SUBHARMONIC FUNCTIONS ON ANNULI. A TWO-PARAMETER APPROACH

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In the paper [1] it has been studied subharmonic functions in the annulus $A_r = \{z: \ 1/r < |z| < r\}$, $r > 1$. In this paper a two-parameter approach for investigation of subharmonic functions in the annulus $A_{s,r} = \{z: \ s < |z| < r\}$, $s < 1 < r$ is suggested. The consideration of functions subharmonic in the annulus $A_{s,r}$ allows to describe their behavior at approaching to the inner and outer boundary circles of such annulus. Nevanlinna characteristic of functions subharmonic in such annulus is introduced. A counterpart of Jensen's theorem for subharmonic functions in such annulus is proved. An estimate of a subharmonic function maximum by its Nevanlinna characteristic is established.

1 Introduction

Extensions of Nevanlinna theory to annuli have been made by many authors [5] – [16]. The main tools the authors used were a lemma on index of meromorphic functions along a circle [7], [8], a decomposition lemma due to G. Valiron [12], argument principle. But these tools are unusable for direct investigation of subharmonic functions on annuli. In the paper [1] it has been

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studied subharmonic functions in the annulus $A_r = \{z : 1/r < |z| < r\}$, $r > 1$, which is invariant with respect to the inversion.

In so doing we apply the Riesz representation theorem ([3], p. 123) and the fact that the integral mean of a harmonic function in an annulus $\frac{1}{2\pi}\int_0^{2\pi} h\left(te^{i\theta}\right) d\theta$ is a linear function of $\log t$ [4] to obtain a counterpart of Jensen's theorem. But these tools and methods are unusable for investigation of subharmonic functions in the annulus $A_{s,r} = \{z : s < |z| < r\}$, $s < 1 < r$, in particular they do not make possible to obtain a counterpart of Jensen's theorem for such annulus.

In this paper we suggest a two-parameter approach for investigation of subharmonic functions in the annulus $A_{s,r}$. The consideration of functions subharmonic in the annulus $A_{s,r}$ gives possibility to describe a behavior of such functions at approaching to the inner and outer boundary circles of the annulus $A_{s,r}$.

We prove a version of Jensen's theorem, we introduce the Nevanlinna characteristic. An estimate of a subharmonic function maximum by its Nevanlinna characteristic is established.

2 A counterpart of Jensen's Theorem

Let $A_{s,r} = \{z : s < |z| < r\}$ and $A^1_{s,r} = \{z : s < |z| \le r\}$, where $s < 1 < r$. Let $u(z)$ be a subharmonic function in $\overline{A_{s,r}}$ and let μ be its Riesz measure. Denote by $I(t,u) = \frac{1}{2\pi}\int_0^{2\pi} u\left(te^{i\theta}\right) d\theta$ the integral mean of a corresponding function $u(z)$ over the circle of radius t. Define

$$N_0(s,r;u) = \frac{1}{\log s}\int_s^1 \frac{n(t)}{t}dt + \frac{1}{\log r}\int_1^r \frac{n(t)}{t}dt, \tag{1}$$

where $n(t)$ is the distribution function of the Riesz measure μ of the function u, i.e. $n(t) = \mu\left(A^1_{1,t}\right)$ if $t > 1$ and $n(t) = -\mu\left(A^1_{t,1}\right)$ if $t < 1$, $n(1) = 0$. Note that $n(t)$ is the continuous on the right.

Theorem 1. *Let $u(z)$ be a subharmonic function in $\overline{A_{s,r}}$ and let μ be its Riesz measure. Then*

$$N_0(s,r;u) = I(r)\frac{1}{\log r} + I(s)\frac{1}{\log(1/s)} - I(1)\frac{\log(r/s)}{\log(1/s)\log(r)}. \tag{2}$$

Proof. Using an explicit representation of Green's function [17], [18] for the annulus $L_q = \{z: \ q < |z| < 1\}$ with $q = 1/r^2$ and making homothety, after some transformations one can get the Green's function for $A_{s,r}$

$$G(z,\zeta) = \frac{\log(|\zeta|/r)}{\log(s/r)} \log\left(\frac{|z|}{r}\right) + \log\left|\frac{r^2 - \bar{\zeta}z}{r(z-\zeta)}\right| - \\ - \sum_{m=1}^{\infty} \frac{1}{m} \frac{s^{2m}}{r^{2m} - s^{2m}} \left(\frac{r^m}{|z|^m} - \frac{|z|^m}{r^m}\right)\left(\frac{r^m}{|\zeta|^m} - \frac{|\zeta|^m}{r^m}\right) \cos m(\sigma - \alpha), \qquad (3)$$

and $z = |z| e^{i\sigma}$, $\zeta = |\zeta| e^{i\alpha}$.

By Poisson-Jensen Theorem ([3], p. 139), for $z \in A_{s,r}$ we have

$$u(z) = h(z) - p(z), \qquad (4)$$

where $h(z)$ is the harmonic continuation of the function $u(z)$ from $\partial A_{s,r}$ into $A_{s,r}$, $p(z) = \int\limits_{A_{s,r}} G(z,\zeta)\, d\mu(\zeta)$.

Using (4), consider the expression

$$\frac{I(t_1,u) - I(1,u)}{\log t_1 - \log 1} - \frac{I(1,u) - I(t_2,u)}{\log 1 - \log t_2}, \qquad (5)$$

$s < t_1 < 1 < t_2 < r$. Thereof that $I(t,h)$ is a linear function of $\log t$ [4] in $A_{s,r}$ we have

$$\frac{I(t_1,h) - I(1,h)}{\log t_1} - \frac{I(1,h) - I(t_2,h)}{-\log t_2} = 0. \qquad (6)$$

Note that $I(t,u)$ is a convex function of $\log t$ in $[s,r]$. Applying the Fubini theorem we obtain

$$I(t,p) = \int\limits_{A_{s,r}} \left(\frac{1}{2\pi} \int\limits_0^{2\pi} G\left(te^{i\theta}, \zeta\right) d\theta\right) d\mu(\zeta). \qquad (7)$$

Since G is subharmonic in $A_{s,r}$, $G = 0$ on $\partial A_{s,r}$, then $I(t,G)$ is continuous on (s,r). Hence $I(t_1,p) \to 0$ as $t_1 \to r-0$, $I(t_2,p) \to 0$ as $t_2 \to s+0$. From (5) using (6) and proceeding to the limit as $t_1 \to r-0$, and then $t_2 \to s+0$ we get

$$\frac{I(r,u) - I(1,u)}{\log r} - \frac{I(1,u) - I(s,u)}{-\log s} = I(1,p) \frac{\log(r/s)}{\log r \log(1/s)}. \qquad (8)$$

Calculating $I(1,p)$ using Fubini's Theorem, we obtain

$$I(r)\frac{1}{\log r}+I(s)\frac{1}{\log(1/s)}-I(1)\frac{\log(r/s)}{\log(1/s)\log r}=$$
$$=\frac{1}{\log(1/s)}\int\limits_{s<|\zeta|\leqslant 1}\log\left(\frac{|\zeta|}{s}\right)d\mu(\zeta)+\frac{1}{\log r}\int\limits_{1<|\zeta|<r}\log\left(\frac{r}{|\zeta|}\right)d\mu(\zeta). \quad (9)$$

The sum of two last integrals is equal to $N_0(s,r;u)$. Thus (9) gives (2).

3 Nevanlinna characteristic

Definition 1. *Let $u(z)$ be a subharmonic function in A_{R_1,R_2}, non identical $-\infty$. The function*

$$T_0(s,r;u)=\frac{1}{\log r}\int\limits_0^{2\pi}u^+\left(re^{i\theta}\right)\frac{d\theta}{2\pi}+\frac{1}{\log(1/s)}\int\limits_0^{2\pi}u^+\left(se^{i\theta}\right)\frac{d\theta}{2\pi}-$$
$$-\frac{\log(r/s)}{\log(1/s)\log r}\int\limits_0^{2\pi}u^+\left(e^{i\theta}\right)\frac{d\theta}{2\pi},\qquad R_1<s<1<r<R_2, \quad (10)$$

is called the Nevanlinna characteristic of $u(z)$, where $u^+=\max(u,0)$.

Theorem 2. *Let u, u_1, u_2 be subharmonic functions in A_{R_1,R_2}, non identical $-\infty$. Then*

1) $T_0(s,r;u_1+u_2)\leqslant T_0(s,r;u_1)+T_0(s,r;u_2)+O(1)$,
   $T_0(s,r;\lambda u)=\lambda T_0(s,r;u)$ *for* $\lambda>0$, *where* $R_1<s<1<r<R_2$.

2) *The function* $\log(1/s)\log r T_0(s,r;u)$ *is nonnegative, increasing and convex with respect to the logarithm of variable* $1<r<R_2$. *As the function of variable s, it is nonnegative, increases when s decreases in the interval* $(R_1,1)$ *and is convex with respect to* $\log(1/s)$.

Proof. The property 1 follows from the inequality $(u_1+u_2)^+\leq u_1^++u_2^+$ and definition (10) of $T_0(s,r;u)$. Since u^+ is subharmonic, applying the counterpart of Jensen's theorem (2) to u^+ we obtain

$$\log(1/s)\log r T_0(s,r;u)=\log(1/s)\log r N_0\left(s,r;u^+\right). \quad (11)$$

Next, fixing s_0, $R_1 < s_0 < 1$, we have

$$r\frac{d}{dr_+}\left(\log\left(1/s_0\right)\log rN_0\left(s_0, r; u^+\right)\right) = -\int\limits_{s_0}^{1}\frac{n\left(t\right)}{t}dt + n\left(r\right)\log\left(1/s_0\right). \quad (12)$$

Fixing r_0, $1 < r_0 < R_2$, we have

$$-s\frac{d}{ds_+}\left(\log\left(1/s\right)\log r_0N_0\left(s, r_0; u^+\right)\right) = -n\left(s\right)\log r_0 + \int\limits_{1}^{r_0}\frac{n\left(t\right)}{t}dt. \quad (13)$$

From (12) and (13) we see that the function $\log\left(1/s\right)\log rN_0\left(s, r; u^+\right)$ satisfies 2).

From (11) we conclude that $\log\left(1/s\right)\log rT_0\left(s, r; u\right)$ possesses the properties listed in 2).

Define

$$m_0\left(s, r; u\right) = \frac{1}{\log r}\int\limits_0^{2\pi}u^-\left(re^{i\theta}\right)\frac{d\theta}{2\pi} + \frac{1}{\log\left(1/s\right)}\int\limits_0^{2\pi}u^-\left(se^{i\theta}\right)\frac{d\theta}{2\pi}, \quad (14)$$

$R_1 < s < 1 < r < R_2$, where $u^- = \left(-u\right)^+$. Now we can rewrite (2) as follows

Theorem 3. *If $u\left(z\right)$ is a subharmonic function in $\overline{A_{R_1,R_2}}$. Then*

$$T_0\left(R_1, R_2; u\right) = N_0\left(R_1, R_2; u\right) + m_0\left(R_1, R_2; u\right) -$$

$$-\frac{\log\left(R_2/R_1\right)}{\log\left(1/R_1\right)\log R_2}\int\limits_0^{2\pi}u^-\left(e^{i\theta}\right)\frac{d\theta}{2\pi}.$$

This is a counterpart of the first fundamental theorem for subharmonic functions on annuli.

4 Relation between $B_0\left(s, r; u\right)$ and $T_0\left(s, r; u\right)$

Set $B_0\left(k, t; u\right) = \max\left\{M\left(k, u\right); M\left(t, u\right)\right\}$, $k < 1 < t$, where $M\left(t, u\right) =$
$= \max\left\{u\left(z\right) : |z| = t\right\}$.

Theorem 4. *If $u(z)$ is a subharmonic function in $\overline{A_{s,r}}$, then for $s < \rho < 1 < \sigma < r$ we have*

$$\frac{\log\sigma\log(1/\rho)}{\log(\sigma/\rho)} T_0(\rho,\sigma;u) \leqslant B_0\left(\rho,\sigma;u^+\right) \leqslant$$

$$\leqslant C_1(s,r,\rho,\sigma)\, T_0(s,r;u) + C_2(s,r,\rho,\sigma)\,\frac{1}{\pi}\int_0^{2\pi} u^+\left(e^{i\theta}\right) d\theta, \qquad (15)$$

where

$$2\frac{\log(1/s)\log r}{\log(r/s)} C_2(s,r,\rho,\sigma) < C_1(s,r,\rho,\sigma) <$$

$$< \max\left\{c_1(s,r,\sigma); c_2(s,r,\rho)\right\} < \max\left\{\frac{r+\sigma}{r-\sigma}\log r; \frac{\rho+s}{\rho-s}\log(1/s)\right\}, \quad (16)$$

$$c_1(s,r,\sigma) = \frac{\log(\sigma/s)}{\log(r/s)}\log r + 2\frac{\sigma}{r-\sigma}\log r,$$

$$c_2(s,r,\rho) = \frac{\log(r/\rho)}{\log(r/s)}\log(1/s) + 2\frac{s}{\rho-s}\log(1/s). \qquad (17)$$

Proof. The left inequality in (15) is obvious. To prove the right inequality we apply Poisson-Jensen formula ([3], p. 139) to the function

$$u^+(z) = \frac{1}{2\pi}\int_0^{2\pi}\left\{u^+\left(re^{i\tau}\right)P_1(z,\tau) + u^+\left(se^{i\tau}\right)P_2(z,\tau)\right\}d\tau -$$

$$-\int_{A_{s,r}} G(z,\zeta)\,d\mu(\zeta), \qquad (18)$$

where $G(z,\zeta)$ is the Green's function (3) for $A_{s,r}$.

Let $\xi_1 = \sigma e^{i\theta}$. Taking into consideration

$$\frac{r^2 - |\xi_1|^2}{|\xi_1 - re^{i\tau}|^2} = 1 + 2\sum_{m=1}^{\infty}\left(\frac{\sigma}{r}\right)^m \cos m(\theta-\tau),$$

$$\frac{|\xi_1|^2 - s^2}{|\xi_1 - se^{i\tau}|^2} = 1 + 2\sum_{m=1}^{\infty}\left(\frac{s}{\sigma}\right)^m \cos m(\theta-\tau),$$

after some calculations we obtain

$$P_1(\xi_1,\tau)=\frac{\log(\sigma/s)}{\log(r/s)}+2\sum_{m=1}^{\infty}\frac{r^m\left(\sigma^{2m}-s^{2m}\right)}{\sigma^m\left(r^{2m}-s^{2m}\right)}\cos m(\theta-\tau), \tag{19}$$

$$P_2(\xi_1,\tau)=\frac{\log(r/\sigma)}{\log(r/s)}+2\sum_{m=1}^{\infty}\frac{s^m\left(r^{2m}-\sigma^{2m}\right)}{\sigma^m\left(r^{2m}-s^{2m}\right)}\cos m(\theta-\tau). \tag{20}$$

From (18), since the contribution from the Riesz mass is negative, using the definition of $T_0(s,r;u)$ we get

$$\begin{aligned}u^+(\xi_1)\leqslant \max\{P_1(\xi_1,\theta)\log r;P_2(\xi_1,\theta)\log(1/s)\}\,T_0(s,r;u)+\\+\frac{1}{\pi}\int_0^{2\pi}u^+\left(e^{i\tau}\right)d\tau\times\left\{\frac{P_1(\xi_1,\theta)+P_2(\xi_1,\theta)}{2}\right\}.\end{aligned} \tag{21}$$

Now let $\xi_2=\rho e^{i\theta}$. Using

$$\frac{r^2-|\xi_2|^2}{|\xi_2-re^{i\tau}|^2}=1+2\sum_{m=1}^{\infty}\left(\frac{\rho}{r}\right)^m\cos m(\theta-\tau),$$

$$\frac{|\xi_2|^2-s^2}{|\xi_2-se^{i\tau}|^2}=1+2\sum_{m=1}^{\infty}\left(\frac{s}{\rho}\right)^m\cos m(\theta-\tau),$$

after some calculations we obtain

$$P_1(\xi_2,\tau)=\frac{\log(\rho/s)}{\log(r/s)}+2\sum_{m=1}^{\infty}\frac{r^m\left(\rho^{2m}-s^{2m}\right)}{\rho^m\left(r^{2m}-s^{2m}\right)}\cos m(\theta-\tau), \tag{22}$$

$$P_2(\xi_2,\tau)=\frac{\log(r/\rho)}{\log(r/s)}+2\sum_{m=1}^{\infty}\frac{s^m\left(r^{2m}-\rho^{2m}\right)}{\rho^m\left(r^{2m}-s^{2m}\right)}\cos m(\theta-\tau). \tag{23}$$

Repeating the above considerations with slight difference to $u^+(\xi_2)$ we come to

$$\begin{aligned}u^+(\xi_2)\leqslant \max\{P_1(\xi_2,\theta)\log r;P_2(\xi_2,\theta)\log(1/s)\}\,T_0(s,r;u)+\\+\frac{1}{\pi}\int_0^{2\pi}u^+\left(e^{i\tau}\right)d\tau\times\left\{\frac{P_1(\xi_2,\theta)+P_2(\xi_2,\theta)}{2}\right\}.\end{aligned} \tag{24}$$

It is easy to verify that $P_1(\xi_2,\theta) < P_1(\xi_1,\theta)$ and $P_2(\xi_1,\theta) < P_2(\xi_2,\theta)$ for $s < \rho < 1 < \sigma < r$. This fact and relations (21), (24) yield that $u^+(\xi_1)$ and $u^+(\xi_2)$ are less than or equal to the value

$$\max\{P_1(\xi_1,\theta)\log r; P_2(\xi_2,\theta)\log(1/s)\}\,T_0(s,r;u) + \\ +\frac{1}{\pi}\int_0^{2\pi} u^+\left(e^{i\tau}\right)d\tau \times C_2(s,r,\rho,\sigma), \tag{25}$$

where

$$C_2(s,r,\rho,\sigma) = \max\left\{\frac{P_1(\xi_1,\theta)+P_2(\xi_1,\theta)}{2}; \frac{P_1(\xi_2,\theta)+P_2(\xi_2,\theta)}{2}\right\}.$$

Choosing the ξ_1 and ξ_2 so that $u^+(\xi_1) = M(\sigma,u^+)$, $u^+(\xi_2) = M(\rho,u^+)$ from (25) we obtain the right inequality of (15), where

$$C_1(s,r,\rho,\sigma) = \max\{P_1(\xi_1,\theta)\log r; P_2(\xi_2,\theta)\log(1/s)\}.$$

From inequalities $P_1(\xi_2,\theta) < P_1(\xi_1,\theta)$ and $P_2(\xi_1,\theta) < P_2(\xi_2,\theta)$ we get

$$\frac{\log(1/s)\log r}{\log(r/s)}P_1(\xi_1,\theta) + \frac{\log(1/s)\log r}{\log(r/s)}P_2(\xi_1,\theta) < C_1(s,r,\rho,\sigma), \tag{26}$$

$$\frac{\log(1/s)\log r}{\log(r/s)}P_1(\xi_2,\theta) + \frac{\log(1/s)\log r}{\log(r/s)}P_2(\xi_2,\theta) < C_1(s,r,\rho,\sigma). \tag{27}$$

From (26) and (27) we obtain

$$2\frac{\log(1/s)\log r}{\log(r/s)}C_2(s,r,\rho,\sigma) < C_1(s,r,\rho,\sigma).$$

The other inequalities in (16) follows from the inequalities

$$P_1(\xi_1,\theta)\log r < \frac{\log(\sigma/s)}{\log(r/s)}\log r + 2\frac{\sigma}{r-\sigma}\log r < \frac{r+\sigma}{r-\sigma}\log r, \tag{28}$$

$$P_2(\xi_2,\theta)\log(1/s) < \frac{\log(r/\rho)}{\log(r/s)}\log(1/s) + 2\frac{s}{\rho-s}\log(1/s) < \\ < \frac{\rho+s}{\rho-s}\log(1/s). \tag{29}$$

Now we compare a growth order of $T_0(s,r;u)$ and $B_0(s,r;u)$. We will consider functions subharmonic in the punctured plane $\mathbb{C}\setminus\{0\}$ with a couple of veritable orders.

A set of veritable orders $Ord\ F$ for nonnegative functions of two variable $F(\tau,r)$, $\tau>1$, $r>1$ was introduced in [2]. If $Ord\ F$ contains only one element, then it is called a couple of veritable orders.

By (Lemma 2, Lemma 3, [2]) if $F(\tau,r)=F_1(\tau)+F_2(r)$ or $F(\tau,r)=\max\{F_1(\tau);F_2(r)\}$, where $F_1(\tau)$, $F_2(r)$ are nonnegative functions, then $F(\tau,r)$ has a couple of veritable orders.

In view of this it is convenient to make change of variable $s=1/\tau$, $\tau>1$, and consider the functions $T_0(\tau,r;u)$, $B_0(\tau,r;u)$ instead of $T_0(s,r;u)$ and $B_0(s,r;u)$.

Before comparing a growth order we will give the following corollary, which is the interesting comparison between $B_0(\tau,r;u)$ and $T_0(\tau,r;u)$ on a certain double sequence. The corresponding result for meromorphic in $\mathbb{C}$ functions was proved by Shimizu T. (see [19], p. 43).

Corollary 1. *Let $u(z)$ be a non-constant subharmonic function in $\mathbb{C}\setminus\{0\}$ and $K>1$. Then*

$$\varliminf_{\substack{\tau\to\infty\\ r\to\infty}}\frac{B_0(\tau,r;u)}{\log\tau\log rT_0(\tau,r;u)\left\{\log T_0(\tau,r;u)\right\}^K}=0. \tag{30}$$

Proof. Set $\phi(y,x)=T_0(e^y,e^x;u)$, $y,x>0$. Fix y_0. We apply Lemma 1.2 ([19], p. 14) with $\phi(y_0,x)$. This is possible since u is non-constant and so unbounded in $\mathbb{C}\setminus\{0\}$. Hence $B_0(\tau_0,r;u)\to\infty$ with r, $B_0(\tau,r_0;u)\to\infty$ with τ and so does $T_0(\tau_0,r;u)$, $T_0(\tau,r_0;u)$ by Theorem 4.

Then we can find a sequence x_n such that $\phi(y_0,x)<K_1\phi(y_0,x_n)$ for $x_n<x<x_n+\left\{\log^+\phi(y_0,x_n)\right\}^{-K_1}$, $x_n\to\infty$ and $\phi(y_0,x_n)\to\infty$ as $n\to\infty$.

Applying Lemma 1.2 ([19], p. 14) again with $\phi(y,x_0)$ we can find a sequence y_m such that $\phi(y,x_0)<K_2\phi(y_m,x_0)$ for $y_m<y<y_m+\left\{\log^+\phi(y_m,x_0)\right\}^{-K_2}$, $y_m\to\infty$ and $\phi(y_m,x_0)\to\infty$ as $m\to\infty$.

Note that $\phi(y,x)=\phi(y_0,x)+\phi(y,x_0)-\phi(y_0,x_0)$. From here and considerations above we have

$$\phi(y,x)<K\phi(y_m,x_n)+(K-1)\phi(y_0,x_0) \tag{31}$$

for $x_n<x<x_n+\left\{\log^+\phi(y_0,x_n)\right\}^{-K_1}$, $y_m<y<y_m+\left\{\log^+\phi(y_m,x_0)\right\}^{-K_2}$, where $K=\max\{K_1,K_2\}>1$.

We choose $\log\sigma = x_n$, $\log r = x_n + \left\{\log^+\phi(y_0, x_n)\right\}^{-K_1}$, $\log\eta = y_m$, $\log\tau = y_m + \left\{\log^+\phi(y_m, x_0)\right\}^{-K_2}$ $(s = 1/\tau, \rho = 1/\eta)$ in Theorem 4. This gives

$$B_0(\eta,\sigma;u) <$$
$$< \max\left\{\frac{\exp\left[\{\log\phi(y_0,x_n)\}^{-K_1}\right]+1}{\exp\left[\{\log\phi(y_0,x_n)\}^{-K_1}\right]-1}\left(x_n + \{\log\phi(y_0,x_n)\}^{-K_1}\right);\right.$$

$$\left.\frac{\exp\left[\{\log\phi(y_m,x_0)\}^{-K_2}\right]+1}{\exp\left[\{\log\phi(y_m,x_0)\}^{-K_2}\right]-1}\left(y_m + \{\log\phi(y_m,x_0)\}^{-K_2}\right)\right\}\times$$
$$\times\{KT_0(\eta,\sigma;u) + (K-1)\,const\}. \tag{32}$$

Using inequalities $\phi(y_0,x_n) < \phi(y_m,x_n)$, $\phi(y_m,x_0) < \phi(y_m,x_n)$ we can obtain

$$B_0(\eta,\sigma;u) <$$
$$< \max\left\{\frac{\exp\left[\{\log\phi(y_0,x_n)\}^{-K_1}\right]+1}{\exp\left[\{\log\phi(y_m,x_n)\}^{-K}\right]-1}\left(x_n y_m + \{\log\phi(y_0,x_n)\}^{-K_1}\right);\right.$$

$$\left.\frac{\exp\left[\{\log\phi(y_m,x_0)\}^{-K_2}\right]+1}{\exp\left[\{\log\phi(y_m,x_n)\}^{-K}\right]-1}\left(y_m x_n + \left\{\log^+\phi(y_m,x_0)\right\}^{-K_2}\right)\right\}\times$$
$$\times\{KT_0(\eta,\sigma;u) + (K-1)\,const\} \sim$$
$$\sim 2K\log\eta\log\sigma T_0(\eta,\sigma;u)\{\log T_0(\eta,\sigma;u)\}^K \tag{33}$$

and $\eta\to\infty$, $\sigma\to\infty$ through the sequences $\exp(y_m)$, $\exp(x_n)$. Thus

$$\underline{\lim}_{\substack{\eta\to\infty\\ \sigma\to\infty}} \frac{B_0(\eta,\sigma;u)}{\log\eta\log\sigma T_0(\eta,\sigma;u)\{\log T_0(\eta,\sigma;u)\}^K} < +\infty.$$

Since we may replace K by $\frac{1}{2}(K+1)$ in the above argument and $T_0(\eta,\sigma;u)\to\infty$ with η, σ. Corollary follows.

Definition 2. *Let u be a subharmonic function in $\mathbb{C}\setminus\{0\}$. A couple of veritable orders of u is called a couple of veritable orders of $T_0(\tau,r;u)$.*

Theorem 5. *If $u(z)$ is a subharmonic function in $\mathbb{C}\setminus\{0\}$ then the couples of veritable orders of the functions $T_0(\tau, r; u)$ and $B_0(\tau, r; u)$ coincide termwise.*

Proof. By (Lemma 2, [2]) $T_0(\tau, r; u)$ has a couple of veritable orders, say $(\lambda_1[u], \lambda_2[u])$, and $\lambda_1[u] = \lambda_1^*[u]$, $\lambda_2[u] = \lambda_2^*[u]$, where

$$\lambda_1^*[u] = \overline{\lim_{\tau\to+\infty}} \frac{\log T_0(\tau, r; u)}{\log \tau} \quad \text{for fixed } r,$$

$$\lambda_2^*[u] = \overline{\lim_{r\to+\infty}} \frac{\log T_0(\tau, r; u)}{\log r} \quad \text{for fixed } \tau.$$

Similarly by Lemma 3, [2] $B_0(\tau, r; u)$ has a couple of veritable orders, say $(\rho_1[u], \rho_2[u])$. Making the change of variable $\rho = 1/\eta$, by theorem 4 we have

$$\frac{\log\sigma\log\eta}{\log(\sigma\eta)} T_0(\eta, \sigma; u) \leqslant B_0(\eta, \sigma; u) \leqslant$$
$$\leqslant C_1(\tau, r, \eta, \sigma) T_0(\tau, r; u) + C_2(\tau, r, \eta, \sigma) \frac{1}{\pi} \int_0^{2\pi} u^+\left(e^{i\theta}\right) d\theta, \tag{34}$$

$\tau > \eta > 1$, $r > \sigma > 1$.

Set $r = \gamma\sigma$ in (34), $\gamma > 1$. Next fix τ_0 and η_0 so that $\tau_0 > \eta_0\gamma$. Using (15), (16), provided $u(z)$ is positive on $|z| = \sigma$ for the certain σ, we obtain from (34)

$$\frac{\log\sigma\log\eta_0}{\log(\sigma\eta_0)} T_0(\eta_0, \sigma; u) \leqslant B_0(\eta_0, \sigma; u) \leqslant C_3(\tau_0, \gamma\sigma, \sigma) T_0(\tau_0, \gamma\sigma; u) +$$
$$+ \frac{\log(\tau_0\gamma\sigma)}{2\log\tau_0\log(\gamma\sigma)} C_3(\tau_0, \gamma\sigma, \sigma) \frac{1}{\pi} \int_0^{2\pi} u^+\left(e^{i\theta}\right) d\theta, \tag{35}$$

where

$$C_3(\tau_0, \gamma\sigma, \sigma) = \frac{\log(\tau_0\sigma)}{\log(\gamma\tau_0\sigma)} \log(\gamma\sigma) + 2\frac{1}{\gamma-1} \log(\gamma\sigma). \tag{36}$$

From (35) we deduce at once that $\lambda_1[u] = \rho_1[u]$. Now set $\tau = \beta\eta$ in (34), $\beta > 1$. Fix r_0, σ_0 so that $r_0 > \beta\sigma_0$. Next as above using (15), (16), provided

$u(z)$ is positive on $|z| = 1/\eta$ for the certain η, we obtain from (34)

$$\frac{\log\sigma_0\log\eta}{\log(\sigma_0\eta)}T_0(\eta,\sigma;u) \leqslant B_0(\eta,\sigma_0;u) \leqslant C_4(\beta\eta,r_0,\eta)\,T_0(\beta\eta,r_0;u) + \\ +\frac{\log(\beta\eta r_0)}{2\log(\beta\eta)\log r_0}C_4(\beta\eta,r_0,\eta)\,\frac{1}{\pi}\int_0^{2\pi} u^+\left(e^{i\theta}\right)d\theta, \tag{37}$$

where

$$C_4(\beta\eta,r_0,\eta) = \frac{\log(r_0\eta)}{\log(r_0\beta\eta)}\log(\beta\eta) + 2\frac{1}{\beta-1}\log(\beta\eta). \tag{38}$$

From (37) we deduce that $\lambda_2[u] = \rho_2[u]$.

5 Conclusion

In the paper an approach for studying subharmonic functions in the annulus $A_{s,r} = \{z:\ s < |z| < r\}$, $s < 1 < r$ is suggested. A counterpart of Jensen's theorem for subharmonic functions in such annulus is proved.

Nevanlinna characteristic of functions subharmonic in the annulus $A_{s,r}$ is introduced, which gives possibility to describe a behavior of such functions at approaching to the inner and outer boundary circles of the annulus $A_{s,r}$. Some elementary properties of this characteristic are established. An estimate of a subharmonic function maximum by its Nevanlinna characteristic is established, which gives possibility to compare a growth order of $T_0(s,r;u)$ and $B_0(s,r;u)$, and their relative growth. The case $K = 1$ in the Corollary 1 is open question.

The obtained results will be used for the further study of subharmonic functions in the annulus $A_{s,r}$. It is planned the further consideration of properties of the introduced Nevanlinna characteristic and dissemination of proposed in the paper methods and tools for study of δ-subharmonic functions in the annulus $A_{s,r}$.

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СУБГАРМОНІЙНІ ФУНКЦІЇ В ПЛОСКИХ КРУГОВИХ КІЛЬЦЯХ. ДВОПАРАМЕТРИЧНИЙ ПІДХІД

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В роботі [1] вивчались субгармонійні функції в кільці A_r, де $A_r = \{z : 1/r < |z| < r\}$, $r > 1$. В цій статті пропонується двопараметричний підхід для дослідження субгармонійних функцій в кільці $A_{s,r} = \{z : s < |z| < r\}$, $s < 1 < r$. Розгляд функцій субгармонійних в кільці $A_{s,r}$ дає можливість описувати їх поведінку при наближенні до внутрішнього і зовнішнього граничних кіл такого кільця. Вводиться характеристика Неванлінни для субгармонійних функцій в такому кільці. Доводиться аналог теореми Йенсена для субгармонійних функцій в такому кільці. Встановлюється оцінка максимуму субгармонійної функції через неванліннівську характеристику.