# THE AKNS HIERARCHY AND THE GUREVICH-ZYBIN DYNAMICAL SYSTEM INTEGRABILITY REVISITED


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A novel approach based upon vertex operator representation is devised to study the AKNS hierarchy. It is shown that this method reveals the remarkable properties of the AKNS hierarchy in relatively simple, rather natural and particularly effective ways. In addition, the connection of this vertex operator based approach with Lie-algebraic integrability schemes is analyzed and its relationship with $\tau$-function representations is briefly discussed. An approach based on the spectral and Lie-algebraic techniques for constructing vertex operator representation for solutions to a Riemann type hydrodynamical hierarchy is devised. A functional representation generating an infinite hierarchy of dispersive Lax type integrable Hamiltonian flows is obtained.

## 1 Introduction

The "miraculous" properties of the AKNS hierarchy related to calculations connected with the integrability of nonlinear dynamical systems have, since the early work of their discoverers [1, 2], been the focus of considerable research. These investigations, such as in [3, 4, 5, 6, 7, 8], have produced further insights into the nature of the AKNS hierarchy and several additional methods of construction. In what follows, we devise an alternative approach to exploring the properties of the AKNS hierarchy based upon its vertex operator representation. It appears that our formulation offers several advantages over existing methods when it comes to simplicity, effectiveness, flexibility and ease of extension, but more detailed confirmation of these observations must await further investigations.

Nonlinear hydrodynamic equations are of constant interest still from classical works by B. Riemann, who had extensively studied them in general three-dimensional case, having paid special attention to their one-dimensional spatial reduction, for which he devised the generalized method of characteristics and Riemann invariants. These methods appeared to be very effective [9] in investigating many types of nonlinear spatially one-dimensional systems of hydrodynamical type and, in particular, the characteristics method in the form of a "reciprocal" transformation of variables has been used recently in studying a so called Gurevich-Zybin system [10, 11] in [12] and a Whitham type system in [13, 14, 15, 16] and [15, 17]. Moreover, this method was further effectively applied to studying solutions to a generalized [15] (owing to D. Holm and M. Pavlov) Riemann type hydrodynamical system

$$D_t^N u = 0, \quad D_t := \partial/\partial t + u\partial/\partial x, \tag{1}$$

where $N \in \mathbb{Z}_+$ and $u \in C^\infty(\mathbb{R}^2;\mathbb{R})$ is a smooth function. Making use of novel methods, devised in [16, 18] and based both on the spectral theory [6, 7, 19, 20] and the differential algebra techniques, the Lax type representations for the cases $N = \overline{1,4}$ were constructed in explicit form.

## 2 The AKNS hierarchy vertex representation analysis

### 2.1 The AKNS hierarchy and its algebraic structure description

To set the stage for our approach, we begin with some fundamentals of the remarkable sequence of Lax integrable dynamical systems that is the focus of this study. We shall analyze the AKNS hierarchy of Lax integrable dynamical systems on a complex $2\pi$-periodic functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C}^2)$, which is well known [1, 2, 4, 6] to be related to the following linear differential spectral problem of Lax type:

$$df/dx - \ell(x;\lambda)f = 0, \quad \ell(x;\lambda) := \begin{pmatrix} \lambda/2 & u \\ v & -\lambda/2 \end{pmatrix}. \tag{2}$$

Here $x \in \mathbb{R}, f \in L^1(\mathbb{R};\mathbb{C}^2)$, the vector function $(u,v)^\top \subset M$, $\top$ denotes the transpose and $\lambda \in \mathbb{C}$ is a spectral parameter. Assume that a vector function $(u,v)^\top \subset M$ depends parametrically on the infinite set $t := \{t_1, t_2, t_3, \ldots\} \in \mathbb{C}^\mathbb{N}$ in such a way that the generalized Floquet spectrum $\sigma(\ell) := \{\lambda \in \mathbb{C} : \sup_{x\in\mathbb{R}} ||f(x;\lambda)||_1 < \infty\}$ of the problem (2) persists in being parametrically iso-spectral, that is $d\sigma(\ell)/dt = 0$. The iso-spectrality condition gives rise to the AKNS hierarchy of nonlinear dynamical systems on the functional manifold $M$ in the general form

$$\frac{d}{dt_j}(u(t), v(t))^\top = K_j[u(t), v(t)], \tag{3}$$

where

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} := \begin{pmatrix} u(x+t_1, t_2, t_3, \ldots) \\ v(x+t_1, t_2, t_3, \ldots) \end{pmatrix} \tag{4}$$

for $t \in \mathbb{C}^{\mathbb{N}}$.

The corresponding vector fields $K_j : M \to T(M), j \in \mathbb{N}$, can be constructed [2, 4, 7, 8, 20, 21] via the following Lie-algebraic scheme: We define the centrally extended affine current $s\ell(2)$-algebra $\hat{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$

$$\tilde{\mathcal{G}} := \{a = \sum_{j\in\mathbb{Z},\, j\ll\infty} a^{(j)} \otimes \lambda^j : a^{(j)} \in C^\infty\left(\mathbb{R}/2\pi\mathbb{Z}; \mathfrak{s}\ell(2;\mathbb{C})\right)\}, \tag{5}$$

endowed with the Lie commutator

$$[(a_1, c_1), (a_2, c_2)] := ([a_1, a_2], \langle a_1, da_2/dx\rangle) \tag{6}$$

with the scalar product

$$\langle a_1, a_2\rangle := \mathrm{res}_{\lambda=\infty} \int_0^{2\pi} \mathrm{tr}(a_1 a_2) dx \tag{7}$$

for any two elements $a_1, a_2 \in \tilde{\mathcal{G}}$, where "res" and "tr" are the usual residue and trace maps, respectively. As the spectrum $\sigma(\ell) \subset \mathbb{C}$ is supposed to be parametrically independent, there is a natural association with flows. These flows are generated by the set $I(\hat{\mathcal{G}}^*)$ of Casimir invariants of the coadjoint action of the current algebra $\hat{\mathcal{G}}$ on a given element $\ell(x;\lambda) \in \tilde{\mathcal{G}}_-^* \cong \tilde{\mathcal{G}}_+$ contained in the space of functionals $\mathcal{D}(\hat{\mathcal{G}})$. Here we have denoted by $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ the natural splitting into two affine subalgebras of positive and negative $\lambda$-expansions. In particular, a functional $\gamma(\lambda)$ is in $I(\hat{\mathcal{G}})$ if and only if

$$[\tilde{S}(x;\lambda), \ell(x;\lambda)] + \frac{d}{dx}\tilde{S}(x;\lambda) = 0, \tag{8}$$

where the gradient $\tilde{S}(x;\lambda) := \mathrm{grad}\gamma(\lambda)(\ell) \in \tilde{\mathcal{G}}_-$ is defined with respect to the scalar product (7) by means of the variation

$$\delta\gamma(\lambda) := \langle \mathrm{grad}\gamma(\lambda)(\ell), \delta\ell\rangle\,. \tag{9}$$

We note here that the determining matrix equation (8) in the case of the element $\ell(x;\lambda) \in \tilde{\mathcal{G}}_-^*$, given by the spectral problem (2), can be easily solved recursively as $\lambda \to \infty$ in the following asymptotic form as

$$\tilde{S}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{S}_j(x)\lambda^{-(j+1)}, \quad \tilde{S}(x;\lambda) = \begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} \\ \tilde{s}_{21} & \tilde{s}_{22} \end{pmatrix}, \tag{10}$$

$$\tilde{S}_0(x) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \tilde{S}_1(x) = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

$$\tilde{S}_2(x) = \begin{pmatrix} -uv & u_x \\ -v_x & vu \end{pmatrix}, \quad \tilde{S}_3(x) = \begin{pmatrix} vu_x - uv_x & u_{xx} - 2u^2v \\ v_{xx} - 2v^2u & uv_x - vu_x \end{pmatrix}, \ldots,$$

and so on, based upon the differential relationships

$$\begin{aligned} \lambda\tilde{s}_{12} &= \tilde{s}_{12,x} + u(\tilde{s}_{11} - \tilde{s}_{22}), \\ -\lambda\tilde{s}_{21} &= \tilde{s}_{21,x} - v(\tilde{s}_{11} - \tilde{s}_{22}), \\ \tilde{s}_{11,x} &= u\tilde{s}_{21} - v\tilde{s}_{12} = -\tilde{s}_{22,x,} \end{aligned} \tag{11}$$

following from (8).

Now we will take into into account that the coadjoint orbits of elements $\ell \in \tilde{\mathcal{G}}_-^*$ with respect to the standard $\mathcal{R}$-structure [4] on the Lie algebra $\hat{\mathcal{G}}$

$$\begin{gathered} [(a_1, c_1), (a_2, c_2)]_{\mathcal{R}} := ([\mathcal{R}a_1, a_2] + [a_1, \mathcal{R}a_2], \langle \mathcal{R}a_1, da_2/dx\rangle - \\ - < da_1/dx, \mathcal{R}a_2 >) \end{gathered} \tag{12}$$

where, by definition, $\mathcal{R} := \frac{1}{2}(P_+ - P_-)$ and $P_\pm\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_\pm$, are Poissonian manifolds [4, 7, 8, 21, 22, 23, 24]. Then the corresponding *a priori* isospectral AKNS flows can be constructed as the commuting Hamiltonian systems on $\tilde{\mathcal{G}}_-^*$

$$\frac{d\ell}{dt_j} := \{\gamma_j, \ell\} = [(\lambda^{j+1}\tilde{S})_+, \ell] + \frac{d}{dx}(\lambda^{j+1}\tilde{S})_+ \tag{13}$$

generated by the Casimir invariants $\gamma_j \in I(\hat{\mathcal{G}}^*), j \in \mathbb{N},$ with respect to the Lie-Poisson structure on $\hat{\mathcal{G}}^*$ defined as

$$\begin{gathered} \{\gamma, \xi\} := \langle \ell, [\mathrm{grad}\gamma(\lambda)(\ell), \mathrm{grad}\xi(\lambda)(\ell)]_{\mathcal{R}}\rangle + \\ + \left\langle \mathcal{R}\mathrm{grad}\gamma(\lambda)(\ell), \tfrac{d}{dx}\mathrm{grad}\xi(\lambda)(\ell)\right\rangle - \left\langle \tfrac{d}{dx}\mathrm{grad}\gamma(\lambda)(\ell), \mathcal{R}\mathrm{grad}\xi(\lambda)(\ell)\right\rangle \end{gathered} \tag{14}$$

for any smooth functionals $\gamma, \xi \in \mathcal{D}(\hat{\mathcal{G}}^*)$. As a result of (13) equation (8) is easily augmented by the commuting hierarchy of evolution equations

$$d\tilde{S}/dt_j = [(\lambda^{j+1}\tilde{S})_+, \tilde{S}] \tag{15}$$

for $j \in \mathbb{N}$, including the determining equation (8) at $j = 1$.

The hierarchies (13) and (15) can be rewritten with respect to the unique $\lambda$-parametric vector field

$$d/dt := \sum_{j\in\mathbb{Z}_+} \lambda^{-j} d/dt_{j+1} \tag{16}$$

on the manifold $M$ as the generating flows

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda^2\tilde{s}_{12,x} + u\lambda^2(\tilde{s}_{11} - \tilde{s}_{22}) \\ \lambda^2\tilde{s}_{21,x} - v\lambda^2(\tilde{s}_{11} - \tilde{s}_{22}) \end{pmatrix} \tag{17}$$

and

$$\frac{d}{dt}\tilde{S}(x;\mu) = [\tilde{S}(x;\mu), \frac{\lambda^3}{\mu - \lambda}\tilde{S}(x;\lambda) + \lambda\tilde{S}_0(x)], \tag{18}$$

where the parameters $\lambda, \mu \to \infty$ in such a way that $|\mu/\lambda| < 1$. Since the flow (17) is, by construction, Hamiltonian on the adjoint space $\tilde{\mathcal{G}}^*_-$, it can be represented also as a Hamiltonian flow on the functional manifold $M$. This will be done in the next two sections with respect to both the evolution vector field (16) and the related vertex vector field mapping $\hat{X}_\lambda : M \to M$ defined as

$$\begin{aligned} &\hat{X}_\lambda := (\hat{X}^+_\lambda, \hat{X}^-_\lambda), \quad \hat{X}^+_\lambda = \exp D_\lambda, \quad \hat{X}^-_\lambda = \exp(-D_\lambda), \\ &D_\lambda := \sum_{j\in\mathbb{Z}_+} \frac{1}{(j+1)}\lambda^{-(j+1)}\frac{d}{dt_{j+1}}, \end{aligned} \tag{19}$$

and satisfying the determining relationship

$$\frac{d}{dt} = \mp\lambda^2 \hat{X}^{\pm,-1}_\lambda \frac{d}{d\lambda}\hat{X}^\pm_\lambda, \tag{20}$$

as $\lambda \to \infty$. These vertex vector field maps and their connections with integrability theory have been studied extensively by a number of researchers, most notably in [2, 3].

## 2.2 Hamiltonian analysis

Consider the Casimir functional $\gamma(\lambda) \in I(\hat{\mathcal{G}})$, $\lambda \in \mathbb{C}$, and its gradient with respect to its dependence on a point $(u, v)^\top \in M$ given by

$$\mathrm{grad}\gamma(\lambda)[u, v] = (\tilde{s}_{21}(x;\lambda), \tilde{s}_{12}(x;\lambda))^\top \in T^*(M), \tag{21}$$

as follows easily from definition (9). By introducing on the manifold $M$ the following two skew-symmetric operators

$$\theta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta := \begin{pmatrix} 2u\partial^{-1}u & \partial - 2u\partial^{-1}v \\ \partial - 2v\partial^{-1}u & 2v\partial^{-1}v \end{pmatrix}, \tag{22}$$

the relationships (11) can be rewritten as

$$\lambda\theta \mathrm{grad}\gamma(\lambda)[u,v] = \eta \mathrm{grad}\gamma(\lambda)[u,v], \tag{23}$$

holding for all $\lambda \in \mathbb{C}$. It follows directly from (17) that

$$\frac{d}{dt}(u,v)^{\top} = -\eta \mathrm{grad}\gamma(\lambda)[u,v], \tag{24}$$

so it is easy to verify that the Casimir invariant $\gamma(\lambda) \in I(\hat{\mathcal{G}})$ simultaneously satisfies the two involutivity conditions

$$\{\gamma(\lambda), \gamma(\mu)\}_{\theta} = 0 = \{\gamma(\lambda), \gamma(\mu)\}_{\eta} \tag{25}$$

for all $\lambda, \mu \in \mathbb{C}$ with respect to two Poissonian structures

$$\{\cdot,\cdot\}_{\theta} := (\mathrm{grad}(\cdot), \theta \mathrm{grad}(\cdot)), \quad \{\cdot,\cdot\}_{\eta} := (\mathrm{grad}(\cdot), \eta \mathrm{grad}(\cdot)) \tag{26}$$

on the manifold $M$, where $(\cdot,\cdot)$ is the standard convolution on the product bundle $T^*(M) \times T(M)$.

As a direct consequence of (24) and (25), one can readily verify that the operators $\theta, \eta : T^*(M) \to T(M)$, defined by (22), are co-symplectic, Nötherian and compatible [7, 20, 21] on $M$. This, in particular, implies that the Lie derivatives [7, 21, 23, 24]

$$L_{\frac{d}{dt}}\theta = 0 = L_{\frac{d}{dt}}\eta, \quad L_{\frac{d}{dt}}\mathrm{grad}\gamma(\lambda)[u,v] = 0 \tag{27}$$

vanish identically on the manifold $M$.

## 2.3 Vertex operator structure analysis

It is well known [4, 6, 7, 20] that the Casimir invariants determining equation (8) allows a general solution representation in the following two important forms:

$$\tilde{S}(x;\lambda) = k(\lambda)F(x+2\pi, x;\lambda) - \frac{\tilde{k}(\lambda)}{2}\mathrm{tr}F(x+2\pi, x;\lambda) \tag{28}$$

and

$$\tilde{S}(y;\lambda) = \tilde{F}(y,x_0;\lambda)\tilde{C}(x_0;\lambda)\tilde{F}^{-1}(y,x_0;\lambda). \tag{29}$$

Here $F(y,x;\lambda)$ and $\tilde{F}(y,x_0;\lambda)$ belong to the space of linear endomorphisms of $\mathbb{C}^2$, End $\mathbb{C}^2$, for all $x, x_0, y \in \mathbb{R}$, and are matrix solutions to the spectral equation (2) satisfying, respectively, the Cauchy problems

$$\frac{d}{dy}F(y,x;\lambda) = \ell(y;\lambda)F(y,x;\lambda), \quad F(y,x;\lambda)|_{y=x} = \mathbf{I}, \tag{30}$$

and

$$\frac{d}{dy}\tilde{F}(y,x_0;\lambda) = \ell(y;\lambda)\tilde{F}(y,x_0;\lambda), \quad \tilde{F}(y,x_0;\lambda)|_{y=x_0} = \mathbf{I} + O(1/\lambda), \tag{31}$$

for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, where $\mathbf{I} \in$ End $\mathbb{C}^2$ is the identity matrix. Here the parameters $\tilde{k}(\lambda) \in \mathbb{C}$ and $\tilde{C}(x_0;\lambda) \in \operatorname{End}\mathbb{C}^2$ are invariants, chosen in such a way that the asymptotic condition

$$\tilde{S}(x;\lambda) \in \tilde{\mathcal{G}}_- \tag{32}$$

as $\lambda \to \infty$ holds for all $x \in \mathbb{R}$.

To construct the solution (28) satisfying condition (32), we find a preliminary partial solution $\tilde{F}(y,x;\lambda) \in \operatorname{End}\mathbb{C}^2$, $x, y \in \mathbb{R}$, to equation (31) at $x_0 = x \in \mathbb{R}$, satisfying the asymptotic Cauchy data

$$\tilde{F}(y,x;\lambda)|_{y=x} = \mathbf{I} + O(1/\lambda) \tag{33}$$

as $\lambda \to \infty$. It is easy to check that

$$\tilde{F}(y,x;\lambda) = \begin{pmatrix} \tilde{e}_1(y,x;\lambda) & -\tilde{u}(y;\lambda)\lambda^{-1}\tilde{e}_2(y,x;\lambda) \\ \tilde{v}(y;\lambda)\lambda^{-1}\tilde{e}_1(y,x;\lambda) & \tilde{e}_2(y,x;\lambda) \end{pmatrix}, \tag{34}$$

is an exact functional solution to (31) satisfying condition (33). Here we have defined

$$\begin{aligned} \tilde{e}_1(y,x;\lambda) &:= \exp\{(y-x)\lambda/2 + \lambda^{-1}\int_x^y u\tilde{v}ds\}, \\ \tilde{e}_2(y,x;\lambda) &:= \exp\{(x-y)\lambda/2 - \lambda^{-1}\int_x^y \tilde{u}vds\}, \end{aligned} \tag{35}$$

where the vector-function $(\tilde{u},\tilde{v})^{\top} \in M$ satisfies the determining functional relationships

$$\tilde{u} = u + \tilde{u}_x\lambda^{-1} - \tilde{u}^2 v\lambda^{-2}, \qquad \tilde{v} = v - \tilde{v}_x\lambda^{-1} - \tilde{v}^2 u\lambda^{-2}, \tag{36}$$

as $\lambda \to \infty$, which were discovered earlier in a very interesting article [25]. It was also shown that exact asymptotic (as $\lambda \to \infty$) functional solutions of these relationships can be easily constructed by means of the standard iteration procedure.

The fundamental matrix $F(y,x;\lambda) \in \operatorname{End}\mathbb{C}^2$ is represented for all $x,y \in \mathbb{R}$ in the form

$$F(y,x;\lambda) = \tilde{F}(y,x;\lambda)\tilde{F}^{-1}(x,x;\lambda). \tag{37}$$

Consequently, if one sets $y = x + 2\pi$ in this formula and defines

$$\tilde{k}(\lambda) := \lambda^{-1}[\tilde{e}_1(x+2\pi,x;\lambda) - \tilde{e}_2(x+2\pi,x;\lambda)]^{-1}, \tag{38}$$

it follows from (37) that the exact matrix representation

$$\tilde{S}(x;\lambda) = \begin{pmatrix} \frac{\lambda^2-\tilde{u}\tilde{v}}{2\lambda(\lambda^2+\tilde{u}\tilde{v})} & \frac{\tilde{u}}{\lambda^2+\tilde{u}\tilde{v}} \\ \frac{\tilde{v}}{\lambda^2+\tilde{u}\tilde{v}} & \frac{\tilde{u}\tilde{v}-\lambda^2}{2\lambda(\lambda^2+\tilde{u}\tilde{v})} \end{pmatrix}, \tag{39}$$

satisfies the necessary condition (32) as $\lambda \to \infty$ .

**Remark 1.** *The invariance of the functional* (37) *with respect to the generating vector field* (16) *on the manifold $M$ derives from the representation* (34)*, the evolution equations* (31) *and*

$$\frac{d}{dt}\tilde{F}(y,x_0;\mu) = \left(\frac{\lambda^3}{\mu-\lambda}\tilde{S}(x;\lambda) + \lambda\tilde{S}_0(x)\right)\tilde{F}(y,x_0;\mu), \tag{40}$$

*which follows naturally from the determining matrix flows* (13) *upon applying the translation $y \to y + 2\pi$.*

The matrix expression (39) coincides as $\lambda \to \infty$ with the asymptotic expansion (10), whose matrix elements satisfy the following important functional relationships:

$$\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}} = \tilde{u}, \ \frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{12}} = \tilde{v}, \tag{41}$$

allowing to introduction in a natural way of the vertex vector field (19). To show this, we need to take the preliminary step of deriving the corresponding evolution equation for the vector function $(\tilde{u},\tilde{v})^\top \in M$ with respect to the generating vector field (16) in the asymptotic form (17) as $\lambda \to \infty$. Before doing this we shall find the form of evolution equation (18) as $\mu,\lambda \to \infty$ :

$$\frac{d}{dt}\tilde{S}(x;\mu) = [\lambda^3 \frac{d}{d\lambda}\tilde{S}(x;\mu) - \lambda\tilde{S}_0(x), \tilde{S}(x;\lambda)], \tag{42}$$

which entails the following differential relationships:

$$\begin{array}{c} d\tilde{s}_{11}/dt = \lambda^3(\tilde{s}_{21}d\tilde{s}_{12}/d\lambda - \tilde{s}_{12}d\tilde{s}_{21}/d\lambda), \\ d\tilde{s}_{22}/dt = \lambda^3(\tilde{s}_{12}d\tilde{s}_{21}/d\lambda - \tilde{s}_{21}d\tilde{s}_{12}/d\lambda), \\ d\tilde{s}_{22}/dt = \lambda^3[\tilde{s}_{12}\frac{d}{d\lambda}(\tilde{s}_{11}-\tilde{s}_{22}) - (\tilde{s}_{11}-\tilde{s}_{22})\frac{d\tilde{s}_{12}}{d\lambda}) - \lambda\tilde{s}_{12}, \\ d\tilde{s}_{11}/dt = \lambda^3[\tilde{s}_{21}\frac{d}{d\lambda}(\tilde{s}_{22}-\tilde{s}_{11}) - (\tilde{s}_{22}-\tilde{s}_{11})\frac{d\tilde{s}_{21}}{d\lambda}) + \lambda\tilde{s}_{21}. \end{array} \tag{43}$$

Using the relationships (43), one can easily obtain by means of simple, but rather cumbersome calculations, the evolution equations for the vector function $(\tilde{u},\tilde{v})^\top \in M$ expressed in the form (41)

$$\begin{array}{c} \frac{d}{dt}[\frac{1-\lambda(s_{11}-s_{22})}{2s_{12}}] = -\lambda^2\frac{d}{d\lambda}[\frac{1-\lambda(s_{11}-s_{22})}{2s_{12}}], \\ \frac{d}{dt}[\frac{1-\lambda(s_{11}-s_{22})}{2s_{12}}] = \lambda^2\frac{d}{d\lambda}[\frac{1-\lambda(s_{11}-s_{22})}{2s_{12}}], \end{array} \tag{44}$$

which hold as $\lambda \to \infty$. As a direct consequence of the differential relationships (44), the following *vertex operator representation* for the vector function $(\tilde{u},\tilde{v})^\top \in M$

$$\begin{array}{l} \tilde{u}(t;\lambda) := u^+(t;\lambda) = \hat{X}^+_\lambda u(t), \\ \tilde{v}(t;\lambda) := v^-(t;\lambda) = \hat{X}^-_\lambda u(t), \end{array} \tag{45}$$

holds. Here we took into account that, owing to the determining functional representations (36), the limits

$$\lim_{\lambda\to\infty}\tilde{u}(t;\lambda) = u(t), \qquad \lim_{\lambda\to\infty}\tilde{v}(t;\lambda) = v(t), \tag{46}$$

exist and the vertex operator $\hat{X}_\lambda : M \to M$ acts on the functional manifold $M$ via the corresponding shift operators defined above by means of the differential relationships (19) and (20). The vertex representation (45) allows, in particular, to readily construct infinite hierarchies of the conservation laws for the generating AKNS integrable vector field (16) as

$$H_+(\lambda) := \int_0^{2\pi} u^+(t;\lambda)v(t)dx, \ H_-(\lambda) := \int_0^{2\pi} v^-(t;\lambda)u(t)dx, \tag{47}$$

which follow from (34), (35), and reasoning from Remark (1). Since the fundamental matrix (37) at $y = x + 2\pi$ defines the solution

$$S(x;\lambda) := \tilde{F}(x+2\pi, x;\lambda)\tilde{F}^{-1}(x,x;\lambda) \tag{48}$$

to the determining equations (8) and (11), its determinant $\det S(x;\lambda)$ is invariant with respect to the generating vector field (16) and equals $\det S(x;\lambda) = \det \tilde{F}(x+2\pi,x;\lambda)\det \tilde{F}^{-1}(x,x;\lambda) = 1$ for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ owing to the condition tr $\ell(x;\lambda) = 0$. Accordingly, based on the matrix representation (34), one finds that the relationships

$$\begin{gathered} \tilde{e}_1(x+2\pi,x;\lambda) := \exp\left[\pi\lambda + \lambda^{-1}H_+(\lambda)\right], \\ \tilde{e}_2(x+2\pi,x;\lambda) := \exp\left[-\pi\lambda - \lambda^{-1}H_-(\lambda)\right], \\ \tilde{e}_1(x+2\pi,x;\lambda)\tilde{e}_2(x+2\pi,x;\lambda) = 1, \\ \tfrac{d}{dt}\tilde{e}_1(x+2\pi,x;\mu) = 0 = \tfrac{d}{dt}\tilde{e}_2(x+2\pi,x;\mu) \end{gathered} \tag{49}$$

hold for all $x \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{C}$. As a consequence of (49), we obtain

$$H_+(\lambda) = H_-(\lambda) \tag{50}$$

for all $\lambda \in \mathbb{C}$; that is, the two hierarchies of conservations law (47) coincide. Concerning the AKNS hierarchy vector fields (16) and the related Hamiltonian flows on the manifold $M$, we can easily derive them from the canonical vertex representations (45), taking into account the recursive functional equations (36). We obtain from that (36) and (47) that

$$\begin{aligned} X_\lambda^+ u = u^+ &= u + \lambda^{-1}u_x + \lambda^{-2}[u_{xx}^+ + (u^+)^2 v] + \lambda^{-3}[(u^+)^2 v]_x = ..., \\ X_\lambda^- v = v^- &= v - \lambda^{-1}v_x - \lambda^{-2}[v_{xx}^- + (v^-)^2 u] + \lambda^{-3}[(v^-)^2 u]_x = ..., \end{aligned} \tag{51}$$

which immediately yield the whole AKNS hierarchy of nonlinear Lax integrable dynamical systems on the functional manifold $M$. For instance,

$$\frac{d}{dt_1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \frac{d}{dt_2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{xx} + 2u^2 v \\ -v_{xx} + 2v^2 u \end{pmatrix}, ..., \tag{52}$$

and so on.

## 2.4 The $\tau$-function representation

The vertex operator representations (34), (39) and (46) can also be naturally associated with the results in [2, 3], based on the generating $\tau$-function approach. The latter makes extensive use of the versatile dual representation (29) for the generating current algebra element $\tilde{S}(x;\lambda) \in \tilde{\mathcal{G}}_-^*$ (as $\lambda \to \infty$) for the AKNS flows with the specially chosen invariant matrix

$$\tilde{C}(x_0;\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \operatorname{End} \mathbb{C}^2. \tag{53}$$

In the context of our approach, the relation with the $\tau$-function representation devised in [2, 3] can be based on the matrix solution (34) and the simple vertex operator mapping properties

$$\hat{X} \begin{pmatrix} \tilde{e}_1(x,y;\lambda) \\ \tilde{e}_2(x,y;\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{e}_2(y,x;\lambda) \\ \tilde{e}_1(y,x;\lambda) \end{pmatrix}, \tag{54}$$

which follow directly from the definitions (35) and (45).

As a result of (34) and (54), the crucial expression for the normalized matrix

$$\begin{aligned} \bar{F}(y,x;\lambda) &:= \left(\det \tilde{F}(x,x;\lambda)\right)^{-1/2} \tilde{F}(y,x;\lambda) = \\ &= \begin{pmatrix} \frac{\lambda \tilde{e}_2^-(x,y;\lambda)}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} & -\frac{u^+(y;\lambda)\tilde{e}_1^+(x,y;\lambda)}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} \\ \frac{v^-(y;\lambda)\tilde{e}_2^+(x,y;\lambda)}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} & \frac{\lambda \tilde{e}_1^+(x,y;\lambda)}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]^{1/2}} \end{pmatrix} := \\ &= \begin{pmatrix} \frac{\tau^-(y,x;\lambda)}{\tau(y,x;\lambda))} & -\frac{u^+(y;\lambda)\tau^+(y,x;\lambda)}{\lambda\tau(y,x;\lambda)} \\ \frac{v^-(y;\lambda)\tau^-(y,x;\lambda)}{\lambda\tau(y,x;\lambda)} & \frac{\tau^+(y,x;\lambda)}{\tau(y,x;\lambda)} \end{pmatrix}, \end{aligned} \tag{55}$$

holds, where we defined the quantities

$$\begin{aligned} \frac{\tau^-(y,x;\lambda)}{\tau(y,x;\lambda)} &:= \frac{\lambda \tilde{e}_2^-(x,y;\lambda)}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]^{1/2}}, \\ \frac{\tau^+(y,x;\lambda)}{\tau(y,x;\lambda)} &:= \frac{\lambda \tilde{e}_1^+(x,y;\lambda)}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]^{1/2}}, \end{aligned} \tag{56}$$

satisfying the compatibility relationship

$$\frac{\tau^+(y,x;\lambda)\tau^-(y,x;\lambda)}{\tau(y,x;\lambda)^2} = \frac{\lambda^2 \exp[\lambda^{-1}\int_x^y (u^+v - uv^-)ds]}{[\lambda^2+u^+(x;\lambda)v^-(x;\lambda)]}. \tag{57}$$

The vertex operator expression (55), as is easily checked, can be readily employed to derive the representation (30), where the exact result (39) entails the additional application of the useful [2] vertex representation

$$\bar{F}(y,x;\lambda)=\frac{1}{\tau(y,x;\lambda)}\begin{pmatrix}\tau^{-}(y,x;\lambda) & -\lambda^{-1}\sigma^{+}(y,x;\lambda)\\ \lambda^{-1}\rho^{-}(y,x;\lambda) & \tau^{+}(y,x;\lambda)\end{pmatrix}, \tag{58}$$

which holds as $\lambda\to\infty$ if $\rho(y,x;\lambda):=v(y)\tau(y,x;\lambda),\sigma(y,x;\lambda):=u(y)\times\tau(y,x;\lambda),\ x,y\in\mathbb{R}$, and mappings $\rho^{-}$ and $\sigma^{+}$ are defined in the obvious fashion. In this regard, it should be noted that the vertex operator representation (58) for the matrix (55) was obtained in [2] as a special normalized solution to the determining equation (31). Taking into account these two dual vertex representations of the AKNS hierarchy of integrable flows on the functional manifold $M$, one can see that the first one — presented in this work — is both technically simpler and more effective in obtaining exact descriptions of such important functional ingredients as conservation laws, symplectic structures and related commuting vector fields.

# 3 The Gurevich-Zybin system vertex representation analysis

## 3.1 The Gurevich-Zybin system and its algebraic structure description

In this Section we are interested in constructing the so called *vertex operator representations* [2, 3, 25, 26, 27] for solutions to the Riemann type hydrodynamical hierarchy (1) for the case $N=2:$

$$\left\{\begin{array}{l} D_t u=u_t+uu_x=v,\\ D_t v=v_t+uv_x=0,\end{array}\right. \tag{59}$$

whose Lax $\ell$-operator equals

$$\ell[u,v;\lambda]:=\begin{pmatrix}-\lambda u_x/2 & -v_x\\ \lambda^2/2 & \lambda u_x/2\end{pmatrix}, \tag{60}$$

where we denoted $v:=D_t u$, and for the case $N=3:$

$$\begin{array}{c} du_1/dt=u_2-u_1u_{1,x},\\ du_2/dt=u_3-u_1u_{2,x},\\ du_3/dt=-u_1u_{3,x},\end{array} \tag{61}$$

whose Lax $\ell$-operator equals

$$\ell[u;\lambda] = \left( \begin{array}{ccc} \lambda u_{2,x} & u_{3,x} & 0 \\ 0 & \lambda u_{N-1,x} & 2u_{3,x} \\ -3\lambda^3 & -\lambda^2 3u_{1,x} & -\lambda^2 2u_{2,x} \end{array} \right), \tag{62}$$

where we denoted $u_1 := u$, and $u_2 := D_t u$ and $u_3 := D_t^2 u$, making use an approach recently devised in [27, 28] for the case of the classical AKNS hierarchy of integrable flows, and which can be easily generalized for treating the problem for arbitrary integers $N \in \mathbb{Z}_+$. We begin with a Lax type linear spectral problem [12, 15, 29] for equation (59) defined on the space of smooth real-valued $2\pi$-periodic functions $(u, v)^\intercal \in M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$ :

$$df/dx = \ell[u, v; \lambda]f, \qquad \ell[u, v; \lambda] := \left( \begin{array}{cc} -\lambda u_x/2 & -v_x \\ \lambda^2/2 & \lambda u_x/2 \end{array} \right), \tag{63}$$

where, by definition, $v := D_t u$, $f \in L_\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is a spectral parameter. Assume that a vector function $(u, v)^\top \in M$ depends parametrically on the infinite set $t := \{t_1, t_2, t_3, \ldots\} \in \mathbb{R}^{\mathbb{Z}_+}$ in such a way that the generalized Floquet spectrum [4, 6, 20] $\sigma(\ell) := \{\lambda \in \mathbb{C} : \sup_{x\in\mathbb{R}} ||f(x; \lambda)||_\infty < \infty\}$ of the linear problem (63) persists in being parametrically iso-spectral, that is $d\sigma(\ell)/dt_j = 0$ for all $t_j \in \mathbb{R}$. The iso-spectrality condition gives rise to a hierarchy of commuting to each other nonlinear bi-Hamiltonian dynamical systems on the functional manifold $M$ in the general form

$$\frac{d}{dt_j}(u(t), v(t))^\top = -\vartheta \operatorname{grad} H_j[u, v] := K_j[u(t), v(t)], \tag{64}$$

where $K_j : M \to T(M)$ and $H_j \in \mathcal{D}(M), j \in \mathbb{Z}_+$, are, respectively, vector fields and conservation laws, on the manifold $M$, which were described before in [15, 18, 29],

$$\vartheta := \left( \begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right) \tag{65}$$

is a Poisson structure on the manifold $M$ and, by definition,

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} := \begin{pmatrix} u(x, t_1, t_{2,} t_{3,} \ldots) \\ v(x.t_1, t_{2,} t_{3,} \ldots) \end{pmatrix} \tag{66}$$

for $t \in \mathbb{R}^{\mathbb{N}}$.

It is well known [4, 6, 7, 20] that the Casimir invariants, determining conservation laws for dynamical systems (64), are generated by the suitably normalized monodromy matrix $\tilde{S}(x;\lambda) \in \text{End } \mathbb{C}^2$ of the linear problem (63)

$$\tilde{S}(x;\lambda) = k(\lambda)S(x;\lambda) - \frac{k(\lambda)}{2}\text{tr}S(x;\lambda), \tag{67}$$

where $F(y,x;\lambda) \in \text{End } \mathbb{C}^2$ is the matrix solution to the Cauchy problems

$$\frac{d}{dy}F(y,x;\lambda) = \ell(y;\lambda)F(y,x;\lambda), \quad F(y,x;\lambda)|_{y=x} = \mathbf{I}, \tag{68}$$

for all $\lambda \in \mathbb{C}$ and $x, y \in \mathbb{R}$, where $\mathbf{I} \in \text{End } \mathbb{C}^2$ is the identity matrix, $S(x;\lambda) := F(x+2\pi, x;\lambda)$ is the usual monodromy matrix for the equation (68). Here the parameter $k(\lambda) \in \mathbb{C}$ is invariant with respect to flows (64) and is chosen in such a way that the asymptotic condition

$$\tilde{S}(x;\lambda) \in \tilde{\mathcal{G}}_- \tag{69}$$

as $\lambda \to \infty$ holds for all $x \in \mathbb{R}$. Here $\tilde{\mathcal{G}}_- \subset \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ is the natural splitting into two affine subalgebras of positive and negative $\lambda$-expansions of the centrally extended [4, 8] affine current $\mathfrak{sl}(2)$-algebra $\hat{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$ :

$$\tilde{\mathcal{G}} := \{a = \sum_{j\in\mathbb{Z},\, j\ll\infty} a^{(j)} \otimes \lambda^j : a^{(j)} \in C^\infty\left(\mathbb{R}/2\pi\mathbb{Z}; \mathfrak{sl}(2;\mathbb{C})\right)\}. \tag{70}$$

The latter is endowed with the Lie commutator

$$[(a_1,c_1),(a_2,c_2)] := ([a_1,a_2], \langle a_1, da_2/dx\rangle), \tag{71}$$

where the scalar product is defined as

$$\langle a_1, a_2\rangle := \text{res}_{\lambda=\infty} \int_0^{2\pi} \text{tr}(a_1 a_2)dx \tag{72}$$

for any two elements $a_1, a_2 \in \tilde{\mathcal{G}}$ with "res" and "tr" being the usual residue and trace maps, respectively. As the spectrum $\sigma(\ell) \subset \mathbb{C}$ of the problem

(63) is supposed to be parametrically independent, flows (64) are naturally associated with evolution equations

$$d\tilde{S}/dt_j = [(\lambda^{j+1}\tilde{S})_+, \tilde{S}] \tag{73}$$

for all $j \in \mathbb{R}$, which are generated by the set $I(\hat{\mathcal{G}}^*)$ of Casimir invariants of the coadjoint action of the current algebra $\hat{\mathcal{G}}$ on a given element $\ell(x;\lambda) \in \tilde{\mathcal{G}}_-^* \cong \tilde{\mathcal{G}}_+$ contained in the space of smooth functionals $\mathcal{D}(\hat{\mathcal{G}})$. In particular, a functional $\gamma(\lambda) \in I(\hat{\mathcal{G}})$ if and only if

$$[\tilde{S}(x;\lambda), \ell(x;\lambda)] + \frac{d}{dx}\tilde{S}(x;\lambda) = 0, \tag{74}$$

where the gradient $\tilde{S}(x;\lambda) := \mathrm{grad}\gamma(\lambda)(\ell) \in \tilde{\mathcal{G}}_-$ is defined with respect to the scalar product (72) by means of the variation

$$\delta\gamma(\lambda) := \langle \mathrm{grad}\gamma(\lambda)(\ell), \delta\ell \rangle \,. \tag{75}$$

To construct the solution to matrix equation (74), we find preliminary a partial solution $\tilde{F}(y,x;\lambda) \in \mathrm{End}\ \mathbb{C}^2$, $x, y \in \mathbb{R}$, to equation (68) satisfying the asymptotic Cauchy data

$$\tilde{F}(y,x;\lambda)|_{y=x} = \mathbf{I} + O(1/\lambda) \tag{76}$$

as $\lambda \to \infty$. It is easy to check that

$$\tilde{F}(y,x;\lambda) = \begin{pmatrix} \tilde{e}_1(y,x;\lambda) & -\frac{\tilde{\beta}(y;\lambda)}{\lambda}\tilde{e}_2(y,x;\lambda) \\ -\frac{\lambda}{\tilde{\alpha}(y;\lambda)}\tilde{e}_1(y,x;\lambda) & \tilde{e}_2(y,x;\lambda) \end{pmatrix}, \tag{77}$$

is an exact functional solution to (68) satisfying condition (76), where we have defined

$$\tilde{e}_1(y,x;\lambda) := \exp\{\frac{\lambda}{2}[u(x) - u(y)] + \lambda\int_x^y \tilde{\alpha}\ dv(s)\}, \tag{78}$$
$$\tilde{e}_2(y,x;\lambda) := \exp\{\frac{\lambda}{2}[u(y) - u(x)] - \frac{\lambda}{2}\int_x^y \tilde{\beta}\ ds\},$$

with the vector-functions $\alpha^\pm \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ satisfying the following determining functional relationships:

$$\begin{aligned} \tilde{\alpha} &= u_x + (u_x^2 - 2v_x + \xi\tilde{\alpha})^{1/2}, \\ \tilde{\beta} &= u_x - (u_x^2 - 2v_x + \xi\tilde{\beta})^{1/2}, \end{aligned} \tag{79}$$

as $\xi := 1/\lambda \to 0$ and existing when the condition $\varphi(x,t) := \sqrt{u_x^2 - 2v_x} \neq 0$ on the manifold $M$ at $t = 0 \in \mathbb{R}^{\mathbb{N}}$.

The fundamental matrix $F(y,x;\lambda) \in \mathrm{End}\ \mathbb{C}^2$ can be represented for all $x, y \in \mathbb{R}$ in the form

$$F(y,x;\lambda) = \tilde{F}(y,x;\lambda)\tilde{F}^{-1}(x,x;\lambda). \tag{80}$$

Consequently, if one sets $y = x + 2\pi$ in this formula and defines the expression

$$k(\lambda) := \lambda^{-1}[\tilde{e}_1(x+2\pi,x;\lambda) - \tilde{e}_2(x+2\pi,x;\lambda)]^{-1}, \tag{81}$$

it follows from (67), (77), and (80) that the exact functional matrix representation

$$\tilde{S}(x;\lambda) = \begin{pmatrix} \frac{[\tilde{\alpha}(x;\lambda)+\tilde{\beta}(x;\lambda)]}{2\lambda[\tilde{\alpha}(x;\lambda)-\tilde{\beta}(x;\lambda)]} & \frac{\tilde{\alpha}\tilde{\beta}}{\lambda^2[\tilde{\alpha}(x;\lambda)-\tilde{\beta}(x;\lambda)]} \\ -\frac{1}{[\tilde{\alpha}(x;\lambda)-\tilde{\beta}(x;\lambda)]} & \frac{[\tilde{\beta}(x;\lambda)+\tilde{\alpha}(x;\lambda)]}{2\lambda[\tilde{\beta}(x;\lambda)-\tilde{\alpha}(x;\lambda)]} \end{pmatrix}, \tag{82}$$

satisfies the necessary condition (69) as $\lambda \to \infty$.

**Remark 2.** *The invariance of the expression* (81) *with respect to the generating vector field* (64) *on the manifold $M$ derives from the representation* (80)*, the equations* (74) *and*

$$\frac{d}{dt}\tilde{F}(y,x_0;\mu) = \frac{\lambda^3}{\mu-\lambda}\tilde{S}(x;\lambda)\tilde{F}(y,x_0;\mu), \tag{83}$$

*which follows naturally from the determining matrix flows* (73) *upon applying the translation $y \to y + 2\pi$.*

The matrix expression (82) gives rise to the following important functional relationships:

$$\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}} = \tilde{\alpha}, \ \frac{-2\lambda^2\tilde{s}_{12}}{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})} = \tilde{\beta}, \tag{84}$$

which allow to introduce in a natural way the vertex operator vector fields

$$X_\lambda^{\pm} = \exp(\pm D_\lambda), \quad D_\lambda := \sum_{j\in\mathbb{Z}_+} \frac{1}{(j+1)}\lambda^{-(j+1)}\frac{d}{dt_{j+1}}, \tag{85}$$

acting on an arbitrary smooth function $\eta \in C^\infty(\mathbb{R}^{\mathbb{Z}_+};\mathbb{R})$ by means of the shifting mappings:

$$\begin{gathered} X_\lambda^\pm \, \eta(x,t_1,t_2,...,t_j,...) := \eta^\pm(x,t;\lambda) = \\ = \eta(x,t_1 \pm 1/\lambda, t_2 \pm /(2\lambda^2), t_3 \pm 1/(3\lambda^3)..., t_j \pm 1/(j\lambda^j),...) \end{gathered} \tag{86}$$

as $\lambda \to \infty$. Namely, we following proposition holds.

**Proposition 1.** *The functional vertex operator expressions*

$$\begin{aligned} \tilde{\alpha}(x,t;\lambda) &= X_\lambda^- \alpha(x,t) = \alpha^-(x,t;\lambda), \\ \tilde{\beta}(x,t;\lambda) &= X_\lambda^+ \beta(x,t)) = \beta^+(x,t;\lambda) \end{aligned} \tag{87}$$

*solve the functional equations* (79), *that is*

$$\begin{aligned} \alpha^- &= u_x + (u_x^2 - 2v_x + \xi\alpha^-)^{1/2}, \\ \beta^+ &= u_x - (u_x^2 - 2v_x + \xi\beta^+)^{1/2}, \end{aligned} \tag{88}$$

*where* $t \in \mathbb{R}^{\mathbb{Z}_+}$ *and* $\xi = 1/\lambda \to 0$.

***Proof.*** To state this proposition it is enough to show that the following relationships hold:

$$\begin{aligned} \frac{d}{d\xi}\left[\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}}\right]_{\lambda=1/\xi} &= \frac{d}{dt}\left[\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}}\right]_{\lambda=1/\xi}, \\ \frac{d}{d\xi}\left[\frac{-8\lambda^2\tilde{s}_{12}}{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}\right]_{\lambda=1/\xi} &= \frac{d}{dt}\left[\frac{-8\lambda^2\tilde{s}_{12}}{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}\right]_{\lambda=1/\xi} \end{aligned} \tag{89}$$

for any parameter $\xi \to 0$, where by definition

$$\frac{d}{dt} := \frac{d}{d\xi} D_\lambda \bigg|_{\lambda=1/\xi} = \sum_{j\in\mathbb{Z}_+} \xi^j \frac{d}{dt_{j+1}} \tag{90}$$

is a generating evolution vector field. Before doing this we find the evolution equation

$$\frac{d}{dt}\tilde{S}(x;\mu) = [\lambda^3 \frac{d}{d\lambda}\tilde{S}(x;\mu), \tilde{S}(x;\lambda)] \tag{91}$$

on the matrix $\tilde{S}(x;\mu)$ as $\mu,\lambda\to\infty$, which entails the following differential relationships:

$$\begin{aligned} d\tilde{s}_{11}/dt &= \lambda^3(\tilde{s}_{21}d\tilde{s}_{12}/d\lambda-\tilde{s}_{12}d\tilde{s}_{21}/d\lambda),\\ d\tilde{s}_{22}/dt &= \lambda^3(\tilde{s}_{12}d\tilde{s}_{21}/d\lambda-\tilde{s}_{21}d\tilde{s}_{12}/d\lambda),\\ d\tilde{s}_{22}/dt &= \lambda^3[\tilde{s}_{12}\tfrac{d}{d\lambda}(\tilde{s}_{11}-\tilde{s}_{22})-(\tilde{s}_{11}-\tilde{s}_{22})\tfrac{d\tilde{s}_{12}}{d\lambda}),\\ d\tilde{s}_{11}/dt &= \lambda^3[\tilde{s}_{21}\tfrac{d}{d\lambda}(\tilde{s}_{22}-\tilde{s}_{11})-(\tilde{s}_{22}-\tilde{s}_{11})\tfrac{d\tilde{s}_{21}}{d\lambda}). \end{aligned} \tag{92}$$

Using these relationships (92), one can easily obtain by means of simple, but rather cumbersome calculations, the needed relationships (89). As their direct consequences the *vertex operator representations* (87) for the vector functions $\tilde{\alpha},\tilde{\beta}\in C(\mathbb{R}^{\mathbb{Z}_+};\mathbb{R})$ hold. $\square$

Now we take into account that, owing to the determining functional representations (79), taht the limits$^\infty$

$$\begin{aligned} \lim_{\lambda\to\infty}\alpha^-(x,t;\lambda) &= u_x(x,t)+\varphi(x,t), && (93)\\ \lim_{\lambda\to\infty}\beta^+(x,t;\lambda) &= u_x(x,t)-\varphi(x,t), && \varphi(x,t):=\sqrt{u_x^2(x,t)-2v_x(x,t)}, \end{aligned}$$

exist on the mamifold $M$. Moreover, having iterated the functional relationships (79), one can find that

$$\begin{aligned} X_\lambda^-\alpha &= \alpha^- = u_x+\varphi+\xi(\frac{u_{xx}}{\varphi}+\frac{\varphi_x}{\varphi})+\\ &+\frac{\xi^2}{2}(\frac{u_{xx}^2+2u_{xx}\varphi_x-u_{3x}\varphi}{\varphi^3}+\frac{\varphi_{xx}\varphi+5\varphi_x^2}{\varphi^3})+O(\xi^3),\\ X_\lambda^+\beta &= \beta^+ = u_x-\varphi-\xi(\frac{u_{xx}}{\varphi}-\frac{\varphi_x}{\varphi})-\\ &-\frac{\xi^2}{2}(\frac{u_{xx}^2-2u_{xx}\varphi_x+u_{3x}\varphi}{\varphi^3}+\frac{\varphi_{xx}\varphi+5\varphi_x^2}{\varphi^3})+O(\xi^3), \end{aligned} \tag{94}$$

which immediately yield the higher Riemann type commuting nonlinear Lax integrable dispersive dynamical systems on the functional manifold $M$. For instance, making use of the relationships

$$\lim_{\lambda\to\infty}[\alpha^-(x,t;\lambda)\pm\beta^+(x,t;\lambda)]/2=\left\{\begin{array}{l} u_x(x,t),\\ \varphi(x,t), \end{array}\right. \tag{95}$$

one easily obtains that

$$\frac{d}{dt_1}\begin{pmatrix}u_x\\ \varphi\end{pmatrix}=\begin{pmatrix}-u_{xx}/\varphi\\ -\varphi_x/\varphi\end{pmatrix},\ \frac{d}{dt_2}\begin{pmatrix}u_x\\ \varphi\end{pmatrix}=\begin{pmatrix}(u_{xx}^2+7\varphi_x^2)/\varphi^3\\ (2u_{3x}\varphi-4u_x\varphi_x)/\varphi^3\end{pmatrix},\ldots, \tag{96}$$

and so on, where $\varphi = \sqrt{u_x^2 - 2v_x}$ and we took into account that the following asymptotic expansiopns hold

$$\begin{array}{c} X_\lambda^- \alpha(x,t;\lambda) = u_x + \varphi - \xi(u_{x,t_1} + \varphi_{t_1}) + \\ + \frac{\xi^2}{2}(u_{x,t_1,t_1} + \varphi_{t_1,t_1} - u_{x,t_2} - \varphi_{t_2}) + O(\xi^3), \\ X_\lambda^+ \beta(x,t;\lambda) = u_x - \varphi + \xi(u_{x,t_1} - \varphi_{t_1}) + \\ + \frac{\xi^2}{2}(u_{x,t_1,t_1} - \varphi_{t_1,t_1} + u_{x,t_2} - \varphi_{t_2}) + O(\xi^3) \end{array} \tag{97}$$

as $\xi = 1/\lambda \to 0$.

It is worth here to mention that the scheme devised above for finding the corresponding vertex operator representations for the Riemann type equation (59) can be similarly generalized for treating other equations of the infinite hierarchy (1) when $N \geq 3$, having taken into account the existence of their suitable Lax type representations found before in recent works [16, 15, 29].

## 4 Concluding remarks

The vertex operator functional representations of the matrix solutions (34) and (39) for the determining equations (30) and (8), respectively, as one can see from the above analysis, are essentially derived from the intrinsic Lie-algebraic structure (13) of the generating vector field (16) on the manifold $M$. As the decisive property of the vertex operator relationships (45) and (47) is fundamentally based on the representations (41) and equations (43), they provide a very straightforward and transparent explanation of many of "miraculous" calculations in [2, 3]. Of course, the results for the AKNS hierarchy in these earlier papers were obtained in a distinctly different manner; namely by means of direct asymptotic power series expansions of solutions to the determining matrix equations (30) and (34).

It should be noted that, in a certain sense, the effectiveness of our approach to studying the vertex operator representation of the AKNS hierarchy owes much to the important exact representation (28) for the solution of the Casimir invariants determining equation (8). This equation entails the extremely effective AKNS hierarchy representation in the simple recursive form (51), which explains several other very interesting results in the literature, such as in [25, 26]. On the other hand, the dual solution representation to (8) in the form (28), used extensively in [2], led naturally to the introduction of the well-known $\tau$-function and made it possible to present the whole AKNS hierarchy in terms of its suitable partial derivatives. Nonetheless, both our vertex operator approach and the $\tau$-function method, as was briefly demonstrated above, are intimately related to each other.

The vertex operator functional representations of the solution to the Riemann type hydrodynamical equations (59) in the form (88) and (93) in the form (94) is crucially based on the corresponding representations (84), (85) and evolution equations (89), (91), which also explains many of vertex operator calculations presented before both in [2, 3] and in [25]. It should be noted that the effectiveness of our approach to studying the vertex operator representation of the Riemann type hierarchy owes much to the important exact representation (82) for the corresponding monodromy matrix, whose properties are described by means of applying the standard [4, 7, 20, 21] Lie-algebraic techniques. As an indication of possible future research, it should also be mentioned that it would be interesting to generalize the vertex operator approach devised in this work to other linear spectral problems such as those related to dynamical systems with a parametrical spectral [13, 20, 30] dependence, spatially two-dimensional [31], Pavlov's and heavenly [32] dynamical systems, and to the BSR systems studied recently in [13, 14, 15, 16].

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# НОВІ АСПЕКТИ ІНТЕГРОВНОСТІ ІЄРАРХІЇ АКНС ТА ДИНАМІЧНОЇ СИСТЕМИ ГУРЄВІЧА-ЗИБІНА

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Запропоновано новий підхід до вивчення ієрархії АКНС та нелінійної динамічної системи Гурєвіча-Зибіна, що грунтується на вершинному операторному представленні. Показано, що цей метод дає інтерпретацію незвичайних властивостей ієрархії АКНС за допомогою ясної, простої, натуральної і ефективної для застосувань конструкції. Також аналізується зв'язок вершинного операторного методу з Лі-алгебраїчною схемою інтегровності та коротко обговорюється підхід на основі $\tau$-функціонального представлення. Розвинуто також підхід, що грунтується на спектральних та Лі-алгебраїчних методах, до знаходження вершинного операторного представлення для гідродинамічної системи Гурєвіча-Зибіна. Знайдено функціональне представлення, що генерує нескінченну ієрархію дисперсійних інтегровних за Лаксом гамільтонових потоків.