimodule [12] is a set X with operations $\bar{\oplus} : X \times X \to X$ and $\bar{*} : L \times X \to X$ such that for all $x, y, z \in X$, $\alpha, \beta \in L$:

(1) $x \mathbin{\bar{\oplus}} y = y \mathbin{\bar{\oplus}} x$;

(2) $(x \mathbin{\bar{\oplus}} y) \mathbin{\bar{\oplus}} z = x \mathbin{\bar{\oplus}} (y \mathbin{\bar{\oplus}} z)$;

(3) there is an (obviously unique) element $\bar{0} \in X$ such that $x \mathbin{\bar{\oplus}} \bar{0} = x$ for all x;

(4) $\alpha \mathbin{\bar{*}} (x \mathbin{\bar{\oplus}} y) = (\alpha \mathbin{\bar{*}} x) \mathbin{\bar{\oplus}} (\alpha \mathbin{\bar{*}} y)$, $(\alpha \oplus \beta) \mathbin{\bar{*}} x = (\alpha \mathbin{\bar{*}} x) \mathbin{\bar{\oplus}} (\beta \mathbin{\bar{*}} x)$;

(5) $(\alpha * \beta) \mathbin{\bar{*}} x = \alpha \mathbin{\bar{*}} (\beta \mathbin{\bar{*}} x)$;

(6) $1 \mathbin{\bar{*}} x = x$; and

(7) $0 \mathbin{\bar{*}} x = \bar{0}$.

Observe that these axioms imply that $(X, \bar{\oplus})$ is an upper semilattice with a bottom element $\bar{0}$, the order is defined as $x \leqslant y \iff x \mathbin{\bar{\oplus}} y = y$, and $\alpha \mathbin{\bar{*}} \bar{0} = \bar{0}$ for all $\alpha \in L$. The operation $\bar{*}$ is isotone in both variables.

Hence an $(L, \oplus, *)$-semimodule is an analogue of a vector space. Similarly, analogues exist for linear and affine mappings. A mapping $f : X \to Y$ between $(L, \oplus, *)$-semimodules is called *linear* if, for all $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in L$, the equality

$$f(\alpha_1 \mathbin{\bar{*}} x_1 \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \alpha_n \mathbin{\bar{*}} x_n) = \alpha_1 \mathbin{\bar{*}} f(x_1) \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \alpha_n \mathbin{\bar{*}} f(x_n)$$

is valid. If the latter equality is ensured only whenever $\alpha_1 \oplus \ldots \oplus \alpha_n = 1$, then f is called *affine*. Observe that an affine mapping f preserves joins, i.e. $f(x_1 \mathbin{\bar{\oplus}} x_2) = f(x_1) \mathbin{\bar{\oplus}} f(x_2)$ for all $x_1, x_2 \in X$. An affine mapping is linear if and only if it preserves the least element.

We call a triple $(X, \bar{\oplus}, \bar{*})$ a *continuous* $(L, \oplus, *)$*-semimodule* if $(X, \bar{\oplus}, \bar{*})$ is an $(L, \oplus, *)$-semimodule such that $(X, \bar{\oplus})$ is a domain, and $\bar{*}$ is infinitely distributive w.r.t. supremum in the both variables (hence is Scott continuous). Observe that such $(X, \bar{\oplus})$ has a least element, a greatest element, and suprema for all subsets, therefore is a continuous lattice. If the poset $(X, \bar{\oplus})$ is a completely distributive lattice, then we call $(X, \bar{\oplus}, \bar{*})$ a *completely distributive* $(L, \oplus, *)$*-semimodule.* This is equivalent to X carrying a compact Hausdorff topology such that the upper semilattice $(X, \bar{\oplus})$ is a distributive Lawson lattice and the operation $\bar{*} : L \times X \to X$ is lower semicontinuous. Therefore we use an equivalent term "*compact Hausdorff Lawson* $(L, \oplus, *)$*-semimodule*".

Let D be a domain. For $m \in \underline{M}_{[L]}D$, we define $\alpha \bar{\odot} m$ to be a least predicate $m' : D \to L$ such that $\alpha * m(b) \leqslant m'(b)$ for all $b \in D$, i.e. $\alpha \bar{\odot} m = (\alpha * m)^u$. Then:

$$(\alpha \bar{\odot} m)(b) = \inf\{\alpha * m(a) \mid a \in D, a \ll b\}.$$

If D has a bottom element, then for $m \in M_{[L]}D$ we need to "adjust" the result:

$$(\alpha \bar{\circledast} m)(b) = (\alpha \bar{\odot} m)(d) \bar{\oplus} \delta_L^D(d) = \begin{cases} (\alpha \bar{\odot} m)(b), & b \neq 0; \\ 1, & b = 0. \end{cases}$$

Proposition 1.1 ([7], 1.7)**.** *The triples* $(\underline{M}_{[L]}D, \bar{\oplus}, \bar{\odot})$ *and* $(M_{[L]}D, \bar{\oplus}, \bar{\circledast})$ *are compact Hausdorff Lawson* $(L, \oplus, *)$*-semimodules.*

We shall consider several categories [13], which can be equivalently defined either in a topological fashion or using order-theoretical properties. In the sequel "semilattice" means "meet-semilattice" if otherwise is not stated, and "semilattice morphism" is a mapping between semilattices that preserves finite meets.

The category of all domains and their Scott continuous mappings is denoted by $\mathcal{D}om$. Its full subcategory with the objects being all domains with bottom elements is denoted $\mathcal{D}om_\perp$. If we also require that bottom elements are preserved by the morphisms, the subcategory $\mathcal{D}om_0$ is obtained. This notation style is applied also to the following categories.

The category that consists of all continuous semilattices and Scott continuous semilattice morphisms is denoted by $\mathcal{CS}em$. The wider category of all continuous semilattices and Scott continuous mappings, which are not necessary meet-preserving, is denoted by $\mathcal{CS}em_\uparrow$. Let $\mathcal{CS}em_0$ and $\mathcal{CS}em_{0\uparrow}$ be the subcategories of $\mathcal{CS}em$ and $\mathcal{CS}em_\uparrow$, which arise when we take only the semilattices with bottom elements and their 0-preserving mappings. Finally, we denote by $\mathcal{CS}em_\perp$ the category of all semilattices with bottom elements and their Scott continuous mapping, not necessary meet- or 0-preserving.

Recall that by the Fundamental Theorem on Compact Semilattices [1, Theorem VI-3.4] a continuous semilattice is complete if and only if it, with some compact Hausdorff topology, is a topological semilattice with local bases consisting of subsemilattices at all points, or a *compact Hausdorff Lawson semilattice* for short; then the topology in question coincides with the Lawson topology. Note that all such semilattices have bottom elements.

Therefore we denote by $\mathcal{LL}aws$ the category of all compact Hausdorff Lawson lower semilattices and their continuous meet-preserving mappings, or, equivalently, of all complete continuous semilattices and all their mappings that preserve all infima and directed suprema. It is a rather narrow category, hence let $\mathcal{LL}aws_{\uparrow}$ be the category with the same objects, but *Scott continuous* mappings as morphisms. Its subcategory that contains only 0-preserving Scott continuous mappings is denoted by $\mathcal{LL}aws_{0\uparrow}$.

Following this notation style, we denote by $(L,\oplus,*)$-$\mathcal{CSM}od_{\uparrow}$ and $(L,\oplus,*)$-$\mathcal{CSA}f\!f_{\uparrow}$ the categories that consist of all continuous $(L,\oplus,*)$-semimodules and their Scott continuous respectively linear and affine maps, which implies preservation of all suprema. By taking only completely distributive $(L,\oplus,*)$-semimodules, we obtain the full subcategories $(L,\oplus,*)$-$\mathcal{L}w\mathcal{SM}od_{\uparrow}$ and $(L,\oplus,*)$-$\mathcal{L}w\mathcal{SA}f\!f_{\uparrow}$, respectively.

Proposition 1.2. *For each Scott continuous mapping $\varphi : D \to K$ from a domain to a continuous L-semimodule there is a unique extension $\Phi : \underline{M}_{[L]}D \to K$ to a morphism in $(L,\oplus,*)$-$\mathcal{CSM}od_{\uparrow}$.*

Proof. For all $\alpha \in L$, $d \in D$ the mapping $\alpha * \eta_{[L]}D(d) : D \to L$ is a monotonic predicate, hence $\alpha * \eta_{[L]}D(d) = \alpha \mathbin{\bar{\odot}} \eta_{[L]}D(d)$. Observe also that $m \in \underline{M}_{[L]}D$ is the least upper bound of the set $\{m(d) * \eta_{[L]}D(d) \mid d \in D\}$.

Therefore if a required extension Φ exists, it must be determined by the formula

$$\begin{aligned}\Phi(m) &= \sup\{\Phi(m(d) \mathbin{\bar{\odot}} \eta_{[L]}D(d)) \mid d \in D\} = \\ &= \sup\{m(d) \mathbin{\bar{\odot}} \Phi(\eta_{[L]}D(d)) \mid d \in D\} = \\ &= \sup\{m(d) \mathbin{\bar{\odot}} \varphi(d) \mid d \in D\}\end{aligned}$$

for all $m \in \underline{M}_{[L]}D$.

Since the suprema in $\underline{M}_{[L]}D$ are calculated pointwise, it is easy to see that the mapping Φ preserves arbitrary suprema. To show $*$-uniformity, extend Φ by the above formula to the set of *all* antitone functions $m : D \to L$.

Obviously $m \leqslant m^u$ implies $\Phi(m) \leqslant \Phi(m^u)$. On the other hand, for all $k \in K$, $k \ll \Phi(m^u)$, there are $d_1, \dots, d_n \in D$ such that $k \ll m^u(d_1) \bar{\odot} \varphi(d_1) \bar{\oplus} \dots \bar{\oplus} m^u(d_n) \bar{\odot} \varphi(d_n)$. Let $d_1' \ll d_1$, ..., $d_n' \ll d_n$, and each d_i' converge to the respective d_i. Then by the Scott continuity of φ and $\bar{\odot}$ we infer that

$$m^u(d_1) \bar{\odot} \varphi(d_1') \bar{\oplus} \dots \bar{\oplus} m^u(d_n) \bar{\odot} \varphi(d_n') \to m^u(d_1) \bar{\odot} \varphi(d_1) \bar{\oplus} \dots \bar{\oplus} m^u(d_n) \bar{\odot}$$
$$\bar{\odot} \varphi(d_n),$$

hence there are $d_1' \ll d_1$, ..., $d_n' \ll d_n$ such that

$$k \leqslant m^u(d_1) \bar{\odot} \varphi(d_1') \bar{\oplus} \dots \bar{\oplus} m^u(d_n) \bar{\odot}$$
$$\bar{\odot} \varphi(d_n') \leqslant m(d_1') \bar{\odot} \varphi(d_1') \bar{\oplus} \dots \bar{\oplus} m(d_n') \bar{\odot} \varphi(d_n') \leqslant \Phi(m).$$

Thus by the continuity of K we obtain $\Phi(m) \geqslant \Phi(m^u)$, and therefore $\Phi(m) = \Phi(m^u)$.

Now, for all $\alpha \in L$, $m \in \underline{M}_{[L]}D$:

$$\Phi(\alpha \bar{\odot} m) = \Phi((\alpha * m)^u) = \Phi(\alpha * m) = \sup\{(\alpha * m(d)) \bar{\odot} \varphi(d) \mid d \in D\} =$$
$$= \sup\{\alpha \bar{\odot}(m(d) \bar{\odot} \varphi(d)) \mid d \in D\} = \alpha \bar{\odot} \sup\{m(d) \bar{\odot} \varphi(d) \mid d \in D\} =$$
$$= \alpha \bar{\odot} \Phi(m).$$

□

Proposition 1.3. *For each Scott continuous mapping $\varphi : D \to K$ from a domain with a bottom element to a continuous L-semimodule there is a unique extension $\Phi : M_{[L]}D \to K$ to a morphism in $(L, \oplus, *)$-$\mathcal{CSA}\mathrm{ff}_{\uparrow}$. It is linear, i.e. it is a morphism in $(L, \oplus, *)$-$\mathcal{CSM}\mathrm{od}_{\uparrow}$, if and only if φ preserves the bottom element.*

Proof is quite analogous, except that the required extension is determined by the formula

$$\Phi(m) = \varphi(0) \bar{\oplus} \sup\{m(d) \bar{\odot} \varphi(d) \mid d \in D\}$$

for all $m \in M_{[L]}D$.

Remark. *Formally speaking, the two latter statements mean that* $\underline{M}_{[L]}D$ *(resp.* $M_{[L]}D$*) is a free object over* D*. Both the domain and the target categories can be chosen differently because* $\underline{M}_{[L]}D$ *and* $M_{[L]}D$ *are completely distributive lattices, and "forgetting" the multiplication makes them not only domains, but also continuous, and even complete, semilattices. Therefore we have the following three "polyvariate" propositions, which are equivalent to the two previous ones.*

Proposition 1.4. *For an object* D *of the category* $\mathcal{D}\mathrm{om}$ *(or of* $\mathcal{CS}\mathrm{em}_{\uparrow}$*, or of* $\mathcal{LL}\mathrm{aws}_{\uparrow}$*) the continuous* L*-semimodule* $\underline{M}_{[L]}D$ *is a free object over* D *in*$(L,\oplus,*)$-$\mathcal{CSM}\mathrm{od}_{\uparrow}$ *(or in* $(L,\oplus,*)$-$\mathcal{L}\mathrm{w}\mathcal{SM}\mathrm{od}_{\uparrow}$*).*

Proposition 1.5. *For an object* D *of the category* $\mathcal{D}\mathrm{om}_{\perp}$ *(or of* $\mathcal{CS}\mathrm{em}_{\perp}$*, or of* $\mathcal{LL}\mathrm{aws}_{\uparrow}$*) the continuous* L*-semimodule* $M_{[L]}D$ *is a free object over* D *in* $(L,\oplus,*)$-$\mathcal{CSA}\mathrm{ff}_{\uparrow}$ *(or in* $(L,\oplus,*)$-$\mathcal{L}\mathrm{w}\mathcal{SA}\mathrm{ff}_{\uparrow}$*).*

Proposition 1.6. *For an object* D *of the category* $\mathcal{D}\mathrm{om}_0$ *(or of* $\mathcal{CS}\mathrm{em}_{0\uparrow}$*, or of* $\mathcal{LL}\mathrm{aws}_{0\uparrow}$*) the continuous* L*-semimodule* $\underline{M}_{[L]}D$ *is a free object over* D *in* $(L,\oplus,*)$-$\mathcal{CSM}\mathrm{od}_{\uparrow}$ *(or in* $(L,\oplus,*)$-$\mathcal{L}\mathrm{w}\mathcal{SM}\mathrm{od}_{\uparrow}$*).*

2 Dual pairs and conjugate operators

For the quantale $L = (L,\oplus,*)$, we denote by L' the quantale $(L,\oplus,*')$ that differs only in the multiplication: $\alpha *' \beta = \beta * \alpha$ for all $\alpha,\beta \in L$.

A little modifying and restricting definitions in [14], we call a pair of an L-semimodule K and an L'-semimodule K a *predual pair* if there is a multiplication $\cdot : K \times K' \to L$ that is distributive and Scott continuous in each variable (hence is jointly Scott continuous), and $(\alpha \mathbin{\bar{\circledast}} k)\cdot(\beta \mathbin{\bar{\circledast}} k') = \alpha * (k \cdot k') * \beta$ for all $k \in K$, $k' \in K'$, and $\alpha,\beta \in L$.

We say that $\cdot$ separates the elements of K if, for all $k_1,k_2 \in K$, $k_1 \nleq k_2$, there is $k' \in K'$ such that $k_1 \cdot k' \nleq k_2 \cdot k'$; similarly for separation of the elements of K'. If $\cdot$ separates the elements of the both semimodules K and K', we say that they form a *dual pair*.

See the latter citation for examples of predual and dual pairs, as well as for an example that K' and $\cdot$ such that K, K' form a dual pair exist even not for every complete L-semimodule K. The most obvious dual pair is $K = K' = L^I$, where I is an arbitrary index set, $\alpha \mathbin{\bar{\circledast}} (a_i)_{i\in I} = (\alpha * a_i)_{i\in I}$ in

K, $\beta \mathbin{\bar{\circledast}} (b_i)_{i\in I} = (b_i * \beta)_{i\in I}$ in K', $(a_i)_{i\in I} \cdot (b_i)_{i\in I} = \sup\{a_i * b_i \mid i \in I\}$. Here we shall construct predual and dual pairs that consist of semimodules of monotonic predicates.

Let D, D' be continuous semilattices with bottom elements. Recall that $M_{[L]}D$ is a continuous L-semimodule, and $M_{[L']}D'$, which in fact is the same as $M_{[L]}D'$ but with different multiplication, is a continuous L'-semimodule.

Fix a Scott continuous mapping $P : D \times D' \to L$ such that $P(d, 0) = P(0, d') = 0$ for all $d \in D$, $d' \in D'$, and define a "scalar-like" product by the formula:

$$(m, m')^*_P = m \cdot m' = \sup\{m(d) * P(d, d') * m'(d') \mid d \in D, d' \in D'\}$$

for all antitone functions $m : D \to L$, $m' : D' \to L$. We use the second notation if P and $*$ are easily guessed. Using arguments similar to used in the proof of Proposition 1.2, we obtain the following lemma.

Lemma 2.1. *For all antitone functions $m : D \to L$, $m' : D' \to L$ the equality $(m^u, m')^*_P = (m, m'^u)^*_P = (m, m')^*_P$ is valid.*

Corollary 2.1. *For all $m \in M_{[L]}D$, $m' \in M_{[L]}D'$, and $\alpha \in L$ we have $(\alpha \mathbin{\bar{\circledast}} m, m')^*_P = \alpha * (m, m')^*_P$.*

Of course, the analogous statement holds for the second argument.

Since joins in $M_{[L]}D$ and $M_{[L]}D'$ are calculated argumentwise, the introduced multiplication is distributive in the both arguments. Therefore Lemma 2.1 implies infinite distributivity, hence separate and joint Scott continuity of the multiplication *of monotonic predicates.*

Now the following statement is at hand.

Proposition 2.1. *Let D, D' be continuous semilattices with bottom elements and $P : D \times D' \to L$ a Scott continuous mapping such that $P(d, 0) = P(0, d') = 0$ for all $d \in D$, $d' \in D'$. Then $M_{[L]}D$ and $M_{[L']}D'$, together with the multiplication $(-, -)^*_P : M_{[L]}D \times M_{[L']}D' \to L$, constitute a predual pair.*

Now we consider a "kernel" $P : D \times D' \to L$ such that, aside from the previously required properties:

(1) P is Scott continuous;

(2) $P(d,0) = P(0,d') = 0$ for all $d \in D$, $d' \in D'$;

the following is satisfied:

(3) P attains only the values in $\mathbf{2} = \{0,1\} \subset L$ and is distributive w.r.t. $\wedge$ (not $\vee$!) in the both variables; and

(4) P separates the elements both of D and D', i.e., if $P(d_1,d') = P(d_2,d')$ for some $d_1,d_2 \in D$ and all $d' \in D'$, then $d_1 = d_2$; analogously for the second argument.

The binary relation $\{(d,d') \in D \times D' \mid P(d,d') = 0\}$ for such P was called a *separating polarity* in [15]; it was also proved there (Proposition 2.6) that, for each *continuous* meet-semilattice D with a bottom element, there is a unique up to isomorphism *continuous* meet-semilattice D', with a bottom element, such that there is $P : D \times D' \to \{0,1\}$ with the above properties. Namely, $D' = D^\wedge$, where $D^\wedge$ is the ordered by inclusion set of all Scott open filters in D, including $\varnothing$, but excluding D itself, hence none of elements of $D^\wedge$ contains the bottom element of D. Recall that a set $F \subset D$ is a Scott open filter if and only if F is a closed under finite meets upper set such that, for any directed subset $A \subset D$ such that $\sup D \in F$, the intersection $A \cap F$ is non-empty. A "canonical" multiplication $P : D \times D^\wedge \to \mathbf{2}$ is determined by the formula

$$P(d,F) = \begin{cases} 0, d \notin F, \\ 1, d \in F, \end{cases} \qquad d \in D, F \in D^\wedge.$$

Proposition 2.2. *Let D, D' be continuous semilattices with bottom elements and $P : D \times D' \to L$ satisfy the above conditions (1)–(4). Then $M_{[L]}D$ and $M_{[L']}D'$, together with the multiplication $(-,-)^*_P : M_{[L]}D \times M_{[L']}D' \to L$, constitute a dual pair.*

Proof. We can assume that $D' = D^\wedge$. Only separation of points is to be verified. Let $m_1 \neq m_2$, e.g. $m_1(d) \nleqslant m_2(d)$ for some $d \in D$. Since $m_2 : D \to L^{op}$ is Scott continuous, there is $d_0 \ll d$ in D such that $m_1(d) \nleqslant m_2(d_0) \geqslant m_2(d)$, which implies $d_0 \neq 0$. By Proposition I-3.3 [1], there is an open filter $F \ni d$ such that $d_0 \ll b$ for all $b \in F$.

Let $m' : D^\wedge \to L$ be defined by the formula

$$m'(d') = \begin{cases} 1, d' \leqslant F, \\ 0, d' \nleqslant F, \end{cases} \qquad d' \in D^\wedge.$$

Then

$$(m_i, m')^*_P = \sup\{m_i(b) \mid b \in F\}, i = 1, 2,$$

therefore

$$(m_1, m')^*_P \geqslant m_1(d) \not\leqslant m_2(d_0) \geqslant \sup\{m_2(b) \mid b \in D, d_0 \ll b\} \geqslant (m_2, m')^*_P,$$

hence $(m_1, m')^*_P \neq (m_2, m')^*_P$. □

Assume that K_1, K_2 are L-semimodules, K'_1, K'_2 are L'-semimodules, multiplications $\cdot : K_1 \times K'_1 \to L$ and $\cdot : K_2 \times K'_2 \to L$ are such that K_1, K'_1 and K_2, K'_2 are dual pairs, and $A : K_1 \to K_2$ is a linear mapping. It is natural to call a linear mapping $A' : K'_2 \to K'_1$ the *(Hermitian) conjugate* to A if $Aa \cdot a' = a \cdot A'a'$ whenever $a \in K_1$, $a \in K'_2$. The separation property implies that, if a conjugate for a given A exists, it is unique. Hence we write $A' = A^*$ in this case, and obviously $A^{**} = A$. It is also immediate that, for the composition $A \circ B$ of linear mappings with conjugates A^* and B^*, respectively, a conjugate exists and is equal to $B^* \circ A^*$.

It is obvious that conjugates exist for linear mappings between (algebraically) free idempotent semimodules [14], moreover, for such mappings conjugation reduces to taking transpose of the respective finite or infinite matrices.

In this paper we consider conjugates to Scott continuous linear mapping between the previously introduced *topologically* free L-idempotent semimodules over continuous semilattices with bottom elements.

Proposition 2.3. *Let D_1, D_2 be continuous semilattices with bottom elements. For each Scott continuous linear mapping $\Phi : M_{[L]}D_1 \to M_{[L]}D_2$, there is a Scott continuous conjugate $\Phi' : M_{[L']}D_2^{\wedge} \to M_{[L']}D_1^{\wedge}$.*

Proof. By Proposition 1.3 the mapping Φ and the required conjugate Φ' must be unique Scott continuous linear extensions of mappings $\varphi : D_1 \to M_{[L]}D_2$, which is the restriction of Φ to D_1, and $\varphi' : D_2^{\wedge} \to M_{[L']}D_1^{\wedge}$, which we can calculate.

The following must hold for all $d_1 \in D_1$, $d'_2 \in D_2^{\wedge}$:

$$\Phi(\eta_{[L]}D_1(d_1)) \cdot \eta_{[L']}D_2^{\wedge}(d'_2) = \eta_{[L]}D_1(d_1) \cdot \Phi'(\eta_{[L']}D_2^{\wedge}(d'_2)), \qquad (*)$$

i.e.

$$\sup\{\varphi(d_1)(d_2) \mid d_2 \in d'_2\} = \sup\{\varphi'(d'_2)(d'_1) \mid d'_1 \ni d_1\}. \qquad (**)$$

The Scott continuity of the function $\varphi'(d_2') : D_1^{\wedge} \to L^{op}$ implies that, for any $d_1' \in D_1^{\wedge}$:

$$\begin{aligned}\varphi'(d_2')(d_1') = \inf\bigl\{\sup\{\varphi'(d_2')(d_1'') \mid d_1'' \in D_1^{\wedge}, d_1'' \ni d_1\} \bigm| d_1 \in d_1'\bigr\} = \\ = \inf\bigl\{\sup\{\varphi(d_1)(d_2) \mid d_2 \in d_2'\} \bigm| d_1 \in d_1'\bigr\},\end{aligned}$$

which we take as a definition of $\varphi'(d_2)$. For each function $\theta : D_1 \to L$, the correspondence $d_1' \mapsto \inf\{\theta(d_1) \mid d_1 \in d_1'\}$ is a normalized monotonic predicate $D_1^{\wedge} \to L$, hence $\varphi'(d_2) \in M_{[L']}D_1^{\wedge}$.

On the other hand, for all $d_1'' \ll d_1'$ in $D_1^{\wedge}$ there is $d_1 \in d_1'$ such that $d_1'' \subset \{d_1\}\uparrow$, hence

$$\begin{aligned}\sup\{\varphi'(d_2'')(d_1'') \mid d_2'' \ll d_2'\} \geqslant \sup\bigl\{\sup\{\varphi(d_1)(d_2) \mid d_2 \in d_2''\} \bigm| d_2'' \ll d_2'\bigr\} = \\ = \sup\{\varphi(d_1)(d_2) \mid d_2 \in d_2'\} \geqslant \inf\bigl\{\sup\{\varphi(d_1)(d_2) \mid d_2 \in d_2'\} \bigm| d_1 \in d_1'\bigr\} = \\ = \varphi'(d_2')(d_1').\end{aligned}$$

Therefore for the function $m' = \sup\{\varphi'(d_2'') \mid d_2'' \ll d_2'\}$ in $M_{[L']}D_1^{\wedge}$ we have $m'(d_1'') \geqslant \varphi'(d_2')(d_1')$ for all $d_1'' \ll d_1'$. This implies $m_1 \geqslant \varphi'(d_2')$, thus

$$\sup\{\varphi'(d_2'') \mid d_2'' \ll d_2'\} = \varphi'(d_2'),$$

i.e. φ' is Scott continuous. Now it is routine but straightforward to show that the Scott continuity of φ implies that φ' satisfies (**), therefore (*). Since $\cdot$ is uniform and infinitely distributive in the both variables, and each element of $M_{[L]}D_1$ and $M_{[L']}D_2^{\wedge}$ is a (probably infinite) supremum of elements of the form $\alpha \mathbin{\bar{\circledast}} \eta_{[L]}D_1(d_1)$ and $\beta \mathbin{\bar{\circledast}} \eta_{[L']}D_2^{\wedge}(d_2')$, respectively, this implies that the unique Scott continuous linear extensions Φ and Φ' of φ and φ' are mutually conjugate. $\square$

Remark. *It is easy to observe that the constructed mapping $\varphi' : D_2 \to M_{[L']}D_1^{\wedge}$ is Scott continuous for all* isotone *$\varphi : D_1 \to M_{[L]}D_2$, but without the Scott continuity of φ the equality (*) can fail.*

3 Discussion of results and open problems

The introduced notion have (obviously non-unique) interpretation in terms of denotational semantics and fuzzy logic. Semimodules of monotonic predicates provide a geometric description of fuzzy knowledge about state of system. This allows to transfer to this field methods of linear algebra and functional analysis, in the spirit of idempotent mathematics [16].

The “kernel” $P : D \times D' \to L$ is related to a case when D and D' consist of descriptions of states of the same system from two “points of view”. Then $P(d, d')$ is a measure of incompatibility of the information portions d and d'. If P attains only values 0 and 1, then $P(d, d') = 0$ means simply that d and d' are compatible, and $P(d, d') = 1$ corresponds to incompatible d, d'. The introduced product $(m, m')^*_P$ then shows how incompatible are L-fuzzy predicates $m : D \to L$ and $m' : D' \to L$. It is also clear how to interpret separation of points.

Similarly linear operators between predicate semimodules are *predicate transformers* [7], and conjugate operators describe the process of “inverse information discovery”. More on applied aspects will be said in our subsequent publication.

There are, hovewer, open questions:

Problem 1. *Describe continuous $(L, \oplus, *)$-semimodules that can be included in a dual pair.*

Problem 2. *Describe Scott continuous linear operators between continuous $(L, \oplus, *)$-semimodules that have conjugate operators.*

[1] *Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M., Scott D.S.* Continuous Lattices and Domains. — Cambridge Univ. Press, 2003. — 591 p.

[2] *D. Dubois, Prade H.* Gradualness, uncertainty and bipolarity: Making sense of fuzzy sets // Fuzzy Sets and Systems — Publ. online 19 Nov 2010, `doi:10.1016/j.fss.2010.11.007`.

[3] *Edalat A.* Domains for computation in mathematics, physics and exact real arithmetic // Bull. Symb. Logic — 1997. — **3**, №4. — P. 401–452.

[4] Raney G.N. Completely distributive complete lattices // Proc. Amer. Math. Soc. — 1952. — **3**. — P. 677–680.

[5] *Heckmann R., Huth M.* A duality theory for quantitative semantics, in: Proceedings of the 11th International Workshop on Computer Science Logic, volume 1414 of Lecture Notes in Computer Science. — Springer Verlag, 1998. — P. 255–274.

[6] *Erné M.* Z-distributive function spaces, preprint, 1998.

[7] *Nykyforchyn O., Repovš D.* L-fuzzy strongest postcondition predicate transformers as L-idempotent linear or affine operators between semimodules of monotonic predicates, preprint, 2011.

[8] *Verwaat W.* Random upper semicontinuous functions and extremal processes// In: Probability and Lattices. Eds: W. Verwaat, H. Holverda. — CWI tract, Centrum voor Wiskunde en Informatica, 1997. — P. 1–56.

[9] *Rosenthal K.* Quantales and Their Applications. — Longman Scientific & Technical, Wiley, Essex, England, New York, 1990. — 165 p.

[10] *Hájek P.* Fuzzy logics with noncommutative conjuctions // J. Logic Computation. — 2003. — **13**, №4. — P. 469–479.

[11] *Bergmann M.* Introduction to Many-Valued and Fuzzy Logic: Semantics, Algebras, and Derivation Systems. — New York: Cambridge Univ. Press, 2008. — 329 p.

[12] *Akian M.* Densities of invariant measures and large deviations // Trans. Amer. Math. Soc. — 1999. — **351**, №11. — P. 4515–4543.

[13] *Mac Lane S.* Categories for the Working Mathematician. 2nd ed. — New York: Springer, 1998. — 314 p.

[14] *Cohen G., Gaubert S., Quadrat J.-P.* Duality and separation theorems in idempotent semimodules // `arXiv: math/0212294v2 [math.FA]`. — 29 Sep 2003.

[15] *Nykyforchyn O., Mykytsey O.* Conjugate measures on semilattices // Visnyk LNU, ser. mech.-mat. — 2010. — **72**. — P. 88–99.

[16] *Kolokoltsov V.N., Maslov V.P.* Idempotent Analysis and Its Applications. — Dordrecht: Kluwer Academic Publ., 1998. — 305 p.

L-ІДЕМПОТЕНТНІ ЛІНІЙНІ ОПЕРАТОРИ МІЖ НАПІВМОДУЛЯМИ ПРЕДИКАТІВ, ДУАЛЬНІ ПАРИ І СПРЯЖЕНІ ОПЕРАТОРИ

Олег НИКИФОРЧИН, Оксана МИКИЦЕЙ

Прикарпатський національний університет ім. Василя Стефаника, вул. Шевченка 57, Івано-Франківськ 76018

e-mail: *oleh.nyk@gmail.com, oksana39@if.ua*

Показано, що множини монотонних предикатів на областях (domains) зі значеннями у цілком дистрибутивних кванталях є вільними цілком дистрибутивними ідемпотентними напівмодулями над цими областями. Також побудовано ідемпотентні дуальні пари і спряжені оператори.