GENERALIZED QUANTUM KINETIC EQUATION FOR INTERACTING PARTICLES WITH QUANTUM STATISTICS

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For initial states of many-particle systems obeying Fermi-Dirac or Bose-Einstein statistics, which are given in terms of a one-particle marginal density operator, the equivalence of the Cauchy problem of the quantum BBGKY hierarchy, and the Cauchy problem of the generalized quantum kinetic equation for bosons and fermions is established. The existence of a strong and a weak solution of the Cauchy problem of stated quantum kinetic equation is proved in the space of trace class operators.

1 Introduction

Experimental advances in the Bose condensation of dilute atomic gases and in the strong correlated Fermi systems have stimulated interesting problems on the quantum theory of many-body systems. Among them it is a description of collective behavior in such systems by quantum kinetic equations, i.e. by the evolution equations for a one-particle marginal density operator [1–4]. Nowadays the considerable progress in the rigorous derivation of quantum kinetic equations in suitable scaling limits [5], in particular

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the nonlinear Schrödinger equation and the Gross-Pitaevskii equation for the Bose condensate [6–11], as well as the quantum Boltzmann equation [12], [13], is observed.

The purpose of this paper is to make a generalization of the results of the paper [14] regarding the description of the evolution of states of quantum many-particle systems in terms of a one-particle marginal density operator, obtained in the case of the Maxwell-Boltzmann statistics on the interacting particles obeying Bose-Einstein or Fermi-Dirac statistics.

We outline the structure of this paper. In the next section we introduce some preliminary definitions and construct a solution of the Cauchy problem of the quantum BBGKY hierarchy for bosons and fermions. In section 3 we formulate the main result related to the describing the evolution of the boson or fermion states in terms of a one-particle marginal density operator, i.e. the link between the quantum BBGKY hierarchy and the generalized quantum kinetic equation for bosons and fermions is established. In the next two sections the formulated results are proved, namely in section 4 we construct the marginal functionals of the states of bosons or fermions, and in section 5 we derive the generalized quantum kinetic equation for bosons and fermions. In section 6 a solution of the Cauchy problem of the obtained generalized quantum kinetic equation is constructed, and the existence of a strong and a weak solution is proved in the space of trace class operators. Finally, in section 7 we conclude with some observations and perspectives for future research.

2 Dynamics of many-particle systems obeying quantum statistics

We consider a quantum many-particle system of identical (spinless) particles with unit mass $m = 1$ in the space $\mathbb{R}^\nu$, $\nu \geq 1$, that obey Fermi-Dirac or Bose-Einstein statistics. Let $\mathcal{H}$ be a one-particle Hilbert space, then the n-particle spaces $\mathcal{H}_n^\pm$ are correspondingly symmetric and antisymmetric tensor products of n Hilbert spaces $\mathcal{H}$ that are associated with the systems of bosons and fermions [15]. We adopt the usual convention that $\mathcal{H}^{\otimes 0} = \mathbb{C}$. We denote by $\mathcal{F}_\mathcal{H}^\pm = \bigoplus_{n=0}^\infty \mathcal{H}_n^\pm$ the Bose or Fermi Fock space over the Hilbert space $\mathcal{H}$.

The Hamiltonian H_n of n-particle system is a self-adjoint operator with domain $\mathcal{D}(H_n) \subset \mathcal{H}_n^{\pm}$:

$$H_n = \sum_{i=1}^{n} K(i) + \sum_{i_1<i_2=1}^{n} \Phi(i_1, i_2), \tag{1}$$

where $K(i)$ is the operator of a kinetic energy of the i-th particle and $\Phi(i_1, i_2)$ is the operator of a two-body interaction potential. The operator $K(i)$ acts on functions ψ_n, that belong to the subspaces $L_0^2(\mathbb{R}^{\nu n}) \subset \mathcal{D}(H_n) \subset L_{\pm}^2(\mathbb{R}^{\nu n})$ of infinitely differentiable symmetric or antisymmetric functions with compact supports according to the formula: $K(i)\psi_n = -\frac{\hbar^2}{2}\Delta_{q_i}\psi_n$, where $h = 2\pi\hbar$ is a Planck constant. Correspondingly we have $\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n$. We shall assume that the function $\Phi(q_{i_1}, q_{i_2})$ is symmetric with respect to permutations of its arguments, translation-invariant and bounded function.

States of many-particle systems of bosons and fermions belong to the corresponding spaces $\mathfrak{L}^1(\mathcal{F}_{\mathcal{H}}^{\pm}) = \oplus_{n=0}^{\infty}\mathfrak{L}^1(\mathcal{H}_n^{\pm})$ of sequences $f = (f_0, f_1, \ldots, f_n, \ldots)$ of trace-class operators $f_n \equiv f_n(1, \ldots, n) \in \mathfrak{L}^1(\mathcal{H}_n^{\pm})$ and $f_0 \in \mathbb{C}$, that satisfy the symmetry condition $f_n(1, \ldots, n) = f_n(i_1, \ldots, i_n)$ for arbitrary $(i_1, \ldots, i_n) \in (1, \ldots, n)$, equipped with the norm:

$$\|f\|_{\mathfrak{L}^1(\mathcal{F}_{\mathcal{H}}^{\pm})} = \sum_{n=0}^{\infty} \|f_n\|_{\mathfrak{L}^1(\mathcal{H}_n^{\pm})} = \sum_{n=0}^{\infty} \mathrm{Tr}_{1,\ldots,n}|f_n(1, \ldots, n)|,$$

where $\mathrm{Tr}_{1,\ldots,n}$ are partial traces over $1, \ldots, n$ particles [3]. The Bose-Einstein and Fermi-Dirac statistics endow the states with additional symmetry properties [15]. We denote by $\mathfrak{L}_0^1$ the everywhere dense set in $\mathfrak{L}^1(\mathcal{F}_{\mathcal{H}}^{\pm})$ of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

The evolution of states is described by the sequences $F(t) = (F_1(t, 1), \ldots, F_s(t, 1, \ldots, s), \ldots)$ of the marginal density operators that satisfy the Cauchy problem of the quantum BBGKY hierarchy

$$\begin{aligned} &\frac{d}{dt}F_s(t, Y) = -\mathcal{N}_s(Y)F_s(t, Y) + \\ &+\frac{1}{v}\sum_{i=1}^{s} \mathrm{Tr}_{s+1}\big(-\mathcal{N}_{\mathrm{int}}(i, s+1)\big)F_{s+1}(t, Y, s+1), \\ &F_s(t)|_{t=0} = F_s^0, \quad s \geq 1, \end{aligned} \tag{2}$$

where $\dfrac{1}{v}$ is the density of particles; $Y \equiv (1, \ldots, s)$ and the operator $\mathcal{N}_s$ is defined on the subspace $\mathfrak{L}_0^1(\mathcal{H}_s^{\pm}) \subset \mathfrak{L}^1(\mathcal{H}_s^{\pm})$ as follows:

$$\mathcal{N}_s f_s \doteq -\frac{i}{\hbar}\big(f_s\, H_s - H_s\, f_s\big) \tag{3}$$

and correspondingly

$$\mathcal{N}_{\mathrm{int}}(i,j) f_s \doteq -\frac{i}{\hbar}\big(f_s\, \Phi(i,j) - \Phi(i,j)\, f_s\big). \tag{4}$$

Hereinafter, we consider initial data satisfying a factorization property or the chaos property [4], which means the absence of correlations at initial time. For a system of identical particles, obeying the Fermi-Dirac or Bose-Einstein statistics, we have

$$F(t)|_{t=0} = F^{(c)} \equiv \big(F_1^0(1), \ldots, \mathcal{S}_s^{\pm} \prod_{i=1}^{s} F_1^0(i), \ldots\big), \tag{5}$$

where the symmetrization operator $\mathcal{S}_n^{+}$ and the anti-symmetrization operator $\mathcal{S}_n^{-}$ on $\mathcal{H}^{\otimes n}$ are defined by the formula:

$$\mathcal{S}_n^{\pm} \doteq \frac{1}{n!} \sum_{\pi \epsilon \mathfrak{S}_n} (\pm 1)^{|\pi|} p_{\pi}. \tag{6}$$

In (6) the operator p_π is a transposition operator of the permutation π from the permutation group $\mathfrak{S}_n$ of the set $(1, \ldots, n)$ and $|\pi|$ denotes the number of transpositions in the permutation π. The operators $\mathcal{S}_n^{\pm}$ are orthogonal projectors, i.e. $(\mathcal{S}_n^{\pm})^2 = \mathcal{S}_n^{\pm}$, ranges of which are correspondingly the symmetric tensor product $\mathcal{H}_n^{+}$ and the antisymmetric tensor product $\mathcal{H}_n^{-}$ of n Hilbert spaces $\mathcal{H}$.

On the spaces $\mathfrak{L}^1(\mathcal{H}_n^{\pm})$ we define the group of operators:

$$\mathcal{G}_n(-t) f_n \doteq e^{-\frac{i}{\hbar} t H_n}\, f_n\, e^{\frac{i}{\hbar} t H_n}. \tag{7}$$

On the spaces $\mathfrak{L}^1(\mathcal{H}_n^{\pm})$ the mapping: $t \to \mathcal{G}_n(-t) f_n$ is an isometric strongly continuous group which preserves positivity and self-adjointness of operators [15]. For $f_n \in \mathfrak{L}_0^1(\mathcal{H}_n^{\pm})$ there exists a limit in the sense of a strong convergence by which the infinitesimal generator of the group of evolution operators (7) is determined as follows:

$$\lim_{t \to 0} \frac{1}{t}\big(\mathcal{G}_n(-t) f_n - f_n\big) = -\mathcal{N}_n f_n, \tag{8}$$

where the operator $(-\mathcal{N}_n)$ is defined by formula (3). Symmetrization and antisymmetrization operators (6) are integrals of motion and as a consequence the equalities hold:

$$\mathcal{G}_n(-t)\mathcal{S}_n^{\pm} = \mathcal{S}_n^{\pm}\mathcal{G}_n(-t),$$

and

$$\mathcal{N}_n\mathcal{S}_n^{\pm} = \mathcal{S}_n^{\pm}\mathcal{N}_n,$$

i.e. the symmetry of states is preserved within evolutionary process [2].

A solution of the quantum BBGKY hierarchy (2) with initial data (5) is represented by the expansion

$$F_s(t,Y) = \sum_{n=0}^{\infty} \frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{s+1,\ldots,s+n}\,\mathfrak{A}_{1+n}(t,\{Y\},s+1,\ldots \qquad (9)$$
$$\ldots,s+n)\,\mathcal{S}_{s+n}^{\pm}\prod_{i=1}^{s+n}F_1^0(i), \quad s\geq 1,$$

where the evolution operator $\mathfrak{A}_{1+n}(t)$, $n\geq 0$, is the $(n+1)th$-order reduced cumulant [3] of the groups of operators (7):

$$\mathfrak{A}_{1+n}(t) = \sum_{k=0}^{n}(-1)^k\,\frac{n!}{k!(n-k)!}\,\mathcal{G}_{s+n-k}(-t). \qquad (10)$$

In case $F_1^0\in\mathfrak{L}^1(\mathcal{H})$ a series (9) converges on the norm of the spaces $\mathfrak{L}^1(\mathcal{H}_s^{\pm})$ for arbitrary $t\in\mathbb{R}^1$ [3].

3 Kinetic dynamics of bosons and fermions

Since we consider initial data (5), which is completely characterized by the one-particle density operator F_1^0, the initial-value problem of the quantum BBGKY hierarchy (2),(5) is not completely well-defined Cauchy problem, because generic initial data is not independent for every unknown operator $F_s(t,1,\ldots,s)$, $s\geq 1$, in the hierarchy of equations. Thus, such initial-value problem can be naturally reformulated as a new Cauchy problem for the operator $F_1(t)$, that corresponds to the independent initial data F_1^0 and the sequence of explicitly defined marginal functionals $F_s\big(t,1,\ldots,s\mid F_1(t)\big)$, $s\geq 2$, of the solution $F_1(t)$ of this Cauchy problem [14].

We define the restated Cauchy problem. The marginal functionals of the state $F_s\big(t,1,\ldots,s \mid F_1(t)\big)$, $s\geq 2$, are represented by the following expansions over products of the one-particle density operator $F_1(t)$:

$$F_s\big(t,Y \mid F_1(t)\big) \doteq \tag{11}$$
$$\doteq \sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{s+1,\ldots,s+n}\,\mathfrak{V}_{1+n}\big(t,\{Y\},X\setminus Y\big)\mathcal{S}^{\pm}_{s+n}\prod_{i=1}^{s+n}F_1(t,i),$$

where $X\setminus Y\equiv(s+1,\ldots,s+n)$ and the $(n+1)th$-order evolution operator $\mathfrak{V}_{1+n}(t)$, $n\geq 0$, is defined as follows:

$$\mathfrak{V}_{1+n}(t,\{Y\},X\setminus Y)\doteq\sum_{k=0}^{n}(-1)^k\sum_{n_1=1}^{n}\ldots\sum_{n_k=1}^{n-n_1-\ldots-n_{k-1}}\frac{n!}{(n-n_1-\ldots-n_k)!}\times$$
$$\times\widehat{\mathfrak{A}}_{1+n-n_1-\ldots-n_k}(t,\{Y\},s+1,\ldots,s+n-n_1-\ldots-n_k)\times$$
$$\times\prod_{j=1}^{k}\sum_{\substack{\mathrm{D}_j\,:\,Z_j=\bigcup_{l_j}X_{l_j},\\ |\mathrm{D}_j|\leq s+n-n_1-\cdots-n_j}}\frac{1}{|\mathrm{D}_j|!}\times$$
$$\times\sum_{i_1\neq\ldots\neq i_{|\mathrm{D}_j|}=1}^{s+n-n_1-\ldots-n_j}\ \prod_{X_{l_j}\subset\mathrm{D}_j}\frac{1}{|X_{l_j}|!}\,\widehat{\mathfrak{A}}_{1+|X_{l_j}|}(t,i_{l_j},X_{l_j}), \tag{12}$$

and $\sum_{\mathrm{D}_j:Z_j=\bigcup_{l_j}X_{l_j}}$ is the sum over all possible dissections of the linearly ordered set $Z_j\equiv(s+n-n_1-\ldots-n_j+1,\ldots,s+n-n_1-\ldots-n_{j-1})$ on no more than $s+n-n_1-\ldots-n_j$ linearly ordered subsets. In (12) we denote by $\widehat{\mathfrak{A}}_{1+n}(t)$ the $(1+n)th$-order reduced cumulant, i.e.

$$\widehat{\mathfrak{A}}_{1+n}(t,\{Y\},X\setminus Y)=\sum_{k=0}^{n}(-1)^k\,\frac{n!}{k!(n-k)!}\,\widehat{\mathcal{G}}_{s+n-k}(t),$$

of the scattering operators:

$$\widehat{\mathcal{G}}_n(t)=\mathcal{G}_n(-t,1,\ldots,n)\prod_{i=1}^{n}\mathcal{G}_1(t,i),\quad n\geq 1. \tag{13}$$

The marginal functionals of the state are represented by converged series (11) under the condition $\dfrac{1}{v}<e^{-2}$ [14], [16].

We observe that the kinetic dynamics of states is described in terms of cumulants of scattering operators (13) in contrast to the evolution of states described by the BBGKY hierarchy (2). We give a few examples of the evolution operators $\mathfrak{V}_n$, $n \geq 1$, of the lower orders:

$$\begin{aligned}
&\mathfrak{V}_1(t,\{Y\}) = \widehat{\mathfrak{A}}_1(t,\{Y\}), \qquad (14)\\
&\mathfrak{V}_2(t,\{Y\},s+1) = \widehat{\mathfrak{A}}_2(t,\{Y\},s+1) - \\
&\qquad\qquad - \widehat{\mathfrak{A}}_1(t,\{Y\}) \sum_{i=1}^{s} \widehat{\mathfrak{A}}_2(t,i,s+1).
\end{aligned}$$

The one-particle density operator $F_1(t)$ is a solution of the following initial-value problem of the generalized quantum kinetic equation for bosons and fermions:

$$\begin{aligned}
&\frac{d}{dt}F_1(t,1) = -\mathcal{N}_1(1)F_1(t,1) + \frac{1}{v}\mathrm{Tr}_2\big(-\mathcal{N}_{\mathrm{int}}(1,2)\big) \times \qquad (15)\\
&\times \sum_{n=0}^{\infty} \frac{1}{v^n}\frac{1}{n!}\mathrm{Tr}_{3,\ldots,n+2}\mathfrak{V}_{1+n}\big(t,\{1,2\},3,\ldots,n+2\big)\mathcal{S}^{\pm}_{n+2}\prod_{i=1}^{n+2}F_1(t,i),
\end{aligned}$$

$$F_1(t,1)|_{t=0} = F_1^0(1), \qquad (16)$$

where the evolution operator $\mathfrak{V}_{1+n}(t)$ is defined by formula (12) and the operator $\mathcal{N}_{\mathrm{int}}(1,2)$ is defined by formula (4). For systems of classical particles such a kinetic equation was formulated in [17] and for quantum systems of particles obeying Maxwell-Boltzman statistics in [14] (see also reviews [18], [19]).

4 Marginal functionals of the states of bosons or fermions

Using kinetic cluster expansions of reduced cumulants of scattering operators (13), we construct the expansions of the marginal functionals of the state $F_s(t,1,\ldots,s \mid F_1(t))$, $s \geq 2$, on the basis of solution expansions (9) of the quantum BBGKY hierarchy. Indeed, taking into account relations:

$$\begin{aligned}
&\widehat{\mathfrak{A}}_{1+n}(t,\{Y\},X\setminus Y) = \qquad (17)\\
&= \sum_{n_1=0}^{n} \frac{n!}{(n-n_1)!}\,\mathfrak{V}_{1+n-n_1}\big(t,\{Y\},s+1,\ldots,s+n-n_1\big) \times \\
&\times \sum_{\substack{\mathrm{D}:\, Z=\bigcup_l X_l,\\ |\mathrm{D}|\leq s+n-n_1}} \frac{1}{|\mathrm{D}|!} \sum_{i_1\neq\ldots\neq i_{|\mathrm{D}|}=1}^{s+n-n_1} \prod_{X_l\subset \mathrm{D}} \frac{1}{|X_l|!}\,\widehat{\mathfrak{A}}_{1+|X_l|}(t,i_l,X_l),
\end{aligned}$$

where $\sum_{\mathrm{D}:Z=\bigcup_l X_l,\,|\mathrm{D}|\leq s+n-n_1}$ is the sum over all possible dissections D of the linearly ordered set $Z\equiv(s+n-n_1+1,\ldots,s+n)$ on no more than $s+n-n_1$ linearly ordered subsets, we represent series over the summation index n and the sum over the summation index n_1 as the two-fold series

$$F_s(t,1,\ldots,s)=\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\sum_{n_1=0}^{\infty}\frac{1}{v^{n_1}}\mathrm{Tr}_{s+1,\ldots,s+n+n_1}\mathfrak{V}_{1+n}\big(t,\{Y\},X\setminus Y\big)\times$$

$$\times\sum_{\substack{\mathrm{D}:\,Z=\bigcup_k X_k,\\ |\mathrm{D}|\leq s+n}}\frac{1}{|\mathrm{D}|!}\sum_{i_1\neq\ldots\neq i_{|\mathrm{D}|}=1}^{s+n}\prod_{X_k\subset\mathrm{D}}\frac{1}{|X_k|!}\mathfrak{A}_{1+|X_k|}(t,i_k,X_k)\times$$

$$\times\prod_{\substack{l=1,\\ l\neq i_1,\ldots,i_{|\mathrm{D}|}}}^{s+n}\mathfrak{A}_1(t,l)\mathcal{S}^{\pm}_{s+n+n_1}\prod_{j=1}^{n+s+n_1}F_1^0(j),$$

where $Z\equiv(s+n+1,\ldots,s+n+n_1)$ is the linearly ordered set and the notations introduced above are used. The series in the right-hand side converges under the condition $\frac{1}{v}<e^{-2}$. In view of the formula

$$\sum_{n_1=0}^{\infty}\frac{1}{v^{n_1}}\mathrm{Tr}_{s+n+1,\ldots,s+n+n_1}\sum_{\substack{\mathrm{D}:\,Z=\bigcup_k X_k,\\ |\mathrm{D}|\leq s+n}}\sum_{i_1<\ldots<i_{|\mathrm{D}|}=1}^{s+n}\prod_{X_k\subset\mathrm{D}}\frac{1}{|X_k|!}\times$$

$$\times\mathfrak{A}_{1+|X_k|}(t,i_k,X_k)\prod_{\substack{l=1,\\ l\neq i_1,\ldots,i_{|\mathrm{D}|}}}^{s+n}\mathfrak{A}_1(t,l)\mathcal{S}^{\pm}_{n+s+n_1}\prod_{j=1}^{n+s+n_1}F_1^0(j)=$$

$$=\mathcal{S}^{\pm}_{s+n}\prod_{i=1}^{s+n}F_1(t,i),\qquad(18)$$

where $\sum_{\mathrm{D}:Z=\bigcup_k X_k,\,|\mathrm{D}|\leq s+n}$ is the sum over all possible dissections D of the linearly ordered set $Z\equiv(s+n+1,\ldots,s+n+n_1)$ on no more than $s+n$ linearly ordered subsets, we identify the series over the summation index n_1 with the products of one-particle density operators, and consequently for $s\geq 2$ the following equality holds:

$$F_s(t,Y)=\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{s+1,\ldots,s+n}\,\mathfrak{A}_{1+n}(t,\{Y\},X\setminus Y)\mathcal{S}^{\pm}_{s+n}\prod_{i=1}^{s+n}F_1^0(i)=$$

$$= \sum_{n=0}^{\infty} \frac{1}{v^n} \frac{1}{n!} \mathrm{Tr}_{s+1,\ldots,s+n} \mathfrak{V}_{1+n}\big(t, \{Y\}, X \setminus Y\big) \mathcal{S}^{\pm}_{s+n} \prod_{i=1}^{s+n} F_1(t,i) =$$
$$= F_s(t \mid F_1(t)),$$

i.e., if kinetic cluster expansions (17) of cumulants of scattering operators (13) hold, then solution expansions (9) for $s \geq 2$ can be represented in the form of marginal functionals of the state (11).

We make a few examples of relations (17) of the kinetic cluster expansions:

$$\widehat{\mathfrak{A}}_1(t, \{Y\}) = \mathfrak{V}_1(t, \{Y\}),$$
$$\widehat{\mathfrak{A}}_2(t, \{Y\}, s+1) = \mathfrak{V}_2(t, \{Y\}, s+1) + \mathfrak{V}_1(t, \{Y\}) \sum_{i_1=1}^{s} \widehat{\mathfrak{A}}_2(t, i_1, s+1).$$

It is evident that solutions of these relations are given by expressions (14), which determine evolution operators (12) of the first, and second order correspondingly in the expansions of marginal functionals of the state (11). In the general case, solutions of recurrence relations (17) are given by expressions (12) [14], [16].

It should be emphasized, that in case under consideration, i.e. the absence of correlations at initial time, the correlations generated by the dynamics of a system are completely governed by evolution operators (12). Typical kinetic properties of constructed marginal functionals of the state (11) are induced by the properties of evolution operators (12).

Summarizing, we observe that in the case of initial data (5), which is completely characterized by the one-particle density operator F_1^0, solution (9) for $s \geq 2$ of the quantum BBGKY hierarchy (2) and marginal functionals of the state (11) give two equivalent approaches for describing the states of quantum many-particle systems obeying quantum statistics.

5 The generalized quantum kinetic equation for bosons and fermions

Let us construct an evolution equation, which satisfies expression (9), (10) in the case of $s = 1$. Taking equality (8) into account and observing the validity of the equalities for reduced cumulants (10) of groups (7) for $f \in \mathfrak{L}^1_0(\mathcal{F}^{\pm}_{\mathcal{H}})$ in the sense of the pointwise convergence (for $n \geq 2$ it is a consequence that we consider a system of particles interacting by a two-body potential (1)):

$$\lim_{t\to 0}\frac{1}{t}\,\mathrm{Tr}_2\,\mathfrak{A}_2(t,1,2)f_2(1,2)=\mathrm{Tr}_2\big(-\mathcal{N}_{\mathrm{int}}(1,2)\big)f_2(1,2),$$
$$\lim_{t\to 0}\frac{1}{t}\,\mathrm{Tr}_{2,\ldots,n+1}\,\mathfrak{A}_{1+n}(t,1,\ldots,n+1)f_{n+1}=0,\quad n\geq 2,$$

we will differentiate over the time variable expression (9), (10) for $s=1$ in the sense of pointwise convergence in the space $\mathfrak{L}^1(\mathcal{H})$. As result it holds:

$$\frac{d}{dt}F_1(t,1)=-\mathcal{N}_1(1)F_1(t,1)+\frac{1}{v}\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}}(1,2))\times$$
$$\times\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\mathrm{Tr}_{3,\ldots,n+2}\,\mathfrak{A}_{1+n}(t,\{1,2\},3,\ldots,n+2)\mathcal{S}^{\pm}_{n+2}\prod_{i=1}^{n+2}F_1^0(i). \tag{19}$$

In the second summand in the right-hand side of equality (19) we expand reduced cumulants (10) of groups (7) into kinetic cluster expansions (17) and represent series over the summation index n and the sum over the summation index n_1 as the two-fold series. Then the following equalities take place:

$$\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{2,\ldots,n+2}(-\mathcal{N}_{\mathrm{int}}(1,2))\mathfrak{A}_{1+n}(t,\{1,2\},3,\ldots,n+2)\times$$
$$\times\mathcal{S}^{\pm}_{n+2}\prod_{i=1}^{n+2}F_1^0(i)=\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{2,\ldots,n+2}(-\mathcal{N}_{\mathrm{int}}(1,2))\sum_{n_1=0}^{n}\frac{n!}{(n-n_1)!}\times$$
$$\times\mathfrak{V}_{1+n-n_1}\big(t,\{1,2\},3,\ldots,n+2-n_1\big)\sum_{\mathrm{D}:Z=\bigcup_l X_l}\frac{1}{|\mathrm{D}|!}\times$$
$$\times\sum_{i_1\neq\ldots\neq i_{|\mathrm{D}|}=1}^{n+2-n_1}\prod_{X_l\subset\mathrm{D}}\frac{1}{|X_l|!}\mathfrak{A}_{1+|X_l|}(t,i_l,X_l)\prod_{\substack{m=1,\\ m\neq i_1,\ldots,i_{|\mathrm{D}|}}}^{2+n-n_1}\mathfrak{A}_1(t,m)\,\mathcal{S}^{\pm}_{n+2}\times$$
$$\times\prod_{i=1}^{n+2}F_1^0(i)=\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}}(1,2))\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\mathrm{Tr}_{3,\ldots,n+2}\mathfrak{V}_{1+n}\big(t,\{1,2\},3,\ldots$$
$$\ldots,n+2\big)\sum_{n_1=0}^{\infty}\sum_{\mathrm{D}:Z'=\bigcup_l X_l}\frac{1}{|\mathrm{D}|!}\sum_{i_1\neq\ldots\neq i_{|\mathrm{D}|}=1}^{n+2}\prod_{X_l\subset\mathrm{D}}\frac{1}{|X_l|!}\mathfrak{A}_{1+|X_l|}(t,i_l,X_l)\times$$
$$\times\prod_{\substack{m=1,\\ m\neq i_1,\ldots,i_{|\mathrm{D}|}}}^{n+2}\mathfrak{A}_1(t,m)\mathcal{S}^{\pm}_{n+2+n_1}\prod_{i=1}^{n+2+n_1}F_1^0(i),$$

where $Z \equiv (n+3-n_1,\dots,n+2)$ and $Z' \equiv (n+3,\dots,n+2+n_1)$ are linearly ordered sets and the notations accepted above are used.

Consequently, applying in the case of $s=2$ formula (18) to the obtained expression, from equality (19) we derive

$$\frac{d}{dt}F_1(t,1) = -\mathcal{N}_1(1)F_1(t,1) + \frac{1}{v}\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}}(1,2)) \times \tag{20}$$
$$\times \sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{3,\dots,n+2}\,\mathfrak{V}_{1+n}\big(t,\{1,2\},3,\dots,n+2\big)\mathcal{S}^{\pm}_{n+2}\prod_{i=1}^{n+2}F_1(t,i).$$

Constructed identity (20) for a one-particle (marginal) density operator defined by (9), (10), we treat as the evolution equation, which governs the one-particle states of many-particle quantum systems obeying the Fermi-Dirac and Bose-Einstein statistics.

We remark that one more approach to the derivation of the generalized quantum kinetic equation consists in its construction on the basis of dynamics of correlations of bosons and fermions [20], [21].

Thus, if initial data is completely defined by a one-particle marginal density operator, then all possible states of infinite-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator without any approximations. In other words, for mentioned states, the evolution of states governed by the quantum BBGKY hierarchy (2) can be completely described by the generalized quantum kinetic equation (15) for bosons and fermions.

6 An existence theorem

For the Cauchy problem (15), (16) in the space $\mathfrak{L}^1(\mathcal{H})$, the following statement is true.

Theorem 1. *The global in time solution of the initial-value problem (15), (16) is determined by the following expansion*

$$F_1(t,1) = \sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{2,\dots,1+n}\,\mathfrak{A}_{1+n}(t,1,\dots,n+1)\mathcal{S}^{\pm}_{n+1}\prod_{i=1}^{n+1}F_1^0(i), \tag{21}$$

where the reduced cumulants $\mathfrak{A}_{1+n}(t)$, $n \geq 0$, are defined by formula (10). If $\frac{1}{v} < e^{-2}$, then for $F_1^0 \in \mathfrak{L}^1_0(\mathcal{H})$ it is a strong (classical) solution and for an arbitrary initial data $F_1^0 \in \mathfrak{L}^1(\mathcal{H})$ it is a weak (generalized) solution.

Proof. Let $F_1^0 \in \mathfrak{L}_0^1(\mathcal{H})$, then the convergent on the norm of the space $\mathfrak{L}^1(\mathcal{H})$ series (21) is a strong solution of initial-value problem (15), (16), if the equality holds:

$$\lim_{\triangle t\to 0}\big\|\frac{1}{\triangle t}\big(F_1(t+\triangle t,1)-F_1(t,1)\big)-\big(-\mathcal{N}_1(1)F_1(t,1)+ \tag{22}$$
$$+\frac{1}{v}\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}}(1,2))\sum_{n=0}^{\infty}\frac{1}{v^n}\frac{1}{n!}\,\mathrm{Tr}_{3,\ldots,n+2}\,\mathfrak{V}_{1+n}\big(t,\{1,2\},3,\ldots$$
$$\ldots,n+2\big)\mathcal{S}^{\pm}_{n+2}\prod_{i=1}^{n+2}F_1(t,i)\big)\big\|_{\mathfrak{L}^1(\mathcal{H})}=0,$$

where abridged notations are applied: the symbols $\mathcal{S}^{\pm}_{n+1}\prod_{i=1}^{n+1}F_1(t,i)$ and $F_1(t,1)$ are implied series (21) and series (18) for $s=1$, respectively.

To prove this fact we use the result of the previous section 5 on the differentiation of expansion (21) over time variable in the sense of the pointwise convergence in the space $\mathfrak{L}^1(\mathcal{H})$ with a little modification. Taking into account formula (8), and that for $n\geq 1$ and $f_{n+1}\in\mathfrak{L}_0^1(\mathcal{H}^{\pm}_{n+1})$, the following equalities are true:

$$\lim_{t\to 0}\big\|\frac{1}{t}\,\mathrm{Tr}_2\,\mathfrak{A}_{1+1}(t,1,2)f_2-\mathrm{Tr}_2\,(-\mathcal{N}_{\mathrm{int}})(1,2)f_2\big\|_{\mathfrak{L}^1(\mathcal{H})}=0,$$
$$\lim_{t\to 0}\big\|\frac{1}{t}\,\mathrm{Tr}_{2,\ldots,n+1}\,\mathfrak{A}_{1+n}(t,1,\ldots,n+1)f_{n+1}\big\|_{\mathfrak{L}^1(\mathcal{H})}=0,$$

in the sense of the norm convergence in the space $\mathfrak{L}^1(\mathcal{H})$ we obtain

$$\lim_{\triangle t\to 0}\big\|\frac{1}{\triangle t}\big(F_1(t+\triangle t,1)-F_1(t,1)\big)-\big(-\mathcal{N}_1(1)F_1(t,1)+ \tag{23}$$
$$+\frac{1}{v}\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}})(1,2)\sum_{k=0}^{\infty}\frac{1}{v^k}\frac{1}{k!}\mathrm{Tr}_{3,\ldots,k+2}\,\mathfrak{A}_{1+k}(t,\{1,2\},3,\ldots$$
$$\ldots,k+2)\mathcal{S}^{\pm}_{k+2}\prod_{i=1}^{k+2}F_1^0(i)\big)\big\|_{\mathfrak{L}^1(\mathcal{H})}=0.$$

In the third summand in the left-hand side of this equality we expand the $(1+k)th$-order reduced cumulants (10) of groups (7) into kinetic cluster expansions (17) and represent series over the summation index n and the sum over the summation index k as the two-fold series. Then, applying formula (18) in the case of $s=n+1$ to the obtained expression, we derive

$$\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}})(1,2)\sum_{k=0}^{\infty}\frac{1}{v^k}\frac{1}{k!}\,\mathrm{Tr}_{3,\ldots,k+2}\,\mathfrak{A}_{1+k}(t,\{1,2\},3\ldots,k+2)\mathcal{S}^{\pm}_{k+2}\times$$

$$\times \prod_{i=1}^{k+2} F_1^0(i) = \sum_{n=1}^{\infty} \frac{1}{v^{n-1}} \frac{1}{(n-1)!} \mathrm{Tr}_{2,\ldots,n+1}(-\mathcal{N}_{\mathrm{int}})(1,2)\times$$
$$\times \mathfrak{V}_{1+n-1}(t,\{1,2\},3,\ldots,n+1)\mathcal{S}^{\pm}_{n+1} \prod_{i=1}^{n+1} F_1(t,i).$$

Under the condition $\frac{1}{v} < e^{-2}$, the series in the right-hand side of this equality converges on the norm of the space $\mathfrak{L}^1(\mathcal{H})$. Hence, according equality (23) and the last equality, we find that for $F_1^0 \in \mathfrak{L}^1_0(\mathcal{H})$ equality (22) is valid.

For the Cauchy problem (15), (16) it can be introduced the notion of a weak solution in the following sense. Consider the functional

$$\big(f, F(t \mid F_1(t))\big) \doteq \sum_{s=0}^{\infty} \frac{1}{v^s} \frac{1}{s!} \,\mathrm{Tr}_{1,\ldots,s}\, f_s\, F_s(t,1,\ldots,s \mid F_1(t)), \quad (24)$$

where $f = (f_0, f_1, \ldots, f_n, \ldots) \in \mathfrak{L}_0(\mathcal{F}^{\pm}_{\mathcal{H}}) \in \mathfrak{L}(\mathcal{F}^{\pm}_{\mathcal{H}})$ is a finite sequence of degenerate bounded operators with infinitely times differentiable kernels with compact supports and elements of the sequence $F\big(t, \mid F_1(t)\big) \doteq \big(I, F_1(t,1), F_2(t,1,2 \mid F_1(t)), \ldots, F_s\big(t,1,\ldots,s \mid F_1(t)\big), \ldots\big)$ are defined by formulas (21) and (11) for the first and other elements correspondingly. If for functional (24) it is valid the equality:

$$\frac{d}{dt}\big(f, F(t \mid F_1(t))\big) = \big(\mathcal{B}^+ f, F(t \mid F_1(t))\big), \quad (25)$$

where $\mathcal{B}^+$ is the dual operator [22] with respect to the generator of the quantum BBGKY hierarchy (2), i.e.

$$(\mathcal{B}^+ f)_s(Y) \doteq \mathcal{N}_s(Y) f_s(Y) + \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{N}_{\mathrm{int}}(j_1,j_2) f_{s-1}(Y \backslash (j_1)), \quad s \geq 1,$$

then expansion (21) is a weak solution in extended meaning of the Cauchy problem (15), (16) of the generalized quantum kinetic equation for bosons or fermions.

To prove equality (25) we transform functional (24) in the following way:

$$\big(f, F(t \mid F_1(t))\big) = \sum_{s=0}^{\infty} \frac{1}{v^s} \frac{1}{s!} \,\mathrm{Tr}_{1,\ldots,s} \sum_{n=0}^{s} \frac{1}{(s-n)!} \times$$
$$\times \sum_{j_1 \neq \ldots \neq j_{s-n}=1}^{s} \;\sum_{Z \subset Y \backslash (j_1,\ldots,j_{s-n})} (-1)^{|Y \backslash (j_1,\ldots,j_{s-n}) \backslash Z|} \times$$

$$\times\mathcal{G}_{s-n+|Z|}(t,(j_1,\ldots,j_{s-n})\cup Z)\,f_{s-n}(j_1,\ldots,j_{s-n})\mathcal{S}_s^{\pm}\prod_{i=1}^{s}F_1^0(i),$$

where $\sum_{Z\subset Y\backslash(j_1,\ldots,j_{s-n})}$ is a sum over all subsets $Z\subset Y\backslash(j_1,\ldots,j_{s-n})$ of the set $Y\backslash(j_1,\ldots,j_{s-n})\subset(1,\ldots,s)$. For $F_1^0\in\mathfrak{L}^1(\mathcal{H})$ and bounded interaction potentials this functional exists. As a result for $f\in\mathfrak{L}_0(\mathcal{F}_{\mathcal{H}}^{\pm})$ the derivative of functional (24) over the time variable in the sense of the $*$-weak convergence in the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}}^{\pm})$ transforms to the functionals [22]

$$\frac{d}{dt}\big(f,F(t\mid F_1(t))\big)=\sum_{s=0}^{\infty}\frac{1}{v^s}\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\big(\mathcal{N}_s(Y)f_s(Y)+$$
$$+\sum_{j_1\neq j_2=1}^{s}\mathcal{N}_{\mathrm{int}}(j_1,j_2)f_{s-1}(Y\backslash(j_1))\big)\,F_s(t,Y\mid F_1(t)).$$

In the sense of above defined notion of a weak solution in extended meaning (25), the last equality means that for arbitrary initial data $F_1^0\in\mathfrak{L}^1(\mathcal{H})$ a weak solution of the initial-value problem of the generalized quantum kinetic equation (15) for bosons or fermions is determined by formula (21). $\square$

We emphasize that intensional Banach spaces for the description of states of infinite-particle systems, that means the description of kinetic dynamics or equilibrium states, are different from the exploit spaces [3], [4].

7 Conclusion

We have proved that in the case of initial data, which is completely defined by a one-particle density operator, all possible states of infinite-particle systems of bosons or fermions at an arbitrary moment of time can be described within the framework of a one-particle density operator, without any approximations together with explicitly defined functionals of this one-particle density operator. One of the advantages of such approach is the possibility to construct the kinetic equations in scaling limits the presence of correlations of particle states at initial time, for instance, correlations characterizing the condensate states of interacting particles obeying Fermi-Dirac or Bose-Einstein statistics [2].

Specific quantum kinetic equations, such as the Boltzmann equation and other, can be derived from constructed generalized quantum kinetic equation in the appropriate scaling limits or as a result of certain approximations. For example, in the mean-field scaling limit [5] we derive the quantum Vlasov kinetic equation for bosons or fermions, in particular in the case of pure states of fermions it reduces to the Hartree-Fock equation or the nonlinear Schrödinger equation.

Observing that in the kinetic (macroscopic) scale of the variation of variables [5] the groups of operators (7) of finitely many particles depend on microscopic time variable $\varepsilon^{-1}t$, where $\varepsilon \geq 0$ is a scale parameter, the dimensionless marginal functionals of the state are represented in the form: $F_s\big(\varepsilon^{-1}t, Y \mid F_1(t)\big)$. Then in the limit $\varepsilon \to 0$ the first two terms of the dimensionless marginal functional expansions (11)

$$
\begin{aligned}
&\widehat{\mathcal{G}}_s(\varepsilon^{-1}t, Y)\mathcal{S}_s^{\pm}\prod_{i=1}^{s} F_1(t,i) + \\
&+\frac{1}{v}\int_0^{\varepsilon^{-1}t} d\tau\, \mathcal{G}_s(-\tau, Y)\mathrm{Tr}_{s+1}\Big(\sum_{i_1=1}^{s}(-\mathcal{N}_{\mathrm{int}}(i_1, s+1))\widehat{\mathcal{G}}_{s+1}(\varepsilon^{-1}t, Y, s+1) - \\
&-\widehat{\mathcal{G}}_s(\varepsilon^{-1}t, Y)\sum_{i_1=1}^{s}(-\mathcal{N}_{\mathrm{int}}(i_1, s+1))\widehat{\mathcal{G}}_2(\varepsilon^{-1}t, i_1, s+1)\Big)\mathcal{S}_{s+1}^{\pm}\times \\
&\times\prod_{i_2=1}^{s+1}\mathcal{G}_1(\tau, i_2)F_1(t, i_2)
\end{aligned}
$$

coincide with corresponding terms constructed by the perturbation method with the use of the weakening of correlation condition by Bogolyubov [4]. Thus, in the kinetic scale, the collision integral of the generalized kinetic equation (15) for bosons or fermions takes the form of Bogolyubov's collision integral [23] and we observe that in a space homogeneous case, the collision integral of the first approximation has a more general form than the collision integral in the Uehling-Uhlenbeck kinetic equation [24], [25].

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УЗАГАЛЬНЕНЕ КВАНТОВЕ КІНЕТИЧНЕ РІВНЯННЯ ДЛЯ ВЗАЄМОДІЮЧИХ ЧАСТИНОК З КВАНТОВИМИ СТАТИСТИКАМИ

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Для початкових станів, які визначаються одночастинковим маргінальним оператором густини, квантових багаточастинкових систем, що задовольняють статистиці Фермі-Дірака або Бозе-Ейнштейна, встановлено еквівалентність задачі Коші для квантової ієрархії ББГКІ та задачі Коші для узагальненого квантового кінетичного рівняння для бозонів та ферміонів. В просторі ядерних операторів доведено існування сильного та слабкого розв'язку задачі Коші для сформульованого квантового кінетичного рівняння.