Математичний Вісник
Наукового товариства
імені Шевченка
2013. — Т.10

Mathematical Bulletin
of the Shevchenko
Scientific Society
2013. — V.10

INTEGRATION OF NONLINEAR EQUATIONS OF THE SOLITON THEORY BY THE PROJECTION METHOD AND DARBOUX-LIKE TRANSFORMATIONS

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O. Chvartatskyi, *Integration of nonlinear equations of the soliton theory by the projection method and Darboux-like transformations*, Math. Bull. Shevchenko Sci. Soc. **10** (2013), 193–202.

The notion of generalized constant of Eiler are considered and investigated for general terms of series which are determined by concrete function. Conditions of existing this constant. Two criterions of existing of Eiler constant are founded. Method of using constant of Eiler for calculating numerical value of partial series. It was proved existing of this constant for divergent series with bounded increasing of general term of series.

О. Чвартацький. *Інтегрування нелінійних рівнянь теорії солітонів методом проектування та перетвореннями типу Дарбу* // Мат. вісник НТШ. — 2013. — Т.10. — С. 193–202.

Проведено порівняння методу інтегрування нелінійних рівнянь, запропонованого В.О. Марченком, та підходу, що грунтується на використанні перетворень типу Дарбу. Отримано в явній формі матричне перетворення Дарбу-Крама-Матвєєва другого типу за допомогою методу проектування В.О. Марченка.

1. Introduction

In the modern theory of nonlinear integrable systems algebraic methods play an important role. Among them there are the Zakharov-Shabat dressing method [1, 2], Marchenko's method [3] and an approach based on the Darboux-Crum-Matveev transformations [4, 5]. Algebraic methods allow us to omit analytical difficulties that arise in the investigation of corresponding direct and inverse scattering problems for nonlinear equations. In paper [6] a connection between V.O. Marchenko's projection method and an approach based on Darboux-Crum-Matveev transformations were investigated. In particular, the general matrix

2010 *Mathematics Subject Classification:* 35Q51, 37K35, 35C08 *УДК:* 517.957, 517.958
Key words and phrases: soliton theory, Darboux transformation, Heisenberg equation
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Darboux-Crum-Matveev transformation of the first type was obtained via V.O. Marchenko's ideas. The aim of this paper is to investigate the connections between the projection method and the differential Darboux-Crum-Matveev transformations of the second type.

This work is organized as follows. In Section 2 we present a short description of the projection method and its applications to the integration of nonlinear integrable systems. As an example we consider the Heisenberg equation. In Section 3 we introduce the Darboux-Crum transformation of the second type and demonstrate its application to the nonlinear equations of mathematical physics. In this section (Subsection 3.1) we also construct the matrix Darboux-Crum-Matveev transformation of the second type via V.O. Marchenko's ideas. It is the main result of this paper which is presented by Theorem 7. This theorem provides us with a possibility to construct solutions of nonlinear equations (including Heisenberg and Ishimori equations) via invariant transformations of the linear differential operators that are involved in Lax pairs. In the final section, we discuss the obtained results and mention problems for further investigations.

2. A projection method and exact solutions of the Heisenberg system

Consider the linear system of the following form:

$$\begin{cases} \alpha_2\varphi_{t_2} + B\varphi_{xx} = 0, \\ B\varphi_x = \varphi A, \end{cases} \tag{1}$$

where φ is a $(2N \times 2N)$-dimensional matrix of functions, $\alpha_2 \in \mathbb{R} \cup i\mathbb{R}$; A, B are $(2N \times 2N)$-constant matrices, $B^2 = I_{2N}$ (I_{2N} denotes $(2N \times 2N)$-identity matrix). The following proposition is proven in [3]:

Proposition 1. $(2N \times 2N)$*-dimensional matrix of functions*

$$S = \Phi^{-1} B \Phi, \tag{2}$$

where $\Phi = \varphi_x\varphi^{-1}$ *and* φ *is a solution of system (1), satisfies the matrix equation:*

$$-4\alpha_2 S_{t_2} = [S, S_{xx}]. \tag{3}$$

In case $N = 1$, $S = S^* = S^{-1}$ equation (3) becomes the Heisenberg equation.

Now we shall consider the structure and properties of the matrix-valued function Φ that arises in formula (2):

$$\Phi = \hat{\varphi}_x \hat{\varphi}^{-1}, \tag{4}$$

where $\hat{\varphi}$ is the $(Nk \times Nk)$-dimensional Wronski matrix of the following form:

$$\hat{\varphi} = \begin{pmatrix} \varphi_1 & \dots & \varphi_N \\ \vdots & \vdots & \vdots \\ \varphi_1^{(N-1)} & \dots & \varphi_N^{(N-1)} \end{pmatrix}, \tag{5}$$

where $\varphi_l = \varphi_l(x) = (\varphi_{ij,l})_{i,j=1}^k$, $l = \overline{1, N}$ are $(k \times k)$-dimensional matrices of functions. Let us recall a proposition from paper [6]:

Proposition 2. *Matrix-valued function $\Phi = \hat{\varphi}_x \hat{\varphi}^{-1}$, where $\hat{\varphi}$ is a Wronski matrix (5), has the following form:*

$$\Phi = \begin{pmatrix} 0 & I_k & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_k \\ \Phi_1 & \Phi_2 & \dots & \Phi_N \end{pmatrix}, \tag{6}$$

where I_k is an identity matrix of dimension $(k \times k)$; Φ_j, $j = \overline{1, N}$ are $(k \times k)$-dimensional matrix-valued functions. The inverse matrix Φ^{-1} has the form:

$$\Phi^{-1} = \begin{pmatrix} -\Phi_1^{-1}\Phi_2 & -\Phi_1^{-1}\Phi_3 & \dots & -\Phi_1^{-1}\Phi_N & \Phi_1^{-1} \\ I_k & 0 & \dots & 0 & 0 \\ 0 & I_k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_k & 0 \end{pmatrix}. \tag{7}$$

In order to find the exact solutions of the Heisenberg equation we will need a Wronski matrix that satisfies system (1) with some matrices A, B and $\alpha_2 = -i$. For this purpose we will consider the (2×2)-dimensional matrix-valued solutions φ_l, $l = \overline{1, N}$ of the systems:

$$i\varphi_{lt_2} - \sigma_3 \varphi_{lxx} = 0, \;\; \varphi_{lx} = \sigma_3 \varphi_l a_l, \tag{8}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$, $a_l \in Mat_{2\times 2}(\mathbb{C})$. Let us put

$$B = \mathrm{diag}(\sigma_3, \sigma_3, \dots, \sigma_3) \in Mat_{2N\times 2N}(\mathbb{C}), \;\; A = \mathrm{diag}(a_1, \dots, a_N) \in Mat_{2N\times 2N}(\mathbb{C}). \tag{9}$$

Then the Wronski matrix $\hat{\varphi}$ satisfies system (1) with matrices B and A, defined by formula (9) and $\alpha_2 = -i$:

$$\begin{cases} i\hat{\varphi}_{t_2} - B\hat{\varphi}_{xx} = 0, \\ B\hat{\varphi}_x = \hat{\varphi} A, \end{cases}$$

Using Proposition 1 (formula (2)) and exact form of functions Φ (6), Φ^{-1} (7) and matrix B (9), we obtain that $(2N \times 2N)$-dimensional function S has form:

$$S = \Phi^{-1} B \Phi = \begin{pmatrix} \Phi_1^{-1}\sigma_3\Phi_1 & \Phi_1^{-1}[\sigma_3, \Phi_2] & \dots & \Phi_1^{-1}[\sigma_3, \Phi_N] \\ 0 & \sigma_3 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_3 \end{pmatrix} \tag{10}$$

and satisfies equation (3) with $\alpha_2 = -i$. By substituting the exact form of matrix-valued function S (10) into equation (3), we notice that its (2×2)-dimensional block

$$S_1 := \Phi_1^{-1}\sigma_3\Phi_1 \tag{11}$$

satisfies the Heisenberg equation:

$$4iS_{1,t_2} = [S_1, S_{1,xx}]. \tag{12}$$

Let us put $a_l = \mathrm{diag}(\lambda_l, -\bar{\lambda}_l)$ in system (8). It is shown in [3] that the matrix-valued function S_1 is Hermitian and regular in case of the following choice of the solution of (8):

$$\varphi_l = \begin{pmatrix} \varphi_{11,l} & \varphi_{12,l} \\ \varphi_{21,l} & \varphi_{22,l} \end{pmatrix}, \; l = \overline{1, N}. \tag{13}$$

where $\varphi_{11,l} = e^{\lambda_l x - i\lambda_l^2 t + \theta_{1l}}$, $\varphi_{12,l} = e^{-\bar{\lambda}_l x - i\bar{\lambda}_l^2 t + \theta_{2l}}$, $\varphi_{21,l} = -\bar{\varphi}_{12,l}$, $\varphi_{22,l} = \bar{\varphi}_{11,l}$, $\theta_{1l}, \theta_{2l} \in \mathbb{C}$.

From the exact form of the matrix S_1 (formula (11)) it follows that S_1 is unitary: $I_2 = \Phi_1\sigma_3\Phi_1^{-1}\Phi_1\sigma_3\Phi_1^{-1} = S_1S_1 = S_1S_1^*$, where I_2 is (2×2)-dimensional identity matrix. If we put $N = 1$ in formula (13), then we obtain the following solution of Heisenberg equation (12):

$$\begin{aligned} S_1 &= \begin{pmatrix} s_{11} & s_{12} \\ \bar{s}_{12} & -s_{11} \end{pmatrix}, \\ s_{11} &= 1 - \frac{\cos^2\gamma_1}{\cosh^2(2\mathrm{Re}(\theta_1))}, \\ s_{12} &= \frac{2\cos\gamma_1 \sinh(2\mathrm{Re}(\theta_1) + i\theta_1)\exp(-2i\mathrm{Im}(\theta_1))}{\cosh^2(2\mathrm{Re}(\theta_1))}, \\ \theta_1 &= \lambda_1 x + i\lambda_1^2 t_2 + \alpha_1, \;\; \lambda_1, \alpha_1 \in \mathbb{C}, \;\; \gamma_1 \in \mathbb{R}. \end{aligned} \tag{14}$$

In the following section we will consider solution generating technique for the Heisenberg equation via differential Darboux-Crum-Matveev transformations of the second type that were investigated in [7].

3. Darboux-Crum-Matveev transformations of the second type

Consider the following pair of operators:

$$L_1 = SD, \quad M_2 = i\partial_{t_2} - SD^2 - \frac{1}{2}S_xD, \tag{15}$$

with (2×2)-dimensional matrix-valued function S, which is unitary and Hermitian: $S = S^*$, $S^{-1} = S^*$. Consider linear problems with operators L_1 and M_2 (15):

$$L_1\{f\} = f\Lambda, \;\; M_2\{f\} = 0, \tag{16}$$

where f is a (2×2)-dimensional matrix-valued function and Λ is a constant matrix with dimension (2×2). The compatibility condition for the system (16) $f_{xt_2} = f_{t_2x}$ results in the Heisenberg equation for S:

$$4iS_{t_2} = [S, S_{xx}]. \tag{17}$$

Let (2×2)-dimensional matrix function φ_1 satisfies linear problems with operators L_1, M_2 (15):

$$L_1\{\varphi_1\} = \varphi_1 A, \;\; M_2\{\varphi_1\} = 0. \tag{18}$$

Consider the following transformation [7]:

$$\tilde{W}_{11} = \Phi_1^{-1}\varphi_1 D\varphi_1^{-1} = \Phi_1^{-1}W_{11} = \Phi_1^{-1}D - I_2, \;\; I_2 = \mathrm{diag}(1,1), \;\; \Phi_1 = \varphi_{1,x}\varphi_1^{-1}. \tag{19}$$

The operator W_{11} in formula (19) is the Darboux-Crum-Matveev transformation of the first type. The following proposition holds:

Proposition 3. *1. Operators $L_1[2]$ and $M_2[2]$ defined by Lax pair L_1, M_2 (15) and operator $\tilde{W}$ (19) via equalities $L_1[2]\tilde{W}_{11} = \tilde{W}_{11}L_1$, $M_2[2]\tilde{W}_{11} = \tilde{W}_{11}M_2$ have the form:*

$$L_1[2] = S[2]D,\ M_2[2] = i\partial_{t_2} - S[2]D^2 - \frac{1}{2}S[2]_x D,\ S[2] = \Phi_1^{-1}S\Phi_1. \tag{20}$$

2. (2×2)-dimensional matrix-valued function $F = \tilde{W}_{11}\{f\}$, where f is an arbitrary solution of linear problem (16), satisfies the system:

$$L_1[2]\{F\} = F\Lambda,\ M_2[2]\{F\} = 0. \tag{21}$$

Proof. Let $L_1[2] = V_1 D + V_0$ and consider the equalities:

$$\begin{aligned} L_1[2]\tilde{W}_{11} - \tilde{W}_{11}L_1 &= (V_1 D + V_0)(\Phi_1^{-1}D - I_2) - (\Phi_1^{-1}D - I_2)SD = \\ &= -V_1\Phi_1^{-1}\Phi_{1,x}\Phi_1^{-1}D + V_1\Phi_1^{-1}D^2 - V_1 D + V_0\Phi_1^{-1}D - V_0 - \Phi_1^{-1}SD^2 - \Phi_1^{-1}S_x D + SD. \end{aligned}$$

By setting coefficients near D^2, D and D^0 equal to zero we obtain the following equations:

$$V_1\Phi_1^{-1} - \Phi_1^{-1}S = 0,\ -V_1\Phi_1^{-1}\Phi_{1,x}\Phi_1^{-1} - V_1 + V_0\Phi_1^{-1} - \Phi_1^{-1}S_x + S = 0,\ \ V_0 = 0. \tag{22}$$

From (22) we get $V_1 = \Phi_1^{-1}S\Phi_1$, $V_0 = 0$ and $-\Phi_1^{-1}S\Phi_{1,x}\Phi_1^{-1} - \Phi_1^{-1}S\Phi_1 - \Phi_1^{-1}S_x + S = 0$. The last equation can be rewritten as $(S\Phi_1)_x = [\Phi_1, S]\Phi_1$. Now we have to verify that the function $\Phi_1 = \varphi_{1,x}\varphi_1^{-1}$ satisfies it. For this purpose we will rewrite equation (18) for function φ_1 in the exact form: $S\varphi_{1,x} = \varphi_1 A$. By multiplying this equation by φ_1^{-1} and differentiating it with respect to x we obtain: $(S\Phi_1)_x = (\varphi_1 A\varphi_1^{-1})_x = \varphi_{1,x}A\varphi_1^{-1} - \varphi_1 A\varphi_1^{-1}\varphi_{1,x}\varphi_1^{-1}$. It remains to notice that $\varphi_{1,x}A\varphi_1^{-1} = \varphi_{1,x}\varphi_1^{-1}\varphi_1 A\varphi_1^{-1} = \Phi_1 S\Phi_1$ and $\varphi_1 A\varphi_1^{-1}\varphi_{1,x}\varphi_1^{-1} = S\Phi_1^2$. In a similar way the exact form of $M_2[2]$ can be found. Finally we notice that $L_1[2]\{F\} = \tilde{W}_{11}L_1\{f\} = \tilde{W}_{11}\{f\}\Lambda = F\Lambda$ and $M_2[2]\{F\} = \tilde{W}_{11}M_2\{f\} = \tilde{W}_{11}\{0\} = 0$. □

We shall notice that under the choice $S = \sigma_3$ systems (18) and (8) coincide. In particular, by putting $A = \mathrm{diag}(\lambda_1, -\bar{\lambda}_1)$ and choosing a solution of system (18) according to (13) with $S = \sigma_3$, we obtain that the (2×2)-dimensional matrix-valued function $S[2] = \Phi_1^{-1}S\Phi_1 = \Phi_1^{-1}\sigma_3\Phi_1$ coincides with the function S_1 (14) and satisfies Heisenberg equation.

3.1. Construction of general matrix Darboux-Crum-Matveev transformation of the second type via the projection method

In this section our aim is to obtain the differential Darboux-Crum-Matveev transformation of the second type via Darboux-Matveev transformation of higher matrix dimension and the projection method. For further purposes we will need the following proposition:

Proposition 4 ([5]). *Let φ be a fixed $(K\times K)$-dimensional matrix solution of equation*

$$L\{\varphi\} := \Big(\alpha\partial_t - \sum_{i=0}^{n} U_i D^i\Big)\{\varphi\} = \varphi\Lambda_1, \tag{23}$$

where U_i are $(K\times K)$-dimensional matrix-valued functions; Λ_1 is a $(K\times K)$-dimensional constant matrix; f is an arbitrary $(K\times M)$-dimensional solution of the equation: $L\{f\} = f\Lambda$, where Λ is a $(M\times M)$-constant matrix. Then the function

$$F := W\{f\} = \varphi D\{\varphi^{-1}f\} = f_x - \varphi_x\varphi^{-1}f \tag{24}$$

satisfies matrix equation:

$$L[2]\{F\} := \Big(\alpha\partial_t - \sum_{i=0}^{n} U_i[2]D^i\Big)\{F\} = F\Lambda, \tag{25}$$

with $U_n[2] = U_n$, $U_{n-1}[2] = U_{n-1} + [A, \varphi_x\varphi^{-1}]$. *The rest of coefficients* $U_j[2]$, $0 \le j \le n-2$, *can be expressed via matrix-valued functions* U_i, $0 \le i < n$, *and the solution* φ *of (23).*

Consider the evolution operator of the following form:

$$L := \alpha\partial_t - \sum_{i=1}^{n} U_i D^i, \quad \alpha \in \mathbb{C}, \tag{26}$$

where U_i are $(K \times K)$-dimensional matrix-valued functions. It should be noticed that the special cases of the operator (26) are operators from Lax pair for Heisenberg equation (15).

Proposition 5. *Let* φ *be a fixed* $(K\times K)$*-dimensional matrix-valued solution of the following equation:*

$$L\{\varphi\} = \varphi\Lambda_1, \tag{27}$$

where Λ_1 *is a* $(K \times K)$*-dimensional constant matrix;* f *is an arbitrary* $(K \times M)$*-dimensional solution of equation* $L\{f\} = f\Lambda$, *where* Λ *is* $(M \times M)$*-constant matrix. Then the function*

$$F := \tilde{W}\{f\} = \Phi^{-1}W\{f\} = (\varphi_x\varphi^{-1})^{-1}W\{f\} = \varphi\varphi_x^{-1}\varphi D\{\varphi^{-1}f\} \tag{28}$$

satisfies matrix equation:

$$L[2]\{F\} := \Big(\alpha\partial_t - \sum_{i=1}^{n} U_i[2]D^i\Big)\{F\} = F\Lambda, \tag{29}$$

where $U_n[2] = \Phi^{-1}U_n\Phi$, $U_{n-1}[2] = \Phi^{-1}U_{n,x}\Phi + n\Phi^{-1}U_n\Phi_x + \Phi^{-1}[U_n,\Phi]\Phi + \Phi^{-1}U_{n-1}\Phi$, *and the rest of coefficients* $U_j[2]$ *can be expressed via* φ *and matrix coefficients* U_i, $1 \le i \le n$.

Proof. Let us define the operator $L[2]$ from equality: $L[2]\tilde{W} - \tilde{W}L = 0$, where the transformation $\tilde{W}$ is defined by formula (28). By setting coefficients near D^i, $1 \le i \le n$, equal to zero we find the exact form of $U_i[2]$, $1 \le i \le n$. In order to show that the coefficient near D^0 is equal to zero in operator $L[2]$, it is sufficient to check that$L[2]\{I_k\} = 0$ where I_k is $(k \times k)$-dimensional identity matrix. It is evident that $\tilde{W}\{I_k\} = I_k$. Thus, $0 = L[2]\tilde{W}\{I_k\} - \tilde{W}L\{I_k\} = L[2]\{I_k\}$. Moreover, the equality $L[2]\tilde{W} - \tilde{W}L = 0$ implies $L[2]\tilde{W}\{f\} = L[2]F = \tilde{W}L\{f\} = F\Lambda$. □

It is evident that the Darboux-Crum-Matveev operator of the second type $\tilde{W}$ defined by (28) provides us with an invariant transformation of operator L (26) into operator $L[2]$ (29).

We will use the last proposition in order to construct Darboux-Crum-Matveev transformation of the second type. Namely, let φ_l, $1 \le l \le N$, be $(k \times k)$-dimensional matrix-valued functions that are fixed solutions of the system:

$$L\{\varphi_l\} = \Big(\alpha\partial_t - \sum_{i=1}^{n} u_i D^i\Big)\{\varphi_l\} = \varphi_l\Lambda_l,\ l \in \{1, \ldots, N\}, \tag{30}$$

where coefficients u_i are $(k\times k)$-dimensional matrix-valued functions; Λ_l are $(k\times k)$-dimensional constant matrices; f is an arbitrary $(k \times m)$-dimensional matrix solution of the equation:

$$L\{f\} = \Big(\alpha\partial_t - \sum_{i=1}^{n} u_i D^i\Big)\{f\} = f\Lambda. \tag{31}$$

with $(m \times m)$-dimensional constant matrix Λ. Let us differentiate each equation of system (30) $N-1$ times. As a result we obtain $N-1$ additional equations. Thus, we obtain $(N\times N)$ equations:

$$\alpha(\varphi_{lt})^{(s)} - \sum_{i=1}^{n}\sum_{j=0}^{s} C_s^j u_i^{(j)} \varphi_l^{(i+s-j)} = \varphi_l^{(s)}\Lambda_l, \quad l\in\{1,\dots,N\},\ s\in\{0,\dots,N-1\}. \tag{32}$$

Equations (32) can be rewritten in the following form:

$$\tilde{L}\{\hat{\varphi}\} := \Big(\alpha\partial_t - \sum_{i=1}^{n} U_i D^i\Big)\{\hat{\varphi}\} = \hat{\varphi}\hat{\Lambda}, \tag{33}$$

where $\hat{\varphi}$ and U_i are matrix $(Nk \times Nk)$-dimensional functions defined by formulae:

$$\hat{\varphi} = \begin{pmatrix} \varphi_1 & \dots & \varphi_N \\ \vdots & \vdots & \vdots \\ \varphi_1^{(N-1)} & \dots & \varphi_N^{(N-1)} \end{pmatrix}, \quad U_i = \begin{pmatrix} u_i & 0 & 0 & \dots & 0 \\ u_i' & u_i & 0 & \dots & 0 \\ u_i'' & 2u_i' & u_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_i^{(N-1)} & C_{N-1}^1 u_i^{(N-2)} & C_{N-1}^2 u_i^{(N-3)} & \dots & u_i \end{pmatrix}. \tag{34}$$

In analogous form we can rewrite equation (31) and its $N-1$ differential consequences:

$$\tilde{L}\{\hat{f}\} := \Big(\alpha\partial_t - \sum_{i=1}^{n} U_i D^i\Big)\{\hat{f}\} = \hat{f}\Lambda, \tag{35}$$

where $\hat{f} := \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(N-1)} \end{pmatrix}$. By applying Proposition 5 we obtain that the function

$$F = \tilde{W}\{\hat{f}\} = \Phi^{-1}\hat{\varphi}D\{\hat{\varphi}^{-1}\hat{f}\} = \Phi^{-1}\hat{f}_x - \hat{f} \tag{36}$$

satisfies the equation:

$$\tilde{L}\{F\} := \alpha F_t - \sum_{i=1}^{n} U_i[2]F^{(i)} = F\Lambda. \tag{37}$$

By using the exact form (7) of matrix-valued function $\Phi^{-1} = \hat{\varphi}\hat{\varphi}_x^{-1}$ (formula (36)) we obtain that

$$\hat{\varphi}\hat{\varphi}_x^{-1} = \begin{pmatrix} -\Phi_1^{-1}\Phi_2 & -\Phi_1^{-1}\Phi_3 & \dots & -\Phi_1^{-1}\Phi_N & \Phi_1^{-1} \\ I_k & 0 & \dots & 0 & 0 \\ 0 & I_k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_k & 0 \end{pmatrix}, \tag{38}$$

where I_k is an identity matrix with dimension $(k \times k)$; Φ_l are $(k \times k)$-dimensional matrix-valued functions. A $(Nk \times m)$-dimensional matrix-valued function F has the form:

$$F = \begin{pmatrix} F_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; F_1 = -f - \sum_{s=1}^{N-1} \Phi_1^{-1}\Phi_{s+1} f^{(s)} + \Phi_1^{-1} f^{(N)} =: \tilde{W}_N\{f\}. \tag{39}$$

By using the form of function F (39) we obtain that equation (37) reduces to the equation for $(k \times m)$-dimensional matrix block F_1 of function F:

$$\alpha(F_1)_t - \sum_{i=1}^{n} (U_i[2])_{11} F_1^{(i)} = F_1 \Lambda,$$

where $(U_i[2])_{11}$ are $(k \times k)$-dimensional blocks of matrices $U_i[2]$ that are situated in the left upper corner.

Remark 6. The operator $\tilde{W}_N$ that is defined by formula (39) has functions φ_j, $1 \le j \le N$, in its kernel (it follows directly from formulae (36) and (38)). Thus $\tilde{W}_N$ is a Darboux-Crum-Matveev transformation of the second type:

$$\tilde{W}_N\{f\} = -f - \sum_{s=1}^{N-1} \Phi_1^{-1}\Phi_{s+1} f^{(s)} + \Phi_1^{-1} f^{(N)} = -f + \sum_{s=1}^{N} \tilde{w}_s f^{(s)}. \tag{40}$$

The previous remark shows that we obtained the exact form of all the coefficients of Darboux-Crum-Matveev transformation operator of the second type in terms of functions Φ_{j+1}, $0 \le j < N$, that belong to Hopf-Cole transformation (5), via Darboux-Matveev transformation (differential operator of the first order) of the second type with a higher matrix dimension.

As a result of previous considerations in this section, we obtained the following generalization of Proposition 5 using V.O. Marchenko's projection method:

Theorem 7. *Let function f be an arbitrary $(k \times m)$-dimensional matrix solution of the equation (31):*

$$L\{f\} = \Big(\alpha\partial_t - \sum_{i=1}^{n} u_i D^i\Big)\{f\} = f\Lambda, \qquad \alpha \in \mathbb{C}, \tag{41}$$

with $(k \times k)$-dimensional matrix-valued functions $u_i = u_i(x,t)$ and $(m \times m)$-constant matrix Λ. Let functions φ_l be fixed $(k \times k)$-dimensional matrix-valued solutions of equations

$$L\{\varphi_l\} = \varphi_l \Lambda_l \tag{42}$$

with $(k \times k)$-constant matrices Λ_l. Assume that the operator $\tilde{W}_N$ is defined by formulae (38)-(39), where $\hat{\varphi}$ is a Wronski matrix constructed by functions φ_l, $l \in \{1, \dots, N\}$. Then, the function $F_1 := \tilde{W}_N\{f\}$ satisfies the equation $\alpha(F_1)_t - \sum_{i=1}^{n} u_i[2] F_1^{(i)} = F_1\Lambda$, with $(k \times k)$-dimensional matrix-valued functions $u_i[2]$ that can be expressed in the exact form via matrix-valued functions u_j, $j \in \{1, \dots, n\}$, and φ_l, $l \in \{1, \dots, N\}$.

In case $N = 1$ the operator $\tilde{W}_N = \tilde{W}_1$ (which is constructed by one solution φ_1 of equation (42)) becomes the differential operator of the first order and Theorem 7 coincides with Proposition 5.

4. Conclusions

In this paper we compared two methods of integration of nonlinear systems that were proposed in [3, 4, 5, 7]. In particular, we investigated a connection between the Darboux-Matveev transformation of the second type that was introduced in [7], and V.O. Marchenko projection method [3]. By combining Darboux-Matveev transformation and the projection method we obtained dressing method for the linear differential operator (41) via Darboux-Crum-Matveev transformations (see Theorem 7). To the special cases of differential operator (41) belong the operators involved in Lax pair for Heisenberg (see formulae (15) and (17)) and Ishimori equations. Thus, Theorem 7 provides us with a solution generating method for the above mentioned nonlinear equations and their "higher" analogues. It should be noticed that the projection method can also be used for integration of the noncommutative generalizations of the famous nonlinear equations of the soliton theory that were considered recently in [8, 9]. In particular, in [8] the noncommutative generalization of the Davey-Stewartson equation was investigated via differential Darboux transformations. The exact form of obtained solution of the latter system can be expressed via quasideterminants that were investigated in [10, 11]. The connection between the theory of quasideterminants and V.O. Marchenko's method was also used for investigation of some noncommutative integrable systems in [12]. The problem of generalization of the Marchenko method to the case of integro-differential Lax pairs remains for further investigations. In particular, such operators arise as a result of the symmetry reductions in the KP hierarchy [13, 14] and their (2+1)-dimensional extensions [15, 16, 17, 18]. We shall also point out that the dressing methods for integro-differential operators from those hierarchies via Darboux transformations were considered in [16, 19, 20].

The results of this paper were approved at conferences dedicated to the 120th anniversary of Stefan Banach [21]

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Received 02.04.2013

Revised 31.07.2013